N-Homeomorphism and N*-Homeomorphism in supra Topological spaces

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ABSTRACT: In this paper, we introduce the concept of strongly supra N-continuous function and perfectly supra N-continuous function and studied its basic properties. Also we introduce the concept of supra N-Homeomorphism and supra N*-Homeomorphism. We obtain the basic properties and the relationship with supra N-closed maps, supra N-continuous maps and supra N-irresolute maps in supra topological spaces.

KEYWORDS: supra N-Homeomorphism, supra N*-Homeomorphism, strongly supra N-continuous function, perfectly supra N-continuous function.

I. INTRODUCTION

In 1983, A.S. Mashhour et al [6] introduced the supra topological spaces and studied, continuous functions and s’ continuous functions. R. Devi [2] have studied generalization of homeomorphisms and also have introduced α-homeomorphisms in topological spaces.

In this paper, we introduce the concept of strongly supra N-continuous function and perfectly supra N-continuous function and studied its basic properties. Also we introduce the concept of supra N-homeomorphism and supra N*-Homeomorphism in supra topological spaces.

II. PRELIMINARIES

Definition 2.1[6]
A subfamily µ of X is said to be supra topology on X if
i) X, φ ∈ µ
ii) If A i ∈ µ ∀ i ∈ j then ∪ A i ∈ µ. (X, µ) is called supra topological space.
The element of µ are called supra open sets in (X, µ) and the complement of supra open set is called supra closed sets and it is denoted by µ^c.

Definition 2.2[6]
The supra closure of a set A is denoted by cl µ(A), and is defined as supra cl(A) = ∩ {B : B is supra closed and A ⊆ B}.
The supra interior of a set A is denoted by int µ(A), and is defined as supra int(A) = ∪ {B : B is supra open and A ⊆ B}.

Definition 2.3[6]
Let (X, τ) be a topological space and µ be a supra topology on X. We call µ a supra topology associated with τ, if τ ⊆ µ.

Definition 2.4[5]
Let (X, µ) be a supra topological space. A set A of X is called supra semi-open set, if A ⊆ cl µ(int µ(A)). The complement of supra semi-open set is supra semi-closed set.

Definition 2.5[4]
Let (X, µ) be a supra topological space. A set A of X is called supra α-open set, if A ⊆ int µ(cl µ(int µ(A))). The complement of supra α-open set is supra α-closed set.

Definition 2.6[7]
Let (X, µ) be a supra topological space. A set A of X is called supra Ω closed set, if sc int µ(A) ⊆ int µ(U), whenever A ⊆ U, U is supra open set. The complement of the supra Ω closed set is supra Ω open set.
Definition 2.7[7]
The supra $\mathcal{C}H$ closure of a set $A$ is denoted by $\mathcal{C}H(A)$, and defined as $\mathcal{C}H(A) = \cap \{B: B$ is supra $\mathcal{C}$ closed and $A \subseteq B\}$. The supra $\mathcal{C}$ interior of a set $A$ is denoted by $\mathcal{C}int(A)$, and defined as $\mathcal{C}int(A) = \cup \{B: B$ is supra $\mathcal{C}$ open and $A \supseteq B\}.$

Definition 2.8
Let $(X, \mu)$ be a supra topological space. A set $A$ of $X$ is called supra regular open if $A = int^{\mu}(cl^{\mu}(A))$ and supra regular closed if $A = cl^{\mu}(int^{\mu}(A))$.

Definition 2.9[9]
Let $(X, \mu)$ be a supra topological space. A set $A$ of $X$ is called supra $N$-closed if $\mathcal{C}H(A) \subseteq U$, whenever $A \subseteq U$, $U$ is supra $\alpha$ open set. The complement of supra $N$-closed set is supra $N$-open set.

Definition 2.10[9]
The supra $N$ closure of a set $A$ is denoted by $Ncl^{H}(A)$, and defined as $Ncl^{H}(A) = \cap \{B: B$ is supra $N$-closed and $A \subseteq B\}$. The supra $N$ interior of a set $A$ is denoted by $Nint^{H}(A)$, and defined as $Nint^{H}(A) = \cup \{B: B$ is supra $N$-open and $A \supseteq B\}.$

Definition 2.11[9]
Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called supra $N$-continuous function if $f^{-1}(\sigma)$ is supra $N$-closed in $(X, \tau)$ for every supra closed set $V$ of $(Y, \sigma)$.

Definition 2.12[9]
Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called supra $N$-irresolute if $f^{-1}(\sigma)$ is supra $N$-closed in $(X, \tau)$ for every supra $N$-closed set $V$ of $(Y, \sigma)$.

Definition 2.13[10]
A map $f:(X, \tau) \rightarrow (Y, \sigma)$ is called supra $N$-closed map(resp. supra $N$-open) if for every supra closed(resp. supra open) $F$ of $X$, $f(F)$ is supra $N$-closed(resp. supra $N$-open) in $Y$.

Definition 2.14[10]
A map $f:(X, \tau) \rightarrow (Y, \sigma)$ is said to be almost supra $N$-closed map if for every supra regular closed $F$ of $X$, $f(F)$ is supra $N$-closed in $Y$.

Definition 2.15[10]
A map $f:(X, \tau) \rightarrow (Y, \sigma)$ is said to be strongly supra $N$-closed map if for every supra $N$ closed $F$ of $X$, $f(F)$ is supra $N$-closed in $Y$.

Definition: 2.16[10]
A supra topological space $(X, \tau)$ is $T^\mu_N$ - space if every supra $N$-closed set in it is supra closed.

III. SOME FORMS OF SUPRA $N$-CONTINUOUS FUNCTIONS

Definition 3.1
A map $f:(X, \tau) \rightarrow (Y, \sigma)$ is called strongly supra $N$-continuous function if the inverse image of every supra $N$-closed set in $(Y, \sigma)$ is supra closed in $(X, \tau)$.

Definition 3.2
A map $f:(X, \tau) \rightarrow (Y, \sigma)$ is called perfectly supra $N$-continuous function if the inverse image of every supra $N$-closed set in $(Y, \sigma)$ is both supra open and supra closed in $(X, \tau)$.

Theorem 3.3
Every perfectly supra $N$-continuous function is strongly supra $N$-continuous function.

Proof
Let $f:(X, \tau) \rightarrow (Y, \sigma)$ be a perfectly $N$-continuous function. Let $V$ be $N$-closed set in $(Y, \sigma)$. Since $f$ is perfectly $N$-continuous function $f^{-1}(V)$ is both supra open and supra closed in $(X, \tau)$. Therefore $f$ is strongly supra $N$-continuous function.

The converse of the above theorem need not be true. It is shown by the following example.

Example 3.4
Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, $\sigma = \{Y, \emptyset, \{a, b\}\}$.
Theorem 3.5
Let \( f : (X, \tau) \to (Y, \sigma) \) be strongly supra N-continuous and \( g : (Y, \sigma) \to (Z, \upsilon) \) be strongly supra N-continuous then their composition \( g \circ f : (X, \tau) \to (Z, \upsilon) \) is a strongly supra N-continuous function.

Proof
Let \( V \) be supra N-closed set in \((Z, \upsilon)\). Since \( g \) is strongly N-continuous, \( g^{-1}(V) \) is supra N-closed in \((Y, \sigma)\). We know that every supra closed set is supra N-closed set, \( g^{-1}(V) \) is supra N-closed in \((Y, \sigma)\). Since \( f \) is strongly N-continuous, \( f^{-1}(g^{-1}(V)) \) is supra closed in \((X, \tau)\), implies \( (g \circ f)(V) \) is supra closed in \((X, \tau)\). Therefore \( g \circ f \) is strongly N-continuous.

Example 3.6
Let \( X = Y = Z = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\} \), \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}, \upsilon = \{Z, \phi, \{ab\}, \{bc\}\} \). \( f : (X, \tau) \to (Y, \sigma) \) be the function defined by \( f(a) = b, f(b) = c, f(c) = a \). Here \( f \) and \( g \) are strongly supra N-continuous and \( g \circ f \) is also strongly supra N-continuous function.

IV. SUPRA N-HOMEOMORPHISM AND SUPRA N'-HOMEOMORPHISM

Definition 4.1
A bijection \( f : (X, \tau) \to (Y, \sigma) \) is called supra N-Homeomorphism if \( f \) is both supra N-continuous function and supra N-closed map (\( f^{-1} \) is N-continuous function).

Definition 4.2
A bijection \( f : (X, \tau) \to (Y, \sigma) \) is called supra N'-Homeomorphism if \( f \) and \( f^{-1} \) are supra N-irresolute.

Theorem 4.3
Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijective supra N-continuous map. Then the following are equivalent
1) \( f \) is an N-open map
2) \( f \) is an N-homeomorphism
3) \( f \) is an N-closed map.

Proof
(i) \( \Rightarrow \) (ii): If \( f \) is a bijective supra N-continuous function, suppose (i) holds. Let \( V \) be supra closed in \((X, \tau)\) then \( V' \) is supra open in \((X, \tau)\). Since \( f \) is supra N-open map, \( f(V') \) is supra N-open in \((Y, \sigma)\). Hence \( f(V) \) is supra N-closed in \((Y, \sigma)\) implies \( f^{-1} \) is supra N-continuous. Therefore \( f \) is an supra N-homeomorphism.

(ii) \( \Rightarrow \) (iii): Suppose \( f \) is an supra N-homeomorphism and \( f \) is bijective supra N-continuous function then from the definition4.1, \( f^{-1} \) is supra N-continuous, implies \( f \) is supra N-closed map.

(iii) \( \Rightarrow \) (i): Suppose \( f \) is supra N-closed map. Let \( V \) be supra open in \((X, \tau)\) then \( V' \) is supra closed in \((X, \tau)\). Since \( f \) is supra N-closed map, \( f(V') \) is supra N-closed in \((Y, \sigma)\). Hence \( f(V) \) is supra N-open in \((Y, \sigma)\). Therefore \( f \) is an supra N-open map.

Remark 4.4
The composition of two supra N-Homeomorphism need not be an supra N-Homeomorphism. Since composition of two supra N-continuous function need be supra N-continuous and composition of two supra N-closed map need not be supra N-closed map. It is seen from the following example

Example 4.5
Let \( X = Y = Z = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{c\}\}, \sigma = \{Y, \phi, \{a\}, \{b\}, \{c\}\}, \upsilon = \{Z, \phi, \{a\}, \{b\}, \{c\}\} \). \( f : (X, \tau) \to (Y, \sigma) \) be the function defined by \( f(a) = b, f(b) = c, f(c) = a \). and \( g : (Y, \sigma) \to (Z, \upsilon) \) be the function defined by \( g(a) = b, g(b) = c, g(c) = a \). Here \( f \) and \( g \) is supra N-Homeomorphism, but its composition is not supra N-closed, since \( g \circ f \{b, c\} = \{a, b\} \) is not supra N-closed in \( Z \). Therefore \( g \circ f \) is not an supra N-Homeomorphism.

Theorem 4.6
Every supra N-Homeomorphism is supra N-continuous.

Proof
It is obvious from the definition 4.1.

The converse of the above theorem need not be true. It is shown by the following example.

Example 4.7
Let \( X = Y = Z = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{c\}\}, \sigma = \{Y, \phi, \{a\}, \{b\}, \{c\}\} \). \( f : (X, \tau) \to (Y, \sigma) \) be the function defined by \( f(a) = b, f(b) = c, f(c) = a \). Here \( f \) is supra N-continuous but not supra N-Homeomorphism, since \( f^{-1} \) is not supra N-continuous.
Theorem 4.8
Every supra N'-Homeomorphism is supra N-irresolute.
Proof
It is obvious from the definition 4.2.
The converse of the above theorem need not be true. It is shown by the following example.

Example 4.9
Let X=Y={a, b, c} and \( \tau = \{X, \phi, \{a\}, \{b, c\} \} \), \( \sigma = \{Y, \phi, \{a, b\}, \{b, c\} \} \). \( f(X, \tau) \rightarrow (Y, \sigma) \) is the function defined by \( f(a)=b, f(b)=c, f(c)=a \). Here \( f \) is supra N-irresolute but not supra N'-Homeomorphism, since \( f^{-1} \) is not supra N-irresolute.

Theorem 4.10
If \( f(X, \tau) \rightarrow (Y, \sigma) \) and \( g(Y, \sigma) \rightarrow (Z, \nu) \) are supra N'-Homeomorphism then the composition \( g\circ f \) is also supra N'-Homeomorphism.
Proof
Let \( V \) be a supra N-closed set in \((Y, \sigma)\). Since \( g \) is supra N'-Homeomorphism \( g \) and \( g^{-1} \) are supra N-irresolute, then \( g^{-1}(V) \) is supra N-closed set in \((Y, \sigma)\). Now \( (g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \). Since \( f \) is N'-Homeomorphism \( f \) and \( f^{-1} \) are supra N-irresolute, then \( f^{-1}(g^{-1}(V)) \) is supra N-closed set in \((X, \tau)\). Thus \( g\circ f \) is supra N-irresolute.
For an supra N-closed set \( V \) in \((X, \tau)\), \( (g\circ f)(V) = g(f(V)) \). By Hypothesis \( f \) is supra N-closed set in \((Y, \sigma)\). Thus \( g(f(V)) \) is supra N-closed set in \((Z, \nu)\). Hence \( (g\circ f)^{-1} \) is supra N-irresolute. Therefore \( g\circ f \) is supra N'-Homeomorphism.

Theorem 4.11
If \( f(X, \tau) \rightarrow (Y, \sigma) \) is a supra N'-Homeomorphism then \( \text{Ncl}(f^{-1}(B)) = f^{-1}(\text{Ncl}(B)) \), for every \( B \subseteq Y \) is supra N-closed.
Proof
Since \( f \) is supra N'-Homeomorphism \( f \) and \( f^{-1} \) are supra N-irresolute. Let \( B \) be supra N-closed set in \((Y, \sigma)\). Since \( f \) is supra N-irresolute \( f^{-1}(B) \) is supra N-closed set in \((X, \tau)\). Since \( B \) is supra N-closed set, \( B = \text{Ncl}(B) \). Therefore \( f^{-1}(\text{Ncl}(B)) \) is supra N-closed set in \((X, \tau)\). Since \( f^{-1}(B) \) is supra N-closed set, \( \text{Ncl}(f^{-1}(B)) = f^{-1}(\text{Ncl}(B)) \) is supra N-closed set in \((X, \tau)\).

Theorem 4.12
If \( f(X, \tau) \rightarrow (Y, \sigma) \) is a supra N'-Homeomorphism then \( \text{Ncl}(f(B)) = f(\text{Ncl}(B)) \), for every \( B \subseteq X \) is supra N-closed.
Proof
Since \( f \) is supra N'-Homeomorphism \( f \) and \( f^{-1} \) are supra N-irresolute. Let \( B \) be supra N-closed set in \((X, \tau)\). Since \( f \) is supra N-irresolute \( f^{-1}(B) \) is supra N-closed set in \((Y, \sigma)\). Since \( B \) is supra N-closed set, \( B = \text{Ncl}(B) \). Therefore \( \text{Ncl}(f(B)) \) is supra N-closed set in \((Y, \sigma)\). Since \( f(B) \) is supra N-closed set, \( \text{Ncl}(f(B)) = f(B) \) is supra N-closed set in \((Y, \sigma)\). Therefore \( \text{Ncl}(f(B)) = f(\text{Ncl}(B)) \) is supra N-closed set in \((Y, \sigma)\).

Theorem 4.13
Every supra N'-Homeomorphism is strongly supra N-closed map.
Proof
Since \( f(X, \tau) \rightarrow (Y, \sigma) \) is supra N'-Homeomorphism \( f \) and \( f^{-1} \) are supra N-irresolute. \( f^{-1} \) is supra N-irresolute implies \( f \) is strongly supra N-closed map.
The converse of the above theorem need not be true. It is shown by the following example.

Example 4.14
Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b, c\} \} \), \( \sigma = \{Y, \phi, \{a, b\}, \{b, c\} \} \). \( f(X, \tau) \rightarrow (Y, \sigma) \) is the function defined by \( f(a)=b, f(b)=c, f(c)=a \). Here \( f \) is strongly supra N-closed but not supra N'-Homeomorphism, since \( f^{-1} \) is supra N-irresolute(strongly supra N-closed map) but \( f \) is not supra N'-Homeomorphism.

Theorem 4.15
If \( (X, \tau) \) and \( (Y, \sigma) \) is a \( T^\mu_N \) space, and if \( f(X, \tau) \rightarrow (Y, \sigma) \) is a supra N-continuous function then \( f \) is supra N-Homeomorphism.
Proof
Let \( V \) be supra closed in \((Y, \sigma)\). Since \( f \) is supra N-continuous \( f^{-1}(V) \) is supra N-closed in \((X, \tau)\). Since \( (X, \tau) \) and \( (Y, \sigma) \) is a \( T^\mu_N \) space, then every supra N-closed set is supra closed set. Let \( B \) be supra closed set in \((X, \tau)\), then \( f(B) \) is supra N-closed in \((Y, \sigma)\), implies \( f \) is N-closed map \( f^{-1} \) is N-continuous function. Hence \( f \) is N-Homeomorphism.

Theorem 4.16
The set \( N^*\text{-h}(X, \tau) \) from \( (X, \tau) \) to itself is a group under the composition of maps.
Proof
Let \( f, g \in N^*\text{-h}(X, \tau) \), then by theorem 4.10 \( g \circ f \in N^*\text{-h}(X, \tau) \), we know that the composition of mapping is associative and the identity \( I: (X, \tau) \rightarrow (X, \tau) \) belonging to \( N^*\text{-h}(X, \tau) \) serves as the Identity element.
If \( f \in N^-h(X, \tau) \) then \( f^{-1} \in N^-h(X, \tau) \) such that \( fof^{-1} = f^{-1}of = I \). Therefore inverse exists for each element of \( N^-h(X, \tau) \). Hence \( N^-h(X, \tau) \) is a group under composition of maps.

**REFERENCE**


