International Journal Of Mathematics And Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
Www.Ijmsi.Org || Volume 2 Issue 10 || November. 2014 || PP-16-18

The Number of Transitive *P* – Groups Of Degree P^3

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ABSTRACT: *In this paper we determine the number of transitive p-groups of degree* p^3

KEYWORDS: *number, transitive, degree, exponent, isomorphism.*

1.1 INTRODUCTION

From our previous result in [2], we can easily deduce that for n ≥ 7, there are, up to isomorphism, 2 non – abelian transitive *p* – groups of degree p^3, exponent p^3 and order p^n while for n = 4, 5 and 6, we have up to isomorphism, one such group.

I. RESULTS

1.2 Proposition: For each odd prime *p*, there are, up to isomorphism, 5 transitive *p* – groups of degree p^3 and order p^3. 3 of these are abelian and of the remaining 2 non – abelian groups, 1 is of exponent p^2 and 1 is of exponent *p*.

Proof: Since each of these groups is transitive, we must have:
$|\alpha^G| = p^3$, $|G_\alpha| = 1$, $\forall\ \alpha \in \Omega$ with $|\Omega| = p^3$,
thus G is regular and two cases arise: (i) G abelian and (ii) G non – abelian.
If G is abelian, then as G is of degree p^3, we have the following possibilities: either

$$G \cong C_{P^3} \text{ or } G \cong C_{P^2} \times C_P \text{ or } G \cong C_p \times C_p \times C_p$$

If G is non – abelian, then as G is a *p* – group of order p^3, it contains a normal *p* – subgroup H of order p^2 which must be abelian, and so either $H \cong C_{p^2}$ or $H \cong C_p \times C_p$. Thus G ≅ < a, b >, where a ∈G is such that

$$a^{p^2} = 1, b \in G - \langle a \rangle, b^p = 1, b^{-1}ab \in \langle a \rangle$$

or $G \cong G^{//}$ = < a, b, c >, where a, b, c ∈G are such that

$$a^p = 1,\ b^p = 1,\ c \in G - < a,\ b >,\ c^p = 1,\ c^{-1}ac,\ c^{-1}bc \in G - < a, b,\ c >,\ c^{-1}ac \neq a,\ c^{-1}bc \neq b$$. This completes the proof.

1.3 Proposition

For each odd prime *p*, there are, up to isomorphism, 5 non – abelian transitive *p* – groups of degree p^3 and exponent *p*.

Proof: Let G be a non – abelian transitive *p* – group of degree p^3 and exponent *p*. Then by [2] we must have $|G| = p^n$, n = 3, 4, 5.
If $|G| = p^3$, then by [2], $G \cong G_1$ = < a, b, c >, where a, b ∈ G are such that

$$a^p = 1,\ b^p = 1,\ c \in G - < a,\ b >,\ c^p = 1,\ c^{-1}ac,\ c^{-1}bc \in G - < a, b,\ c >,\ c^{-1}ac \neq a,\ c^{-1}bc \neq b.$$

If $|G| = p^4$, then G contains a normal *p* – subgroup H of order p^3 which is either abelian in which case $H \cong C_p \times C_p \times C_p$, and so $G \cong G_2$ = < a, b, c, d >, where a, b, c ∈ G are such that

$$a^p = 1, b^p = 1, c^p = 1,\ d \in G - \langle a \rangle \times \langle b \rangle \times \langle c \rangle,\ d^p = 1,\ d^{-1}ad,\ d^{-1}bd,\ d^{-1}cd \in \langle a \rangle \times \langle b \rangle \times \langle c \rangle,$$

$$d^{-1}ad \neq a,\ d^{-1}bd \neq b,\ d^{-1}cd \neq c.$$

or non – abelian in which case $H \cong G_1$, so that $G \cong G_3 < G_1, d >$, where $d \in G - G_1$ is such that $d^p = 1$, $G_1 \trianglelefteq G_3$.
If $|G| = p^5$, then G contains a normal *p* – subgroup H of order p^4. Suppose H is non – abelian, then by the case $|G| = p^4$ above, either $H \cong G_2$ or $H \cong G_3$. These yield the following possibilities for G: either $G \cong < G_2, e >$,

where $e \in G - G_2$, $e^{p} = 1$, $G_2 \trianglelefteq G$, or $G \cong \langle G_3, f \rangle$, where $f \in G - G_3$, $f^{p} = 1$, $G_3 \trianglelefteq G$. If H were abelian, we would have
$H \cong C_p \times C_p \times C_p \times C_p$ or $H \cong \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, where $a^{p} = 1$, $b^{p} = 1$, $c^{p} = 1$, $d^{p} = 1$. But $C(a) = \langle a \rangle$, $C(b) = \langle b \rangle$, $C(c) = \langle c \rangle$, and $C(d) = \langle d \rangle$, so that
$\langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle = C(a) \cap C(b) \cap C(c) \cap C(d)$, and hence
$|H| = p^{4} = |\langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle| = |C(a) \cap C(b) \cap C(c) \cap C(d)| \leq |C(a)| = p$, which is impossible. Thus H is not abelian.

1.4 Proposition : For each odd prime p, there are, up to isomorphism, ($2p(p+1) - 7$) non – abelian transitive p – groups of degree p^3 and exponent p^3.

Proof: Since G is transitive of degree p^3, two cases arise: either G contains exactly one generator of order p^3 and the remaining generators are each of order p (by [1]) or G contains no generator of order p^3 but 3 generators each of order p^2 and other generators each of order p. By [2], such a group G exists only for n = 4, 5, . . . , $p(p+1)+1$, with $|G| = p^{n}$ and (by our opening remark in **1.1** above), for each n = 7, 8, . . . , $p(p+1)+1$, we have two non – isomorphic such groups. Thus their total number is $2[p(p+1)+1-6]$. And for each n = 4, 5, 6, there is, up to isomorphism, one such group. So that the total number of non – abelian transitive p – groups of degree p^3 and exponent p^3 is $2[p(p+1)+1-6] + 3 = 2p(p+1)-7$

1.5 Remark: (i) For the case $p = 2$, we have, up to isomorphism, 6 non – abelian transitive 2 – groups of degree $2^3 = 8$ and exponent $2^3 = 8$, by [1]. (ii) From Proposition1.2 we deduce that there are, up to isomorphism, 2 non – abelian transitive p – groups of degree p^3 and exponent p: one containing the unique abelian transitive p – group $C_p \times C_p \times C_p$ as a normal subgroup and the other containing the unique non – abelian transitive p – group of exponent p mentioned in the Lemma. But by [2], for any such a group G, the rank $r(G) = 3$, 4 or 5. Of interest to us is the case $r(G) = 5$. By [1], if $G^{/}$ is a transitive p – group of degree p^3 and rank 6 containing G as a normal subgroup, then $G^{/}$ must be of exponent p^2. Consequently, we obtain two new non – isomorphic non – abelian transitive p – groups of degree p^3, exponent p^2 and order p^6 which are not generated by any generators of order p^2.
Applying the same argument above to our result in[2], we get two new non – isomorphic non – abelian transitive p – groups of degree p^3, exponent p^2 and order p^7 (obtained from $G^{/}$) with one isomorphic to the unique non – abelian transitive p – groups of degree p^3, exponent p^2 and order p^7 generated by a generator of order p^2 . An important implication of the above is that there is no non – abelian transitive p – group G of degree p^3, exponent p and rank (G) with $6 \leq \text{rank}(G) \leq p(p+1)$.

1.6 Proposition : For each odd prime p, there are, up to isomorphism, ($p(p+1) + 2$) non – abelian transitive p – groups of degree p^3 and exponent p^2.

Proof: If G is a non – abelian transitive p – groups of degree p^3 and exponent p^2, then either G is generated by a generator of order p^2 or not. In the first case, by Remark1.5, for each
n = 3, 4, . . . , $p(p+1)+1$ with $|G| = p^{n}$, there is, up to isomorphism, one such group. Hence their total number is $p(p+1)+1-2 = p(p+1)-1$.
In the second case, by Remark1.5., we must have $|G| = p^6$ or $|G| = p^7$. In the first case,
we have 2 non – isomorphic non – abelian such groups (by Remark1.5(ii)) and in the second case, we have 1 such group, up to isomorphism. Hence in all, we have
$p(p+1) - 1 + 2 + 1 = p(p+1) + 2$ such groups./.

1.7 Remark : For the case $p = 2$, we have, up to isomorphism, 10 non – abelian transitive 2 – groups of degree $2^3 = 8$ and exponent $2^2 = 4$, by [1].
For non – abelian transitive p – groups of degree p^3, we have:

1.8 Theorem : For each odd prime p, there are, up to isomorphism, $3(p(p+1)+1)$ different transitive p – groups of degree p^3. Three of these are abelian. Of the $3p(p+1)$ non – abelian groups, we have that $2p(p+1) - 7$ are of exponent p^3, $p(p+1) + 2$ are of exponent p^2 and the remaining 5 are of exponent p.

Proof : By Proposition1.4., the number of non – abelian transitive p – groups of degree p^3 and exponent p^3 is ($2p(p+1) - 7$),
And by Proposition1.6, the number non – abelian transitive p – groups of degree p^3 and exponent p^2 is $p(p+1) + 2$,

By [1]., the number of non – abelian transitive *p* – groups of degree p^3 and exponent *p* is 5 while by Proposition1.2, there are three abelian such groups. Adding these numbers together we get the result.

1.9 Remark: For the case $p = 2$, we have, up to isomorphism, 19 transitive 2 – groups of degree $2^3 = 8$ by [1].

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