International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767, P-ISSN: 2321 – 4759
www.Ijmsi.Org || Volume 2 Issue 10 || November. 2014 || PP.19-23

A Fifth-Order Iterative Method for Solving Nonlinear Equations

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ABSTRACT : *In this paper, we suggest a new one-step, fifth-order iterative method for solving nonlinear equations, which is an improvement on methods introduced by Hosseini [10]. Several numerical examples are given and compared to other well known methods of the same order, illustrating the efficiency and performance of the proposed method.*

KEYWORDS: *Taylor's series expansion, Non-linear equation, Computational Local Order of Convergence, Iterative methods.*

I. INTRODUCTION

We consider iterative methods to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function and continuously differentiable, and I is a neighborhood of the root α. By iterative method we mean a sequence $\{x_n\}_{n\in N}$, defined by

$$x_{n+1} = \varphi\left(x_n; x_{n-1}, x_{n-2}; \dots, x_{n-j}\right), \qquad n \geq 0 \tag{1}$$

where φ is the iteration function.

There are several existing methods to compute the root α of a nonlinear equation $f(x) = 0$ (see [1-16] and the reference therein). The most famous of these methods is the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

starting from some initial value x_0 with convergence order 2. This method is based on Taylor's series expansion. Hosseini [10] extended the procedure of obtaining (2) and obtained a cubic iterative method of order 3 given as:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \quad n = 0,1,2,\dots \tag{3}$$

and a quartic iterative methods of order 4 given as

$$x_{n+1} = x_n - \left.\frac{f(2f'^2 - ff'')^2}{f^i\left((f'^2 - ff'')(2f'^2 - 2ff'') + \frac{2}{3}f^2 f' f'''\right)}\right|_{x=x_n}, \quad n = 0,1,2,\dots \tag{4}$$

The methods (3) and (4) converges from both left side $(x_0 < \alpha)$ and right side $(x_0 > \alpha)$ towards the root α when tested for most functions, whereas the other methods work usually well from one side only. In this work, we introduced a new iterative method of order 5 by extending (3) and (4).

II. DEVELOPMENT OF THE METHOD

Let $f(x) = 0$ be a nonlinear equation. The Taylor's series expansion around a given initial point $x = \theta$, assuming θ being close enough to the simple root $x = \alpha$ is given as follows:

$$f(x) = f(\theta) + f'(\theta)(x-\theta) + \frac{f''(\theta)}{2!}(x-\theta)^2 + \frac{f'''(\theta)}{3!}(x-\theta)^3 + \frac{f''''(\theta)}{4!}(x-\theta)^4 + HOT = 0 \tag{5}$$

where HOT denotes the higher order terms. Then the nonlinear equation becomes

$$f(x) = f(\theta) + f'(\theta)(x-\theta) + \frac{f''(\theta)}{2!}(x-\theta)^2 + \frac{f'''(\theta)}{3!}(x-\theta)^3 + \frac{f''''(\theta)}{4!}(x-\theta)^4 + HOT \tag{6}$$

When θ is close enough to α, equation (6) becomes

$$f(x) = f(\theta) + f^{'}(\theta)(x-\theta) + \frac{f^{''}(\theta)}{2!}(x-\theta)^2 + \frac{f^{'''}(\theta)}{3!}(x-\theta)^3 + \frac{f^{''''}(\theta)}{4!}(x-\theta)^4 \approx 0 \qquad (7)$$

hence;

$$f(\theta) + f^{'}(\theta)(x-\theta) + \frac{f^{''}(\theta)}{2!}(x-\theta)^2 + \frac{f^{'''}(\theta)}{3!}(x-\theta)^3 + \frac{f^{''''}(\theta)}{4!}(x-\theta)^4 \approx 0 \qquad (8)$$

$$f(\theta) + (x-\theta)\left[f^{'}(\theta) + \frac{f^{''}(\theta)}{2!}(x-\theta) + \frac{f^{'''}(\theta)}{3!}(x-\theta)^2 + \frac{f^{''''}(\theta)}{4!}(x-\theta)^3\right] \approx 0 \qquad (9)$$

The relation for obtaining (4) in Hosseini [10] is given as:

$$(x-\theta) \approx \frac{-3f(\theta)[2f^{'2}(\theta) - f(\theta)f^{''}(\theta)]^2}{12f^{'5}(\theta) + 6f^2(\theta)f^{'}(\theta)f^{''2}(\theta) - 18f(\theta)f^{'3}(\theta)f^{''}(\theta) + 2f^2(\theta)f^{'2}(\theta)f^{'''}(\theta)} \qquad (10)$$

Using the relation (10) in (9) we have the algorithm stated below:

Algorithm 1.1

Assume that the function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ has a single root $\alpha \in I$, where I is an open interval. Assume furthermore that f(x) is a sufficiently differentiable function in the neighborhood of α, and let $x_0 = \theta$ be close enough to α a simple zero of $f(x)$, the approximate solution x_{n+1} by the one step iteration scheme is achieved by the following steps:

INPUT initial approximation x_0; tolerance ε and maximum number of iteration N_{max}.
OUTPUT number of Iteration N and approximate solution x_{n+1}, or a message of failure.
Step 1: Set $n = 0$ and $N = 1$
Step 2: While $N \leq N_{max}$ do steps 3 to 5
Step 3: Calculate

$$j_n = f^{'}f^2f^{'''} \qquad (11a)$$

$$L_n = 2f^{'2} - 2ff^{''} \qquad (11b)$$

$$y_n = 2f^{'2} - ff^{''} \qquad (11c)$$

$$z_n = y_nL_n + \tfrac{2}{3}j_n \qquad (11d)$$

$$x_{n+1} = x_n - \left.\frac{ff^{'3}z^3}{f^{'4}z^3 - \frac{1}{2}y^2ff^{'2}f^{''}z^2 + \frac{1}{6}y^4f^2f^{'}f^{'''}z - \frac{1}{24}y^6f^3f^{''''}}\right|_{x=x_n} \qquad (11e)$$

Step 4 If $|x_{n+1} - x_n| < \varepsilon$, then OUTPUT (x_{n+1}); stop.
Step 5 Set $n = n+1, N = N+1$ and go to Step 2.
Step 6: OUPUT (“Method failed after N_{max} iterations, N_{max} =” N_{max}.

It will be shown that the computational order of convergence of the proposed method is five and hence it has fifth-order convergence.

III. COMPUTATIONAL ORDER OF CONVERGENCE OF THE METHOD

After the work of Weerakoon and Fernando [14], many other authors have considered the Computational Order of Convergence (COC) in their research (see Grau and Noguera [5], and Grau-Sanchez, Noguera and Gutierrez [8] and references therein). In all those papers the COC is used to test numerically the order of convergence of the methods presented. In view of this, Grau-Sanchez et al. [4] provided a new parameter with low cost than COC. Here, we also established the order of convergence of (11) by computation. Since the proposed method is a one-point iterative method for solving nonlinear equation $f(x) = 0$, it will generate a sequence of approximation of the root $\{x_n\}_{n\in N}$, defined by (1) which we uses in determining its order.

Definition 1. (See Dennis and Schnable [1]) Let $\alpha \in \mathbb{R}$, $x_n \in \mathbb{R}$, $n = 0,1,2, \ldots$ Then, the sequence $\{x_n\}$ is said to converge to α if

$$\lim_{n\to\infty}|x_n - \alpha| = 0 \qquad (12)$$

If, in addition, there exists a constant $c \geq 0$, an integer $x_0 \geq 0$, and $p \geq 0$ such that for all $n \geq x_0$,

$$|x_{n+1} - \alpha| \leq c|x_n - \alpha|^p \qquad (13)$$

then $\{x_n\}$ is said to converge to α with q-order at least p. If $p = 5$, the convergence is said to be of order 5.

Definition 2 (See Grau-Sanchez et al. **[4])** The computational local order of convergence, $\overline{\rho_n}$, (CLOC) of a sequence $\{x_n\}_{n\geq 0}$ is defined by

$$\overline{\rho_n} = \frac{log|e_n|}{log|e_{n-1}|}, \tag{14}$$

where x_{n-1} and x_n are two consecutive iterations near the roots α and $e_n = x_{n-1} - \alpha$.

The local order of convergence of an iterative method in a neighborhood of a root is the order of the corresponding sequence. If it is ρ, then the method approximately multiplies by ρ the number of correct decimals after each iteration. That is, from (14) we get $\log_{10}|e_n| \approx \overline{\rho_n}\log_{10}|e_{n-1}|$, for n large enough. Grau-Sanchez et al [4] have shown that (12) is a variant of Computational Order of Convergence (COC) and for sequence $\{x_n\}$ converging to α, with starting points $x_{-j}, \dots, x_{-1}, x_0$ close enough to α, the value of $\overline{\rho_n}$ converges to ρ (the order of the method), when $n \to \infty$.

IV. NUMERICAL EXAMPLES

The accuracy of our contribution is tested on numerous numerical problems. Our goal is fulfilled in this section by comparison of our method with the other existing fifth-order methods. These include Kou and Li method (KM) [11] given by

$$x_{n+1} = z_n - \left(1 + \frac{M(x_n)}{1+M(x_n)}\right)\frac{f(z_n)}{f'(z_n)} \tag{15}$$

where

$$t(x_n) = \frac{f''(x_n)f(f'(x_n))}{f^{i2}(x_n)} \tag{16}$$

$$M(x_n) = \frac{f''(x_n)(f(x_n)-f(z_n))}{f'^2(x_n)} \tag{17}$$

YoonMee and Changbum's method (YCM) [15] with $D = -1, A = 1, B = 3, C = 5$ is defined by

$$x_{n+1} = y_n - \frac{f'^2(y_n) + 3f'(x_n)}{5f'(y_n) + f'(y_n)} \times \frac{f(y_n)}{f'(x_n)} \tag{18}$$

where

$$y_n = x_n - \frac{f(y_n)}{f'(x_n)} \tag{19}$$

Grau and Diaz-Barrero's method (GM) [7] defined by

$$x_{n+1} = x_n - \left(1 + \frac{f''(x_n)(f(x_n) + f(z_n))}{2f'^2(x_n)}\right)\frac{f(x_n) + f(z_n)}{f'(x_n)} \tag{20}$$

where

$$z_n = x_n - \left(1 + \frac{1}{2}\frac{f''(x_n)f(x_n)}{f'^2(x_n)}\right)\frac{f(x_n)}{f'(x_n)} \tag{21}$$

Noor and Noor's method (NNM) [12] defined by

$$x_{n+1} = x_n - \frac{2[f(x_n) + h(z_n)]f'(x_n)}{2f'^2(x_n) - [f(x_n) + h(z_n)]f''(x_n)} \tag{22}$$

$$h(x) = f(x) - f(x_n) - (x - x_n)f'(x_n) - \frac{1}{2}(x - x_n)^2 f''(x_n) \tag{23}$$

$$z_n = x_n - \left(1 + \frac{t(x_n)}{2 - t(x_n)}\right)\frac{f(x_n)}{f'(x_n)} \tag{24}$$

$$t(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)} \tag{25}$$

Ezzati and Azadegan's method (EAM) [2] defined by

$$x_{n+1} = z_n - \frac{f(z_n) + f\left(z_n - \frac{f(z_n)}{f^{'}(x_n)}\right)}{f^{'}(x_n)} \quad (26)$$

where

$$z_n = x_n - \frac{f(x_n) + f\left(x_n - \frac{f(x_n)}{f^{i}(x_n)}\right)}{f^{'}(x_n)} \quad (27)$$

We used the following test functions and display the approximate zero x_* found up to the 15th decimal places.

$$\begin{cases} f_1 = x^3 + 4x^2 - 10 \ [2,3,15], & x_* = 1.365230013414100 \\ f_2 = \ln x + \sqrt{x} - 5 \ [2], & x_* = 8.309432694231570 \\ f_3 = x^2 - e^x - 3x + 2 \ [2], & x_* = 0.257530285439861 \\ f_4 = (x+2)e^x - 1 \ [2,10], & x_* = -0.4428544010023889 \\ f_5 = sin^2 x - x^2 + 1 \ [2,15], & x_* = 1.404491648215340 \\ f_6 = xe^{x^2} - sin^2 x + 3cosx + 5 \ [2,3,10], & x_* = -1.207647827130920 \\ f_7 = (x-1)^3 - 2 \ [2], & x_* = 2.259921049894870 \end{cases}$$

All calculations were done using Microsoft EXCEL VISUAL BASIC for Application (VBA) using 15 digit floating arithmetic. We use the following stopping criteria for computer programs: $|x_{n+1} - x_n| < \varepsilon$, $|f(x_n)| < \varepsilon$ and so, when the stopping criterion is satisfied, x_{n+1} is taken as a computed value of the exact root. For numerical illustration in this section we use the fixed stopping criterion $\varepsilon = 1.E^{-15}$, where ε represents tolerance.

We present below some iteration results for some selected functions given above with their corresponding CLOC.

Table 1. OOM iterations results with their CLOC.

Iteration	$f_3(x), x_0 = 1$	$f_4(x),\ x_0 = 2$	$f_5(x), x_0 = 1$	$f_7(x),\ x_0 = 3$
1.	0.256010825834790	0.407745500839142	1.395127685808690	2.271993950088230
2.	0.257530285439861	-0.415213763430005	1.404491648197400	2.259921049960910
3.	0.257530285439861	-0.442854398907566	1.404491648215340	2.259921049894870
4.		-0.442854401002389	1.404491648215340	2.259921049894870
5.		-0.442854401002389		
CLOC	5.767859	5.568893427	5.297457	5.307191192

From Table 1 we observed that the computational local order of convergence (CLOC) of the method on all the test functions is at least 5.

Table 2. Comparison of the number of iterations (NIT) in some existing methods and OOM .

Functions	**NIT**					
	GM	NNM	KM	YCM	EAM	OOM
$f_1(x), x_0 = 1$	4	6	4	3	4	2
$f_2(x), x_0 = 7$	2	3	2	2	2	2
$f_3(x), x_0 = 1$	4	5	4	3	3	2
$f_4(x), x_0 = 2$	5	9	Failed	5	5	4
$f_5(x), x_0 = 1$	Failed	7	5	4	6	3
$f_6(x), x_0 = -1$	4	6	Failed	4	4	3
$f_7(x), x_0 = 3$	4	7	4	4	4	3

Table 3. Comparison of number of functions evaluation (NOFE) of different iterative methods.

Functions	**NOFE**					
	GM	NNM	KM	YCM	EAM	OOM
$f_1(x), x_0 = 1$	16	36	24	12	20	10
$f_2(x), x_0 = 7$	8	18	12	8	10	10
$f_3(x), x_0 = 1$	16	30	24	12	15	10
$f_4(x), x_0 = 2$	20	54	ND	20	25	20
$f_5(x), x_0 = 1$	ND	42	30	16	30	15
$f_6(x), x_0 = -1$	16	36	ND	16	20	15
$f_7(x), x_0 = 3$	16	42	24	16	20	15

ND – Not defined NOFE – Number of Functions Evaluation

The computational results shows that OOM requires less iterations and number of functions evaluations (NOFE) than GM, NNM, KM, YCM, and EAM as far as the numerical results are concerned. Therefore, the new method (OOM) is of practical interest.

V. CONCLUSION

We have shown that OOM is at least fifth-order convergent provided the first, second, third and fourth derivatives of the function exist. Computed results (Table 1) support the fifth- order convergence, and for some functions the Computational Local Order of Convergence (CLOC) is even more than five. We have also observed that, OOM needs less total function evaluation at iteration convergence points than some existing methods of same order compared with; it is evident by the computed results in (Table 3). Finally, it is hoped that this study makes a contribution to solve nonlinear equation.

Competing Interests: The authors declare that no competing interests exist.

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