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New Two-Step Method with Fifth-Order Convergence for Solving Nonlinear Equations

[1,]Ogbereyivwe Oghovese, [2,]Atoma O. Johnson
Department of Mathematics and Statistics, Delta State Polytechnic, Ozoro, Nigeria

ABSTRACT : *In this paper, a new iterative method for solving nonlinear equations is presented. The method is a modified Newton's method. Per iteration the presented method require two evaluations of the function and two evaluations of the first-order derivatives. The convergence order of the method is established to five and the efficiency index is 1.4953. Numerical comparisons are made with several other existing methods to show the performance of the presented methods.*

KEYWORDS: *Convergence order, Taylor's series expansion, Nonlinear equations*

I. INTRODUCTION

Solving nonlinear equations is an important issue in pure and applied mathematics. Researchers have developed various effective methods to find a single root x^* of the nonlinear equation $f(x) = 0$, where $f: D \subset R \to R$ is a scalar function on an open interval D. Newton's method is one of the best iterative methods to find x^* by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

that converges quadratically in some neighborhood of x^* [1].

There are plenty of modified iterative methods to improve the order of convergence or to simplify the computation in open literatures. For more details, see [1-17] and the references therein. Chun [2] developed two one-parameter fourth-order methods, which are given by:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) + 2\beta f(y_n)^2} \dfrac{f(y_n)}{f'(x_n)} \end{cases} \tag{2}$$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^3}{f(x_n)^2 S_2(x_n, y_n) + 2\beta f(y_n)^2 S_\beta(x_n, y_n)} \dfrac{f(y_n)}{f'(x_n)} \end{cases} \tag{3}$$

where $S_\beta = f(x_n) - \beta f(y_n)$ and $\beta \epsilon R$ is a constant. We note that the methods defined by (2) and (3) reduce to the Traub-Ostrowski method [3] when $\beta = 0$. These two-point methods are special cases of the more general family of two-point methods

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - G(t) \dfrac{f(y_n)}{f'(x_n)} \end{cases} \tag{4}$$

with particular cases

$$G(t) = \frac{1}{1 - 2t + 2\beta t^2} \tag{5}$$

and

$$G(t) = \frac{1}{1 - 2t + 2\beta t^2(1 - \beta t)} \tag{6}$$

where

$$t = \frac{f(y_n)}{f'(x_n)} \tag{7}$$

Ostrowski's method [4], given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)}{f(x_n) - 2f(y_n)} \dfrac{f(y_n)}{f'(x_n)} \end{cases} \quad (8)$$

is an improvement of (1). The order increases by at least two at the expense of additional function evaluation at another point iterated by the Newton's method. This is also a special case of (4) with particular case

$$G(t) = \frac{1}{1-2t} \quad (9)$$

To improve the local order of convergence and efficiency index, many more modified methods of (8) have been developed. These include Chun and Ham [5], Kou et al [6], Bi et al [7] and reference therein.

In this paper, we develop new Newton-type iterative method to find a single root of nonlinear equations. The method is a special case of (4) with better computational order and efficiency.

II. THE METHOD AND THE ANALYSIS OF CONVERGENCE

In this section we present the new fifth order iterative methods. We begin with the following theorem and definitions.

Definition 1: *Let $\alpha \in R, x_n \in R, n = 0,1,2,\dots$ Then the sequence $\{x_n\}$ is said to converge to α if*

$$\lim_{n\to\infty} |x_n - \alpha| = 0 \quad (10)$$

In addition, there exists a constant $c \geq 0$, an integer $n_0 \geq 0$ and $p \geq 0$ such that for all $n \geq n_0$

$$|x_{n+1} - \alpha| \leq |x_n - \alpha|^p \quad (11)$$

then $\{x_n\}$ is said to converge to α of order p.

Definition 2: *The computational efficiency of an iterative method of order p, requiring k function evaluations per iteration [8], is calculated by*

$$\sqrt[k]{p}. \quad (12)$$

Theorem 1. *Let $\varphi_1(x), \varphi_2(x), \dots, \varphi_s(x)$ be iterative functions with the orders $r_1, r_2, \dots, r_s$, respectively. Then the composition of iterative functions*

$$\varphi(x) = \varphi_1(\varphi_2(\dots\dots(\varphi_s(x))\dots)) \quad (13)$$

defines the iterative method of the order $r_1, r_2, \dots, r_s$ [9].

We now present the method. Consider the following iteration scheme

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(y_n)} \end{cases} \quad (14)$$

Equation (14) is a composite Newton method and by Theorem 1, has fourth-order convergence. To achieve iteration with (14), it requires four evaluations. Hence by definition 2, the computational efficiency of (14) is $\sqrt[4]{4}$ which does not increases the computational efficiency of the Newton's method that is $\sqrt[2]{2}$. Our aim is to improve the order and computational efficiency of (14) by presenting a fifth-order method of special case (4). We achieve this by introducing a weight function expressed as

$$H(t) = (1 - t^2)^{-1}, \quad (15)$$

where t is as defined in (7).

Then the improved iteration scheme (14) becomes

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - H(t)\dfrac{f(y_n)}{f'(y_n)} \end{cases} \quad (16)$$

Equation (16) is based on composition of two steps, the Newton's method step which is a predictor-type and weighted Newton step which is a corrector-type. As a consequence, the order of convergence is improved from four for double Newton method to five for the new method. In order to establish the fifth-order convergence of the proposed method (16), we state the following theorem.

Theorem 2: Let α be a simple zero of sufficiently differentiable function $f: R \to R$ for an open interval I. If x_0 is sufficiently close to , then the two-step method defined by (16) has convergence at least of order five.

Proof

Let $e_n = x_n - \alpha$ be the error in the iterate x_n. Using Taylor's series expansion, we get

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O({e_n}^7)] \quad (17)$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O({e_n}^8)] \quad (18)$$

where $c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}$ for $k \in N$..

Now

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ +(-88_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + (16c_2^5 - 52c_2^3c_3 + \\ 28c_2^2c_4 - 17c_3c_4 + c_2(33c_3^2 - 13c_5) + 5c_6)e_n^6 + O(e_n^7) \quad (19)$$

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ -2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 + (28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 \\ -17c_3c_4 + c_2(37c_3^2 - 13c_5)) + 5c_6)e_n^6 + O(e_n^7)] \quad (20)$$

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2 e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + c_2(8c_2^3 - 11c_2c_3 + 6c_4)e_n^4 \\ -4(c_2(4c_2^4 - 7c_2^2c_3 + 5c_2c_4 - 2c_5))e_n^4 + 2(16c_2^6 - 34c_2^4c_3 + 6c_3^3 \\ +30c_2^3c_4 - 13c_2^2c_5 + c_2 - 8c_3c_4 + 5c_6))c_n^6 + O(e_n^7) \quad (21)$$

$$\frac{f(y_n)}{f'(x_n)} = c_2 e_n + 2(2c_3 - 3c_2^2)e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e_n^3 + O(e_n^4) \quad (22)$$

$$H(t) = 1 + c_2^2 e_n^2 + 4c_2(c_3 - c_2^2)e_n^3 + 2c_2(7c_2^3 - 10c_2c_3 + 3c_4 + c_2^3)e_n^4 + O(c_n^5) \quad (23)$$

$$\frac{f(y_n)}{f'(y_n)} = c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (3c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ -(18c_2^4 - 16c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_2^5)e_2^5 + O(c_n^6) \quad (24)$$

$$H(t)\frac{f(y_n)}{f'(y_n)} = c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ -2(4c_2^4 - 11c_2^2c_3 + 3c_3^2 + 5c_2c_4 - c_5)e_2^5 + O(e_n^6) \quad (25)$$

using (17) to (25) in (16), we get

$$e_{n+1} = 2c_2^2(2c_2^2 - c_3)e_n^5 + O(e_n^6) \quad (26)$$

which shows that the method is at least fifth order convergent method.

We now discuss the efficiency index of the method by using definition 2. The number of function evaluations per iteration of the method is 4. Therefore from definition 2 the efficiency index of the method is 1.4953, which is better than Newton's method $2^{\frac{1}{2}} = 1.4142$, Homeier's method $3^{\frac{1}{3}} = 1.4422$ [10], Wang's method $6^{\frac{1}{5}} = 1.4310$ eq. (8) in [11], Siyyam's method $5^{\frac{1}{5}} = 1.3792$ [12] and equivalent to Kou's method $5^{\frac{1}{4}} = 1.4953$ [13] and Sharma's method $5^{\frac{1}{4}} = 1.4953$ eq. (22) in [14].

III. APPLICATIONS

Now, consider some test problems to illustrate the efficiency of the developed method namely OM and compare it with classical Newton Method (NM), the method of Abbasbandy [17] (AM), the method of Homeier [10] (HM), and the mehods of Chun [15] (CM2 is referred to method 10 in Chun [15] with fourth-order convergence) and (CM3 is referred to method 11 in Chun [15] with fifth-order convergence), the method of Noor and Noor [16] (NNM), the Siyyam's method [12] (SM), and the method of Sharma [14] (M2 is referred to method 22 in Sharma [14]). The comparisons are given in Table 1.The test examples used are chosen from Chun [15]. They are given below:

$$\begin{cases} f_1(x) = sin^2x - x^2 + 1 \\ f_2(x) = x^2 - e^x - 3x + 2 \\ f_3(x) = \cos x - x \\ f_4(x) = (x-1)^3 - 1 \\ f_5(x) = x^3 - 10 \\ f_6(x) = xe^{x^2} - sin^2x + 3\cos x + 5 \\ f_7(x) = e^{x^2+7x-30} - 1 \end{cases} \quad (27)$$

All calculation was done using 15-digit floating-point arithmetic. The following stopping criteria are used for computations.

$$i. \quad |x_{n+1} - x_n| < \epsilon \qquad ii. \quad |f(x_{n+1})| < \epsilon \quad (28)$$

where $\epsilon = 10^{-15}$.
Table 1 shows the calculated root and the number of iterations necessary to reach the root up to the desired accuracy by each method.

Table 1: Comparison of number of iterations of various iterative method

Functions	x_0	NM	AM	HM	CM2	CM3	NNM	SM	M2	OM
f_1	1	7	5	4	5	4	4	5	4	4
f_2	2	6	5	5	4	4	4	3	4	3
f_3	1.7	5	4	4	4	3	3	3	4	2
f_4	3.5	8	5	5	5	5	5	4	Failed	3
f_5	1.5	7	5	4	5	5	4	4	4	4
f_6	-2	9	6	6	6	5	5	5	Failed	4
f_7	3.5	13	7	8	8	7	7	7	Failed	6

IV. CONCLUSION

A fifth order two-step method is proposed to solve nonlinear equations without evaluation of second derivative of the function and it requires three functions and one first derivative per iteration. Its efficiency index is 1.4953 which is better than some existing methods. With the help of some test problems, comparison of obtained results with some existing methods is also given.

Competing Interests: The authors declare that no competing interests exist.

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