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Finslerian Nonholonomic Frame For Matsumoto (α,β)-Metric

Mallikarjuna Y. Kumbar, Narasimhamurthy S.K. , Kavyashree A.R. .
Department of P.G. Studies and Research in Mathematics,Kuvempu University, Shankaraghatta - 577451,Shimoga, Karnataka, INDIA.

***ABSTRACT*:** The whole point of classical dynamics is to show how a system changes in time; in other words, how does a point on the configuration space change once we give initial conditions? For a system with nonholonomic constraints, the state after some time evolution depends on the particular path taken to reach it. In other words, one can return to the original point in configuration space but not return to the original state. The main aim of this paper is, first we determine the Finsler deformations to the Matsumoto (α,β) −metric and we construct the Finslerian nonholonomic frame. Further we obtain the Finslerian nonholonomic frame for special (α,β)- metric i.e., $F = c_1\beta + \frac{\alpha^2}{\beta}$, $(c_1 \geq 0)$.

KEY WORDS:
$Finsler\ space, (\alpha,\beta) - metrics, GL - metric, Finslerian\ nonholonomic\ \ frame.$

I. *INTRODUCTION*

In 1982, P.R. Holland ([1][2]), studies a unified formalism that uses a nonholonomic frame on space-time arising from consideration of a charged particle moving in an external electromagnetic field. In fact, R.S. Ingarden [3] was first to point out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space called Randers space. The author Beil R.G. ([5][6]), have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries. The above authors used the common Finsler idea to study the existence of a nonholonomic frame on the vertical subbundle $V\,TM$ of the tangent bundle of a base manifold M.

Consider $a_{ij}(x)$, the components of a Riemannian metric on the base manifold M,$a(x,y) > 0$ two functions on TM and $B(x,y) = B_i(x,y)dx^i$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x)B_j(x) \tag{1.1}$$

is a generalized Lagrange metric, called the Beil metric . We say also that the metric tensor g_{ij} is a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x,y) = \exp\big(2\sigma(x,y)\big)$. The case $a(x,y) = 1$ with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6].

In this paper, the fundamental tensor field might be taught as the result of two Finsler deformations. Then we can determine a corresponding frame for each of these two Finsler deformations. Consequently, a Finslerian nonholonomic frame for a Matsumoto (α,β) −metric and special (α,β) −metric i.e., $F = c_1\beta + \frac{\alpha^2}{\beta}$ $(c_1 \geq 0)$ will appear as a product of two Finsler frames formerly determined. As if $c_1 = 0$ then it takes form of Kropina metric case.

II. *PRELIMINARIES*

An important class of Finsler spaces is the class of Finsler spaces with (α,β) −metrics [11]. The first Finsler spaces with (α,β) −metric were introduced by the physicist G. Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested to consider a more general case, the class of Lagrange spaces with (α,β) −metric, which was discussed in [12]. A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P.R. Holland [1][2].If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β) −metric. Next we look for some different Finsler space with (α,β) −metrics.

Definition 2.1. : A Finsler space $F^n = (M, F(x,y))$ is called (α,β) −metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F: TM \to R$ is given by,

$$F^2(x,y) = L(\alpha(x,y), \beta(x,y)),$$

where $\alpha^2\,(x,y) = a_{ij}\,(x) y^i y^j$, α is a Riamannian metric on M, and $\beta(x,y) = b_i(x) y^i$ is a $1 - form$ on M .

Consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i\,\partial y^j}$ the fundamental tensor of the Randers space (M,F), taking into account the homogeneity of α and F we have the following formulae:

$$p^i = \frac{1}{\alpha} y^i = a_{ij}\frac{\partial\alpha}{\partial y^j}; \qquad p_i = a_{ij} p^j = \frac{\partial\alpha}{\partial y^i};$$

$$l^i = \frac{1}{L} y^i = g_{ij}\frac{\partial L}{\partial y^j}; \qquad l_i = g^{ij}\frac{\partial L}{\partial y^j} = p_i + b_i;$$

$$l^i = \frac{1}{L} p^i; \qquad l^i\, l_j = p^i p_i = 1; \qquad l^i\, p_i = \frac{\alpha}{L};$$

$$p^i l_i = \frac{L}{\alpha}; \qquad b_i p^i = \frac{\beta}{\alpha}\ ; \qquad b_i l^i = \frac{\beta}{L}.$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij} = \frac{L}{\alpha} a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha} p_i p_j = \frac{L}{\alpha}\left(a_{ij} - p_i p_j\right) + l_i l_j .$$

Theorem 2.1: For a Finsler space (M,F) consider the matrix with the entries:

$$Y_j^i = \sqrt{\frac{\alpha}{L}}\left(\delta_j^i - l_i l_j + \sqrt{\frac{\alpha}{L}} p^i p_j\right) \qquad (2.4)$$

defined on TM. Then $Y_j = Y_j^i \left(\frac{\partial}{\partial y^i}\right), j \in 1,2, \ldots, n$ is an nonholonomic frame.

Theore m 2.2: With respect to frame the holonomic components of the Finsler metric tensor $(a_{\alpha\beta})$ is the Randers metric (g_{ij}).

$$i.e \qquad g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}. \qquad (2.5)$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ i.e., $y_i = a_{ij} y^j$, $b^i = a^{ij} b_j$, and so on. For a Finsler space with (α,β) −metric $F^2(x,y) = L\big(\alpha(x,y), \beta(x,y)\big)$ we have the Finsler invarienrs [13].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial\alpha}; \quad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial\beta^2}; \quad \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \quad \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial\alpha}\right): \quad (2.6)$$

where, subscripts $-2, -1, 0, 1$ gives us the degree of homogeneity of these invariants. For a Finsler space with (α,β) −metric

we have: $$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \tag{2.7}$$
with respect to these notations, we have that the metric tensor g_{ij} of a Finsler space with (α, β) −metric is given by [13]:
$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\left(b_i(x)y_j + b_j(x)y_i\right) + \rho_{-2}y_i y_j \tag{2.8}$$

From (2.8) we can see that g_{ij} is the result of two Finsler deformations:
$$i)\ a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$$
$$ii)\ h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_i b_j. \tag{2.9}$$
The Finslerian nonholonomic frame that corresponds to the first deformation (2.9) is, according to the Theorem 7.9.1 in [10], given by:
$$x_j^i = \sqrt{\rho}\delta_j^i - \frac{1}{B^2}\left(\sqrt{\rho} \pm \sqrt{\rho + \frac{B^2}{\rho_{-2}}}\right)\ \left(\rho_{-1}b^i + \rho_{-2}y^i\right)(\rho_{-1}b_j +_{-2}y_j) \tag{2.10}$$
where
$$B^2 = a_{ij}\left(\rho_{-1}b^i + \rho_{-2}y^i\right)\left(\rho_{-1}b^j + \rho_{-2}y^j\right) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}.$$
The metric tensors a_{ij} and h_{ij} are related by:
$$h_{ij} = X_i^k X_j^l a_{kl} \tag{2.11}$$
Again the frame that corresponds to the second deformation (2.9) is given by:
$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\right) b^i b_j, \tag{2.12}$$
where
$$C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2.$$
The metric tensors h_{ij} and g_{ij} are related by the formula:
$$g_{mn} = Y_m^i Y_n^j h_{ij}\ . \tag{2.13}$$

Theorem 2.3: Let $F^2(x,y) = L\big(\alpha(x,y), \beta(x,y)\big)$ be the metric function of a Finsler space with (α, β) −metric for which the condition (2.7) is true. Then
$$v_j^i = X_k^i Y_j^k$$
is a Finslerian nonholonomic frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

III. FINSLERIAN NONHOLONOMIC FRAME FOR $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ −metric

In this section, we consider two Finsler spaces with (α, β) −metrics, such as Matsumoto metric and special (α, β) −metric $i.e., F = c_1\beta + \frac{\alpha^2}{\beta}$ then we construct Finslerian nonholonomic frame for these.

3.1 FINSLERIAN NONHOLONOMIC FRAME FOR MATSUMOTO $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ −METRIC:

In the first case, for a Finsler space with the fundamental function $L = F^2 = \frac{\alpha^4}{(\alpha-\beta)^2}$, the Finsler invariants (2.6) are given by:
$$\rho_1 = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3},\ \ \rho_0 = \frac{3\alpha^3}{(\alpha-\beta)^4}\quad \rho_{-1} = \frac{\alpha^2(\alpha-4\beta)}{(\alpha-\beta)^4},\ \ \rho_{-2} = \frac{\beta(4\alpha-\beta)}{(\alpha-\beta)^4}. \tag{3.1}$$
$$B^2 = \frac{\alpha^2(\alpha-4\beta)^2(b^2\alpha^2-\beta)}{(\alpha-\beta)^8}$$

Using (3.1) in (2.10) we have,

$$X_j^i = \sqrt{\frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}}\delta_j^i - \frac{\alpha^2}{b^2\alpha^2-\beta^2}\left\{\sqrt{\frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}} \pm \sqrt{-\frac{\alpha^2(-\alpha\beta^2+2\alpha\beta^2+2\beta^3+b^2\alpha^3-4b^2\alpha^2\beta)}{(\alpha-\beta)^4\beta}}\right\}\left(b^i - \frac{\beta y^i}{\alpha^2}\right)\left(b_j - \frac{\beta y_j}{\alpha^2}\right). \quad (3.2)$$

Again using (3.1) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \frac{\sqrt{\beta(\alpha-\beta)^3 C^2}}{\alpha^2}\right) b^i b_j; \quad (3.3)$$

where

$$C^2 = \frac{\alpha^2(\alpha-2\beta)b^2}{(\alpha-\beta)^3} - \frac{(\alpha-4\beta)\left(b^2\alpha^2-\beta^2\right)^2}{\beta(\alpha-\beta)^4}.$$

Theorem 3.4: Consider a Finsler space $L = \frac{\alpha^4}{(\alpha-\beta)^2}$, for which the condition (2.7) is true. Then

$$V_j^i = X_k^i Y_j^k$$

is a Finslerian nonholonomic frame with X_k^i and Y_j^k are given by (3.2) and (3.3) respectively.

3.2 FINSLERIAN NONHOLONOMIC FRAME FOR SPECIAL $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ −METRIC *i.e.*, $\boldsymbol{F = c_1\beta + \frac{\alpha^2}{\beta}}$:

In the second case, for a Finsler space with the fundamental function $L = F^2 = \Big(c_1\beta +$ *α2β2,* the Finsler invariants (2.6) are given by:

$$\rho_1 = \frac{2(c_1\beta^2+\alpha^2)}{\beta^2}, \quad \rho_0 = \frac{c_1^2\beta^4+3\alpha^4}{\beta^4},$$
$$\rho_{-1} = -\frac{4\alpha^2}{\beta^3}, \rho_{-2} = \frac{4}{\beta^2}, \quad (3.4)$$
$$B^2 = \frac{16\alpha^2(\alpha^2b^2-\beta^2)}{\beta^6}$$

Using (3.4) in (2.10) we have,

$$X_j^i = \sqrt{\frac{2(c_1\beta^2+\alpha^2)}{\beta^2}}\delta_j^i$$

$$-\frac{1}{16}\left[\left\{\frac{\beta^6\left(\sqrt{\frac{2(c_1\beta^2+\alpha^2)}{\beta^2}} \pm \sqrt{\frac{2(\beta^4c_1-\alpha^2\beta^2+2\alpha^4b^2)}{\beta^4}}\right)}{\alpha^2(\alpha^2b^2-\beta^2)}\right\} \left[\left(\frac{4y^i}{\beta^2} - \frac{4\alpha^2b^i}{\beta^3}\right)\left(\frac{4y_j}{\beta^3} - \frac{4a^2b_j}{\beta^3}\right)\right]\right] \quad (3.5)$$

Again using (3.4) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 - \frac{C^2\beta^4}{\alpha^4 - c_1^2\beta^4}}\right) b^i b_j \quad (3.6)$$

where

$$C^2 = \frac{2(c_1\beta^2+\alpha^2)}{\beta^2} + \frac{4(\alpha^2b^2-\beta^2)^2}{\beta^4}.$$

Theorem 3.5: Consider a Finsler space L= $\left(c_1\beta + \frac{\alpha^2}{\beta}\right)^2$, for which the condition (2.7) is true.Then

$$V_j^i = X_k^i Y_j^k$$

is a Finslerian nonholonomic frame with X_k^i and Y_j^k are given by (3.5)and (3.6) respectively.

IV. CONCLUSION

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli P.L., Bucataru I. ([7][8]), has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α, β) −metric, it appears a natural question: Does how many Finsler space with (α, β) −metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with (α, β) −metrics.

In this work, we consider the two special Finsler metrics and we determine the Finsleriannonholonomic frames. Each of the frames we found here induces a Finsler connection on TM with torsion and no curvature. But, in Finsler geometry, there are many (α, β) −metrics, in future work we can determine the frames for them also.

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