International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 4 ǁ April. 2014ǁ PP-04-14

An Exposition on Solution Process and Nature of Solutions of Simple Delay Differential Equations

Ukwu Chukwunenye
Department of Mathematics University of Jos P.M.B 2084 Jos, Nigeria

ABSTRACT: *This article gave an exposition on the solutions of single-delay autonomous linear differential equations with algorithmic presentations of the solution processes of two solution methods, with illustrative examples verifying the consistency of both solution methods. The article went further to examine the nature of solutions, obtaining in the process a complete characterization of all non-oscillatory solutions of a problem instance in the absence of initial functions. In the sequel the article proved that there is no nontrivial oscillatory solution if a constant initial function is specified for the delay differential equation in question and hence for delay differential equations in general.*

KEYWORDS: *Algorithmic, Continuity, Delay, Nature, Process.*

I. INTRODUCTION

The method of steps and forward continuation recursive procedure are prevalent in the literature on functional differential equations. However literature on detailed illustrative examples and robust analyses of problem instances is quite sparse, leaving very little room for enhanced understanding and appreciation of this class of differential equations. This article, which makes a positive contribution in the above regard, is motivated by Driver (1977) and leverages on some given problems there to conduct detailed analyses on the process and structure of solutions and thorough analyses of some problem instances. The algorithmic format used in the two solution methods, the verification of the consistency of both methods, the investigation of appropriate feasibility conditions on the solutions and the lucidity of our presentations are quite novel in this area where presentations are skeletal for the most part, forcing readers to plough through several materials and figure out a lot of things on their own with the attendant opportunity cost in time. See also Hale (1977) for more discussion on method of steps.

II. PRELIMINARIES

For an arbitrary initial function problem:

$$\dot{x}(t) = ax(t) + bx(t-h) \qquad (1)$$

$$x(t) = \phi(t),\ t \in [-h, 0] \qquad (2)$$

a, b scalar constants, ϕ continuous, it is clear that the integrating factor, $I(t) = e^{-at}$

If we denote the solution on the interval $J_k = [(k-1)h,\ kh],\ k = 1,\ 2, \ldots.$ by $y_k(t)$, then on J_1, (1) becomes $\dot{x}(t) = ax(t) + b\phi(t-h)$, from which it is clear the integrating factor, $I(t) = e^{-at}$.

Hence $y_1(t)$ can be obtained from the relation:

$$y_1(t) = e^{at}\left[\int b e^{-at}\ \phi(t-h)\,dt + C_1\right]. \qquad (3)$$

$y_1(t)$ exists, since $b e^{-at}\ \phi(t-h)$ is integrable, in view of the fact that e^{-at} and ϕ are continuous, $t-h$ being in $[-h,\ 0]$.

The solutions $y_k(t),\ k = 1,\ 2, \ldots,$ are given recursively by:

$$y_k(t) = e^{at}\left[\int b\, e^{-at} y_{k-1}(t-h)\,dt + C_k\right] \qquad (4)$$

where the continuity of y_{k-1} guarantees the integrability of $be^{-at} y_{k-1}(t-h)$ and hence the continuity of y_k, assuring the existence of y_k, for $k = 1, 2, \cdots$

Above solutions are unique, arising from the unique determination of the C_ks from the relation:

$$y_{k-1}([k-1]h) = y_k([k-1]h) \tag{5}$$

Note that if $j \in \{1, 2, ..., k-1\}$, there is no general expression for the $c_{k\,j}$ appearing in (6) below. Moreover if the initial function in (2) is not constant, it is impossible to express y_k, in the form, $y_k(t) = \left[d_k + \left(\sum_{j=1}^{k} d_{k\,j} t^{j-1} \right) e^{at} \right] \phi(t).$ However, if $a \neq 0$ and $\phi(t)$ is a polynomial of degree *m*, say, $m \geq 0$, then the y_k's take the form:

$$y_k(t) = \sum_{j=0}^{m} d_{k\,j} t^j + \left(\sum_{j=0}^{k-1} c_{k\,j} t^j \right) e^{at} \tag{6}$$

for some constants $d_{k\,j}$, $j = 0, 1, \quad m;$ $c_{k\,j}$, $j = 0, 1, \cdots, k-1.$ In other words, $y_k(t)$ is the sum of some polynomial of degree m and the product of e^{at} and some polynomial $P_{k-1}(t)$ of degree $k-1$, where $y_0(t) = \phi(t)$.

The process of getting the above coefficients is easy, but the computations get rather unwieldy. The computational procedure is set out in the following algorithmic steps, noting that:

$$y_k(t) = x(t) \text{ for}$$

$$t \in [(k-1)h, \quad kh]. \tag{7}$$

Method 1: Algorithmic Steps:

[1] Set $k = 1$ in (6)

[2] Plug in $y_1(t)$ for $x(t)$, $\dot{y}_1(t)$ for $\dot{x}(t)$ and $\phi(t-h)$ for $x(t-h)$ in (1).

[3] Compare the coefficients of the resulting left and right hand expressions to secure the $c_{k,j}$s and $d_{k,j}$s in the relevant ranges for j, and obtain $y_1(t)$. Verify that the condition $y_1(0) = \phi(0)$ is satisfied, where $y_0(t) = \phi(t)$.

[1] Set $k = 2$ in (6).

[2] Substitute $y_2(t)$ for $x(t)$, $\dot{y}_2(t)$ for $\dot{x}(t)$ and $y_1(t-h)$ for $x(t-h)$ in (1)

[3] Compare the coefficients of the resulting left and right hand expressions to secure the $c_{k,j}$s and $d_{k,j}$s in the relevant ranges for j, and hence obtain $y_2(t)$. Check that the condition $y_2(h) = y_1(h)$ is satisfied.

This implementation process continues, so that at the k^{th} stage, $y_k(t)$ is substituted for $x(t)$, $\dot{y}_k(t)$ for $\dot{x}(t)$ and the already secured $y_{k-1}(t-h)$ substituted for $x(t-h)$ in (1). Then the coefficients of the resulting left and right hand expressions are compared to yield the $c_{k,j}$s and $d_{k\,j}$s for $j = 0, 1, 2, ..., m, m+1,$ and hence $y(t)$, for $k \geq 3$. Verify that the continuity condition $y_k([k-1]h) = y_{k-1}([k-1]h)$ holds.

Alternatively, proceed as follows:

Method 2: Algorithmic Steps:

[1] Set $k = 1$ in (76) and then $y_0(t-h) = \phi(t-h)$ and evaluate the integral.

[2] Set $y_1(0) = \phi(0) \equiv y_0(0)$ to secure C_1, following the computation of the integral on the right of (4)

Having initialized the process in step 1, $y_k(t)$ is obtained by using (4) in conjunction with the condition $y_k([k-1]h) = y_{k-1}([k-1]h),\ k = 2,3,\cdots$. The integral in (4) can be determined since $y_{k-1}(t-h)$ is already known. The condition $y_k([k-1]h) = y_{k-1}([k-1]h)$ secures C_k .

Example 1

Solve the initial function problem:

$\dot{x}(t) = ax(t) + bx(t-1)$ on [0, 2], $a \in \mathbf{R}$.

$x(t) = \phi(t) = 1 + t,\ t \in [-1,\ 0]$

Solution:

Method 1

$$h = 1, k = 1, c_1 = c_{10}, d_1 = d_{10} \Rightarrow y_1(t) = d_1 + d_{11}\, t + c_1\, e^{at};$$

$$\dot{y}_1(t) = a\ y_1(t) + b\ y_0(t-h)$$

$$\Rightarrow d_{11} + a\, c_1\, e^{at} = a\, d_1 + a\, d_{11}\, t + b(1 + t - 1) + a\, c_1\, e^{at}$$

$$\Rightarrow (d_{11} - a\, d_1) - (a\, d_{11} + b)t = 0 \Rightarrow d_{11} = \frac{-b}{a}, d_1 = \frac{1}{a} d_{11} = \frac{-b}{a^2}.$$

If $a \neq 0,$ then $c_1 = 1 - d_1 = 1 + \frac{b}{a^2,}\ a \neq 1.$ The condition $y_1(0) = \phi(0) \Rightarrow d_1 + c_1 = 1$.

Verification

$$c_1 + d_1 = 1 + \frac{b}{a^2} - \frac{b}{a^2} = 1,\ a \neq 0.$$

Therefore, $y_1(t) = -\frac{b}{a^2} - \frac{b}{a}\, t + \left(1 + \frac{b}{a_2}\right) e^{at}$, if $a \neq 0;$ that is

$$x(t) = -\frac{b}{a^2} - \frac{b}{a} t + \left(1 + \frac{b}{a^2}\right) e^{at} \text{ for } t \in [0 \quad 1],\ \text{if } a \neq 0.$$

The condition $b \neq 0$ precludes degeneracy of (1) to an ordinary differential equation.

$$y_2(t) = d_2 + d_{21}\, t + (c_2 + c_{21}\, t) e^{at}, \text{ where } d_2 \equiv d_{2,0}$$

$$\dot{y}_2(t) = a\ y_2(t) + b\ y_1(t-1)$$

$$\Rightarrow d_{21} + ac_2 e^{at} + c_{21} e^{at} + ac_{21} t\, e^{at} = a\, d_2 + a\, d_{21}\, t + ac_2 e^{at} + ac_{21} t e^{at} + b\left[d_1 + d_{11}(t-1) + c_1 e^{a(t-1)}\right]$$

$$\Rightarrow d_{21} - (a\, d_2 + b\, d_1 - d_{11}\, b) - (a\, d_{21} + b\, d_{11}) t + (c_{21} - bc_1 e^{-a}) e^{at} = 0$$

We claim that the members of the set $\{1,\ t,\ e^{at}\}$ are linearly independent for $a \neq 0.$

Proof:

$$\left\| \begin{pmatrix} 1 & t & e^{at} \\ 0 & 1 & ae^{at} \\ 0 & 0 & ae^{at} \end{pmatrix} \right\| = ae^{at} \neq 0, \text{ for } a \neq 0.$$

This proves the claim. Indeed $1, t, e^{at}$, and te^{at} are linearly independent as the Wronskian turns out to be $a^4 \neq 0$ for $a \neq 0$.

By the above claim it is valid to set each of the above coefficients to zero. Consequently,

$$d_{21} = -\frac{b\,d_{11}}{a} = -\frac{b}{a}\left(-\frac{b}{a}\right) = \frac{b^2}{a^2},\ c_{21} = c_1\, b\, e^{-a} = b\left(1+\frac{b}{a^2}\right)e^{-a},$$

$$d_2 = \frac{d_{21} - b\,d_{11} + b\,d_1}{a} = \frac{1}{a}\left(\frac{b^2}{a^2} + \frac{b^2}{a^2} - \frac{b^2}{a}\right) = 2\frac{b^2}{a^3} - \frac{b^2}{a^2}.$$

The requirement $y_2(1) = y_1(1)$ must be enforced.

$$y_2(1) = y_1(1) \Rightarrow d_2 + d_{21} + (c_2 + c_{21})e^a = d_1 + d_{11} + c_1 e^a = -\frac{b^2}{a^2} - \frac{b}{a} + e^a + \frac{b}{a^2}e^a$$

$$\Rightarrow c_2 = \left(\frac{-b}{a^2} - \frac{b}{a} + e^a + \frac{b}{a^2}e^a - 2\frac{b^2}{a^3} + \frac{b^2}{a^2} - \frac{b^2}{a^2} - b - \frac{b^2}{a^2}\right)e^{-a}.$$ Hence,

$$x(t) = y_2(t) = 2\frac{b^2}{a^3} - \frac{b^2}{a^2} + \frac{b^2}{a^2}t - \frac{b}{a^2}e^{a(t-1)} - \frac{b}{a}e^{a(t-1)} + e^{at} + \frac{b}{a^2}e^{at}$$
$$-2\frac{b^2}{a^3}e^{a(t-1)} - be^{a(t-1)} - \frac{b^2}{a^2}e^{a(t-1)} + b\left(1+\frac{b}{a^2}\right)te^{a(t-1)}, \quad \text{for } t \in [1,\ 2].$$

$$y_2(1) = 2\frac{b^2}{a^3} - \frac{b^2}{a^2} + \frac{b^2}{a^2} - \frac{b}{a^2} - \frac{b}{a} + e^a + \frac{b}{a^2}e^a - 2\frac{b^2}{a^3} - b - \frac{b^2}{a^2} + b + \frac{b}{a^2} = -\frac{b}{a^2} - \frac{b}{a} + e^a + \frac{b}{a^2}e^a$$

$y_1(1) = -\frac{b}{a^2} - \frac{b}{a} + e^a + \frac{b}{a^2}e^a$. Therefore the condition $y_2(1) = y_1(1)$ is satisfied.

We proceed to show that $y_2(t)$ satisfies the delay differential equation on $[1,\ 2]$.

$$\dot{y}_2(t) = \frac{b^2}{a^2} - \frac{b}{a}e^{a(t-1)} - be^{a(t-1)} + ae^{at} + \frac{b}{a}e^{at} - \frac{2b^2}{a^2}e^{a(t-1)} - abe^{a(t-1)} - \frac{b^2}{a}e^{a(t-1)} + be^{a(t-1)}$$
$$+\frac{b^2}{a^2}e^{a(t-1)} + abte^{a(t-1)} + \frac{b^2}{a}te^{a(t-1)}.$$

$$ay_2(t) + by_1(t-1) = 2\frac{b^2}{a^2} - \frac{b^2}{a^2} + \frac{b^2}{a}t - \frac{b}{a}e^{a(t-1)} - be^{a(t-1)} + ae^{at} + \frac{b}{a}e^{at} - 2\frac{b^2}{a^2}e^{a(t-1)} - abe^{a(t-1)}$$
$$-\frac{b^2}{a}e^{a(t-1)} + abte^{a(t-1)} + \frac{b^2}{a}te^{a(t-1)} + \left(-\frac{b^2}{a^2} - \frac{b^2}{a}t + \frac{b^2}{a} + be^{a(t-1)} + \frac{b^2}{a^2}e^{a(t-1)}\right).$$

Therefore $\dot{y}_2(t) = ay_2(t) + by_1(t-1).$ This completes the proof that $y_2(t)$ solves the initial function problem on the interval $[1,\ 2]$.

Method 2: Algorithmic Steps

Set k = 1 in (4) and then $y_0\left(t-h\right) = \phi\left(t-h\right)$, evaluate the integral.
Set $y_1(0) = \phi(0) \equiv y_0(0)$ to secure C_1, following the computation of the integral on the right of (4)
Having initialized the process in step 1, $y_k(t)$ is obtained by using (4) in conjunction with the condition $y_k\left(\left[k-1\right]h\right) = y_{k-1}\left(\left[k-1\right]h\right),\ k = 2, 3, \cdots$. The integral in (4) can be determined since $y_{k-1}\left(t-h\right)$ is already known. The condition $y_k\left(\left[k-1\right]h\right) = y_{k-1}\left(\left[k-1\right]h\right)$ secures C_k.
Consider the interval $J_1 = [0,\ 1]$. Set $y_0(t-1) = \phi(t-1) = 1 + t - 1 = t,$ on $J_0 = [-1,\ 0]$. Apply (4) with $k = 1,\ h = 1$ to get $y_1\left(t\right) = e^{a\,t}\left[\int b\, e^{-a\,t}\, t\, dt + C_1\right] = e^{at}\left[-\frac{b}{a}t\, e^{-at} - \frac{b}{a^2}e^{-at} + C_1\right].$

$$y_1(0)=\phi(0)=1 \Rightarrow -\frac{b}{a^2}+C_1=1 \Rightarrow C_1=1+\frac{b}{a^2} \Rightarrow y_1(t)=-\frac{b}{a}t-\frac{b}{a^2}+\left(1+\frac{b}{a^2}\right)e^{at}$$

$$\Rightarrow y_1(t)=-\frac{b}{a^2}-\frac{b}{a}t+\left(1+\frac{1}{a^2}\right)e^{at}.$$

Consider the interval $J_2=[1,\ 2]$. Set $y_2(1)=y_1(1)=-\frac{b}{a^2}-\frac{b}{a}+e^a+\frac{b}{a^2}e^a$. Apply (4), with $k=2,$ to get

$$y_2(t)=e^{at}\left[\int b\,e^{-at}y_1(t-1)\,dt+C_2\right]=e^{at}\left[\int b\,e^{-at}\left(-\frac{b}{a^2}-\frac{b}{a}(t-1)+\left(1+\frac{b}{a^2}\right)e^{a(t-1)}\right)dt+C_2\right]$$

$$\Rightarrow y_2(t)=e^{at}\left[-\frac{b^2}{a^2}\int e^{-at}dt-\frac{b^2}{a}\int e^{-at}t\,dt+\frac{b^2}{a}\int e^{-at}dt+e^{-a}b\int dt+e^{-a}\frac{b^2}{a^2}\int dt+C_2\right]$$

$$\Rightarrow y_2(t)=e^{at}\left[\frac{b^2}{a^3}e^{-at}-\frac{b^2}{a^2}e^{-at}+e^{-a}bt+e^{-a}\frac{b^2}{a^2}t-\frac{b^2}{a}\left(-\frac{1}{a}te^{-at}-\frac{1}{a^2}e^{-at}\right)+C_2\right]$$

$$y_2(1)=-\frac{b}{a^2}-\frac{b}{a}+e^a+\frac{b}{a^2}e^a=e^a\left[\frac{b^2}{a^3}e^{-a}-\frac{b^2}{a^2}e^{-a}+e^{-a}b+e^{-a}\frac{b^2}{a^2}-\frac{b^2}{a}\left(-\frac{1}{a}e^{-a}-\frac{1}{a^2}e^{-a}\right)+C_2\right]$$

$$\Rightarrow -\frac{b}{a^2}-\frac{b}{a}+e^a+\frac{b}{a^2}e^a=\frac{b^2}{a^3}-\frac{b^2}{a^2}+b+\frac{b^2}{a^2}+\frac{b^2}{a^2}+\frac{b^2}{a^3}+C_2e^a$$

$$\Rightarrow -\frac{b}{a^2}-\frac{b}{a}+e^a+\frac{b}{a^2}e^a=2\frac{b^2}{a^3}+b+\frac{b^2}{a^2}+C_2e^a$$

$$\Rightarrow C_2=-\frac{b}{a^2}e^{-a}-\frac{b}{a}e^{-a}+1+\frac{b}{a^2}-2\frac{b^2}{a^3}e^{-a}-be^{-a}-\frac{b^2}{a^2}e^{-a}$$

$$\Rightarrow y_2(t)=e^{at}\left[\frac{b^2}{a^3}e^{-at}-\frac{b^2}{a^2}e^{-at}+e^{-a}bt+e^{-a}\frac{b^2}{a^2}t-\frac{b^2}{a}\left(-\frac{1}{a}te^{-at}-\frac{1}{a^2}e^{-at}\right)\right]$$
$$+\ e^{at}\left(-\frac{b}{a^2}e^{-a}-\frac{b}{a}e^{-a}+1+\frac{b}{a^2}-2\frac{b^2}{a^3}e^{-a}-be^{-a}-\frac{b^2}{a^2}e^{-a}\right)$$

$$\Rightarrow y_2(t)=\frac{b^2}{a^3}-\frac{b^2}{a^2}+e^{a(t-1)}bt+\frac{b^2}{a^2}te^{a(t-1)}+\frac{b^2}{a^2}t+\frac{b^2}{a^3}$$
$$+\ e^{at}\left(-\frac{b}{a^2}e^{-a}-\frac{b}{a}e^{-a}+1+\frac{b}{a^2}-2\frac{b^2}{a^3}e^{-a}-be^{-a}-\frac{b^2}{a^2}e^{-a}\right)$$

$$\Rightarrow y_2(t)=\frac{b^2}{a^3}-\frac{b^2}{a^2}+e^{a(t-1)}bt+\frac{b^2}{a^2}te^{a(t-1)}+\frac{b^2}{a^2}t+\frac{b^2}{a^3}$$
$$-\frac{b}{a^2}e^{a(t-1)}-\frac{b}{a}e^{a(t-1)}+e^{at}+\frac{b}{a^2}e^{at}-2\frac{b^2}{a^3}e^{a(t-1)}-be^{a(t-1)}-\frac{b^2}{a^2}e^{a(t-1)}$$

$$\Rightarrow y_2(t)=\frac{b^2}{a^3}-\frac{b^2}{a^2}+e^{a(t-1)}bt+\frac{b^2}{a^2}te^{a(t-1)}+\frac{b^2}{a^2}t+\frac{b^2}{a^3}$$
$$-\frac{b}{a^2}e^{a(t-1)}-\frac{b}{a}e^{a(t-1)}+e^{at}+\frac{b}{a^2}e^{at}-2\frac{b^2}{a^3}e^{a(t-1)}-be^{a(t-1)}-\frac{b^2}{a^2}e^{a(t-1)}$$

$$=2\frac{b^2}{a^3}-\frac{b^2}{a^2}+\frac{b^2}{a^2}t+\left(1+\frac{b}{a^2}\right)e^{at}-\left(b+\frac{b}{a}+\frac{b}{a^2}+\frac{b^2}{a^2}+2\frac{b^2}{a^3}\right)e^{a(t-1)}+\left(b+\frac{b^2}{a^2}\right)te^{a(t-1)}$$

$$\Rightarrow x(t)=2\frac{b^2}{a^3}-\frac{b^2}{a^2}+\frac{b^2}{a^2}t+\left(1+\frac{b}{a^2}\right)e^{at}-\left(b+\frac{b}{a}+\frac{b}{a^2}+\frac{b^2}{a^2}+2\frac{b^2}{a^3}\right)e^{a(t-1)}+\left(b+\frac{b^2}{a^2}\right)te^{a(t-1)}$$

is the solution of the initial function problem on the interval $[1, 2]$. This is consistent with the result obtained using Method 1.

The case $a=0 \Rightarrow \dot{x}(t)=bx(t-1),\ t\geq 0;\ x(t)=\phi(t)=1+t,\ t\in[-1,0].$ Hence

$\dot{x}(t)=bt \Rightarrow x(t)=\frac{1}{2}bt^2+d_1, t\in[0,1].$ The continuity condition $x(0)=\phi(0)=1$

$\Rightarrow d_1=1 \Rightarrow x(t)\equiv y_1(t)=\frac{1}{2}bt^2+1$ on $[0,1]$.

On (1,2), $\dot{x}(t)=\frac{1}{2}b^2(t-1)^2+b \Rightarrow x(t)\equiv y_2(t)=\frac{1}{6}b^2(t-1)^3+bt+d_2.$

The continuity condition $y_2(1)=y_1(1)\Rightarrow d_2=1-\frac{1}{2}b \Rightarrow x(t)=y_2(t)=\frac{1}{6}b^2(t-1)^3+bt-\frac{b}{2}+1.$

Using the relation $y_k(t)=e^{at}\left[\int b\, e^{-at}y_{k-1}(t-h)\,dt+C_k\right]$, with $a=0, h=1$ and $k=1$ yields

$$y_1(t)=\int by_0(t-h)\,dt+C_1=\int b\phi(t-1)\,dt+C_1=\int btdt+C_1=b\frac{t^2}{2}+C_1;\ y_1(0)=\phi(0)=1\Rightarrow C_1=1$$

$$\Rightarrow y_1(t)=x(t)=b\frac{t^2}{2}+1,\ t\in[0,1].$$

$$y_2(t)=\int by_1(t-h)\,dt+C_2=\int b\left[\frac{1}{2}b(t-1)^2+1\right]dt+C_2=\frac{1}{6}b^2(t-1)^3+bt+C_2;$$

$$y_2(1)=y_1(1)\Rightarrow b+C_2=\frac{1}{2}b+1\Rightarrow C_2=1-\frac{1}{2}b\Rightarrow y_2(t)=x(t)=\frac{1}{6}b^2(t-1)^3+bt+1-\frac{1}{2}b, t\in[1,2].$$

This also agrees with the preceding result.

Nature of nontrivial oscillatory solutions

One can appreciate the tedium involved in extending the solutions beyond the interval [1, 2]. Equation (1) is a first order scalar, linear, homogeneous delay differential equation with real coefficients. Solutions obtained thus far have been real non-oscillatory.

In general oscillatory solutions of (2) can be obtained in the absence of initial function specification by assuming solutions of the form:

$$x(t)=e^{\lambda t}, \qquad (8)$$

to obtain the transcendental equation:

$$\lambda-a-de^{-\lambda h}\quad 0 \qquad (9)$$

This transcendental equation is analyzed completely to determine the nature of nontrivial solutions of oscillatory type.

We assert that in general real solutions do not exist in the form $x(t)=e^{\lambda t}$.

Our analysis and conclusion in the next example will validate the above assertion.

Example 2

Prove that: $\dot{x}(t)=-x(t-\pi/2)$ (10)

admits non-trivial oscillating solutions and determine the nature of such solutions.

Prove also that there are no nontrivial oscillatory solutions if a constant initial function is specified for the delay differential equation (10).

Proof

Our proof will rely partly on the following result which we proceed to establish:

$$x \ge \ln(1+x)\ \forall\ x > -1 \qquad (11)$$

To establish this result, observe that the result is equivalent to:

$$e^x \ge 1+x \qquad (12)$$

Set:

$$f(x) = e^x - (1+x) \qquad (13)$$

Then, $$f'(x) = \begin{cases} e^x - 1 \ge 0,\ \forall\ x \ge 0 \\ \le 0,\ \forall\ x \le 0 \end{cases}$$

$f(0) = 0$, $f'(x) = 0$ iff $x = 0$. Thus (0, 0) is the only stationary point of *f(x)*.

Now $f''(x) = e^x,\ \forall\ x$. In particular $f''(0) > 0 \Rightarrow (0,0)$ is a global minimum of $f(x)$.

Hence $f(x) \ge 0\ \forall\ x$; that is $e^x \ge 1+x,\ \forall\ x$; this translates to $x \ge \ln(1+x)$.

Assume a solution of the form $x(t) = e^{\lambda t}$ in the example. Then $\lambda = -e^{-\frac{\lambda\pi}{2}}$, noting that

$$a = 0, b = -1 \text{ and } h = \frac{\pi}{2}.$$

The equation:

$$\lambda = -e^{-\frac{\lambda\pi}{2}} \qquad (14)$$

is not satisfied for any real $\lambda \ge 0$, since $-e^{\frac{-\lambda\pi}{2}} < 0$ if λ is real. Therefore if λ is real, then $\lambda < 0$, so that $-\lambda = e^{-\lambda\pi/2} \Rightarrow \ln(-\lambda) = -\lambda\pi/2$

But $\ln(-\lambda) \le -\lambda - 1\quad \forall\ \lambda < 0$, by an appeal to (11).

Therefore, $-\lambda - 1 \ge -\lambda\pi/2$, or $\left(\frac{\pi}{2} - 1\right)\lambda \ge 1$. Since $\pi/2 - 1 > 0$, we conclude that $\lambda \ge \frac{2}{\pi - 2} > 0$

This contradicts the fact that λ must be negative. Therefore, the equation $-\lambda = e^{-\lambda\pi/2}$ cannot be satisfied by any real λ. In other words, any solution of the form $x(t) = e^{\lambda t}$ must yield $\lambda = \alpha + \beta i$, when α and β are real numbers such that $\beta \ne 0$. Plugging this λ into the transcendental equation yields:

$$\alpha + \beta i = -e^{-\frac{\pi\alpha}{2}}\left[\cos\left(\frac{\beta\pi}{2}\right) + i\sin\left(\frac{\beta\pi}{2}\right)\right], \qquad (15)$$

from which we infer that:

$$\alpha = -e^{-\frac{\pi\alpha}{2}}\cos\left(\frac{\beta\pi}{2}\right) \qquad (16)$$

and:

$$\beta = -e^{-\frac{\pi\alpha}{2}} \sin\left(\frac{\beta\pi}{2}\right) \qquad (17)$$

If $\alpha = 0$, the above relations imply that:

$$\cos\left(\frac{\beta\pi}{2}\right) = 0 \qquad (18)$$

and:

$$\beta = -\sin\left(\frac{\beta\pi}{2}\right) \qquad (19)$$

From (18) we infer that: $\frac{\beta\pi}{2} = (2k+1)\frac{\pi}{2}$, k integer

$$\Rightarrow \beta = 2k+1 \qquad (20)$$

$\Rightarrow \beta$ is an odd integer, positive or negative.

$(19) \Rightarrow$ the relation $|\beta| > 1$ is infeasible.

Therefore: $|\beta| \le 1$ (21)

The relations (20) and (21) imply that $|\beta| = 1$, thus $\beta = -1$ or 1; hence if $\alpha = 1$, then $\beta = -1$ or 1 and

$$x(t) = c_1 \cos t + c_2 \sin t, \qquad (22)$$

for real constants c_1 and c_2.

(22) can be rewritten in the form:

$$x(t) = \sqrt{c_1^2 + c_2^2} \sin(t - \delta) \qquad (23)$$

for some real δ with $0 < \delta < \pi/2$. Clearly if c_1 and c_2 do not vanish at the same time, then $\sqrt{c_1^2 + c_2^2} > 0$, so that (23) represents non-trivial oscillatory solutions, being sinusoidal.

Remarks

If $\beta = 0$, $(8) \Rightarrow \alpha = -e^{-\frac{\pi\alpha}{2}} \Rightarrow \alpha \ne 0$ since $-e^{-\frac{\pi\alpha}{2}} < 0$.

If $\alpha \ne 0$, then $\beta \ne 0$, since real solutions do not exist in the form (8)

If $\alpha > 0$, then $\cos\left(\frac{\beta\pi}{2}\right) < 0 \Rightarrow (1+4k)\frac{\pi}{2} < \beta\frac{\pi}{2} < (3+4k)\frac{\pi}{2}$

$\Rightarrow 1+4k < \beta < 3+4k$, for any integer k. If $\alpha < 0$, then $\cos\left(\beta\frac{\pi}{2}\right) > 0$

$\Rightarrow 4k < \beta < 1+4k$ (1st quadrant) or $3+4k < \beta < 4+4k$ (4th quadrant for any integer k).

(16) and (17) imply that $-\alpha e^{\frac{\pi\alpha}{2}} = \cos\left(\beta\frac{\pi}{2}\right)$ and $-\beta e^{\frac{\pi\alpha}{2}} = \sin\left(\beta\frac{\pi}{2}\right)$

$$\Rightarrow \alpha^2 e^{\pi\alpha} + \beta^2 e^{\pi\alpha} = 1 \Rightarrow (\alpha^2 + \beta^2) e^{\pi\alpha} = 1 \Rightarrow \alpha^2 + \beta^2 = e^{-\pi\alpha} \Rightarrow \beta^2 = e^{-\pi\alpha} - \alpha^2$$

$\Rightarrow \beta = \pm\sqrt{e^{-\pi\alpha} - \alpha^2} \Rightarrow$ solutions to the transcendental equation (9) are given by $\lambda = \alpha + i\beta$ and the set $\left\{(\alpha,\beta): \beta = \pm\sqrt{e^{-\pi\alpha} - \alpha^2}, \beta > 0\right\}$. There is an infinity of oscillatory solutions. These solutions are given by:

$$x(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right), \tag{24}$$

for all real α, β for which $\beta^2 = e^{-\pi\alpha} - \alpha^2$ and $e^{-\pi\alpha} > \alpha^2$. These solutions are nontrivial if $c_1{}^2 + c_2{}^2 > 0$. Furthermore, if $\alpha \neq 0$ then $\beta \neq 0$ and $\tan\left(\pi/2\,\beta\right) = -\beta/\alpha$. If $\alpha > 0$, then $1+4k < \beta < 3+4k$ and if $\alpha < 0$, then, $4k < \beta < 1+4k$, for any integer *k*. If $\alpha < 0$, then $-\pi\alpha > 0$ and $e^{-\pi\alpha} = 1 + (-\pi\alpha) + \frac{\pi^2}{2}\alpha^2 + \sum_{j=3}^{\infty}\frac{(-\pi\alpha)^j}{j!} > \frac{\pi^2}{2}\alpha^2 > \alpha^2$. Therefore $\beta^2 > 0$ if $\alpha < 0$.

If $\alpha > 0$, the relation $\beta^2 = e^{-\pi\alpha} - \alpha^2$ is infeasible for $\alpha \geq 1$, since $\alpha^2 + \beta^2 = e^{-\pi\alpha} < 1$. In fact it is feasible for $0 < \alpha < 0.45445$. In other words, $\beta^2 > 0$, for $0 < \alpha < 0.45445$. Clearly $e^{-\pi\alpha} < 1$ for $\alpha > 0$. In particular $0 < \beta^2 < 1$ for $0 < \alpha < 0.45445$. Hence, $\beta \neq 0, -1 < \beta < 1$ for this set of α -values.

From the preceding analysis, we have obtained a complete characterization of all non-oscillatory solutions of (10) in the absence of initial functions.

Next we will prove that there is no nontrivial oscillatory solution if a constant initial function is specified for the delay differential equation (10).

Suppose ϕ is a continuous initial function specified by $x(t) = \phi(t),\ t \in \left[-\pi/2,\ 0\right]$ with respect to (10). Then (24) $\Rightarrow x(0) = c_1 = \phi(0)$. If $\alpha = 0$, so that $\beta = \pm 1$, then

$x(t) = c_1 \cos t + c_2 \sin t \Rightarrow x\left(-\pi/2\right) = -c_2 = \phi\left(-\pi/2\right)$ and $x\left(-\pi/4\right) = \frac{\sqrt{2}}{2}(c_1 - c_2) = \phi\left(-\pi/4\right)$.

Suppose $\phi(t) = \phi_0$, a constant. Then, $c_1 = -c_2 = \frac{\sqrt{2}}{2}(c_1 - c_2)$

$\Rightarrow c_1 = \frac{\sqrt{2}}{2}(c_1 + c_1) = c_1\sqrt{2} \Rightarrow c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow x(t) = 0\ \forall\ t \geq -\pi/2 \Rightarrow$ trivial solution.

Suppose $\alpha < 0$, so that $4k < \beta < 1+4k$ or $3+4k < \beta < 4+4k$, *k* integer.

Recall that $x(t) = e^{\alpha t}\left[c_1\cos(\beta t) + c_2\sin(\beta t)\right], t \in \left[\frac{-\pi}{2}, \infty\right);\ x(t) = \phi_0,\ \forall t \in \left[\frac{-\pi}{2}, 0\right]$.

Let us evaluate $x(t)$ for all t such that $\beta t \in \left\{-\frac{\pi}{4}, -\frac{\pi}{2}\right\} \Rightarrow t \in \left\{-\frac{\pi}{4\beta}, -\frac{\pi}{2\beta}\right\}$. Since $t \in \left[-\frac{\pi}{2}, \infty\right)$, these requirements are simultaneously satisfied only by $\beta \geq 1$. Therefore,

for feasibility take $k > 0$ such that $\frac{-\pi}{2\beta} \in \left[-\frac{\pi}{2},\ 0\right]$ and $\frac{-\pi}{4\beta} \in \left[-\frac{\pi}{2},\ 0\right]$. From the preceding results,

$x\left(\frac{-\pi}{2\beta}\right) = e^{\frac{-\pi}{2\beta}\alpha}\left(0 - c_2\right) = \phi_0$ and $x\left(\frac{-\pi}{4\beta}\right) = e^{\frac{-\pi}{4\beta}\alpha}\left[c_1 - c_2\right]\frac{\sqrt{2}}{2} = \phi_0$

But $x(0) = c_1 = \phi_0 \Rightarrow c_2 = e^{\frac{-\pi\alpha}{4\beta}}c_2\frac{\sqrt{2}}{2} + e^{\frac{\pi\alpha}{4\beta}}c_2\frac{\sqrt{2}}{2} \Rightarrow \left[1 - \frac{\sqrt{2}}{2}\left(e^{\frac{-\pi\alpha}{4\beta}} + e^{\frac{\pi\alpha}{4\beta}}\right)\right]c_2 = 0,$

$\Rightarrow c_2 = 0$ or $\left[1 - \frac{\sqrt{2}}{2}\left(e^{\frac{-\pi\alpha}{4\beta}} + e^{\frac{\pi\alpha}{4\beta}}\right)\right] = 0.$ But $\left[1 - \frac{\sqrt{2}}{2}\left(e^{\frac{-\pi\alpha}{4\beta}} + e^{\frac{\pi\alpha}{4\beta}}\right)\right] = 0$ if and only if

$$e^{\frac{-\pi\alpha}{4\beta}} + e^{\frac{\pi\alpha}{4\beta}} = \frac{\sqrt{2}}{2}.$$

Let us examine the function $w = e^{v} + e^{-v} - \frac{\sqrt{2}}{2}.$ Clearly, $\frac{dw}{dv} = e^{v} - e^{-v}\begin{cases} < 0, \text{ if } v < 0 \\ = 0, \text{ if } v = 0 \\ > 0, \text{ if } v > 0 \end{cases}$

Therefore w is decreasing for $v < 0$ and increasing for $v > 0$; w has global minimum at $v = 0$, with minimum value $2 - \sqrt{2} \approx 0.585786.$ (Note: $w''(v) > 0\ \forall v \in \mathbf{R}$). We deduce that w is never less that $2 - \sqrt{2}.$ This proves that $\left[1 - \frac{\sqrt{2}}{2}\left(e^{\frac{-\pi\alpha}{4\beta}} + e^{\frac{\pi\alpha}{4\beta}}\right)\right] \neq 0,$ proving that $c_2 = 0$; hence $c_1 = 0.$

We conclude that $x(t) = 0, \forall t \geq \frac{-\pi}{2},$ which implies that only the trivial solution exists.

For the case $\alpha > 0,$ we earlier proved that $0 < \alpha < 0.45445$ and $-1 < \beta < 1,$ for feasibility.

We wish to evaluate $x(t)$ for all t such that $\beta t \in \left\{-\frac{\pi}{4}, -\frac{\pi}{2}\right\} \Rightarrow t \in \left\{-\frac{\pi}{4\beta}, -\frac{\pi}{2\beta}\right\}.$ Since $t \in \left[-\frac{\pi}{2}, \infty\right),$ these requirements are simultaneously satisfied only by $\beta \geq 1.$ Since $-1 < \beta < 1,$ the case $\alpha > 0$ is infeasible. These prove that there are no nontrivial oscillatory solutions if a constant initial function is specified for the delay differential equation (10).

Remarks:

One could also reason along the same lines as in Driver (1977) to establish the existence of infinitely many complex roots for the equation $\lambda = -e^{-\lambda\frac{\pi}{2}}.$ To achieve this, set $z = \frac{1}{\lambda}, w(z) = 0,$ where $w(z) = 1 + ze^{-\frac{h}{z}},$ and h is an arbitrary delay. Clearly $w(z) \neq 1,$ for $z \neq 0.$ But $w(z)$ has an isolated singularity at $z = 0.$ Thus by Picard's theorem, in every neighbourhood of $z = 0, w(z)$ takes on every value infinitely often with the exception of 1. Hence $w(z) = 0$ infinitely often and so the equation $\lambda = -e^{-\lambda\frac{\pi}{2}}$ must have infinitely many complex roots: $\lambda = \alpha + \beta i; \alpha, \beta$ real. However the subsequent analyses undertaken above are imperative for solving the problem.

III. CONCLUSION

This article gave an exposition on how single-delay scalar differential equations could be solved using two methods presented in algorithmic formats. The results, which relied on the continuity of solutions, verified the consistency of both methods using the same problem, as well as gave a clue on the associated level of difficulty. It went on to conduct detailed and rigorous analyses on the nature, characterization and flavor of solutions for some problem instances. The article revealed that functional differential equations are difficult to solve, even for the most innocent of problems-such as the autonomous linear delay type with further simplifying restrictions; this difficulty is largely attributable to the many issues that must be dealt with, two such issues being the transcendental nature of the associated characteristic equations and the prescribed class of initial functions.

REFERENCE

[1] Driver, R. D. (1977). Ordinary and Delay Differential Equations. Applied Mathematical Sciences 20, Springer-Verlag, New York.

[2] Hale, J. K. (1977). Theory of functional differential equations. *Applied Mathematical Science, Vol. 3, Springer-Verlag, New York.*