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Alternative Optimal Expressions for the Structure and Cardinalities of Determining Matrices for a Class Of Double – Delay Control Systems

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ABSTRACT : *This paper exploited the results in Ukwu [1] to obtain the cardinalities, computing complexity and alternative optimal expressions for the determining matrices of a class of double – delay autonomous linear differential systems through a sequence of theorems and corollaries and the invocation of key facts about permutations. The paper also derived a unifying theorem for the major results in [1].The proofs were achieved using ingenious combinations of summation notations, the multinomial distribution, and greatest integer functions, change of variables techniques and compositions of signum and max functions. The computations were mathematically illustrated and implemented on Microsoft Excel platform for some problem instances.*

KEYWORDS:*Cardinalities, Determining, Double, Platform, Structure.*

I. INTRODUCTION

The importance of determining matrices stems from the fact that they constitute the optimal instrumentality for the determination of Euclidean controllability and compactness of cores of Euclidean targets. See Gabasov and Kirillova [2] and [3 and 4]. In sharp contrast to determining matrices, the use of indices of control systems on the one hand and the application of controllability Grammians on the other, for the investigation of the Euclidean controllability of systems can at the very best be quite computationally challenging and at the worst mathematically intractable. Thus, determining matrices are beautiful brides for the interrogation of the controllability disposition of delay control systems. See [1]. However up-to-date review of literature on this subject reveals that there was no result on the structure of determining matrices for double-delay systems prior to [1]. This could be attributed to the severe difficulty in identifying recognizable mathematical patterns needed for inductive proof of any claimed result. This paper extends and embellishes the main results in [1] by effectively resolving ambiguities in permutation infeasibilities and obviating the need for explicit piece-wise representations of $Q_k(jh)$, as well as conducting careful analyses of the computational complexity and cardinalities of the determining matrices, thus filling the yawning gaps in [1] and much more.

II. MATERIALS AND METHODS

The derivation of necessary and sufficient condition for the Euclidean controllability of system (1) on the interval $[0,\ t_1]$, using determining matrices depends on the following:

[1] obtaining workable expressions for the determining equations of the $n \times n$ matrices $Q_k(jh)$, for $j: t_1 - jh > 0, k = 0, 1, \cdots$

[2] showing that $\Delta X^{(k)}(t_1 - jh, t_1) = (-1)^k Q_k(jh)$, for j: $t_1 - jh > 0, k = 0, 1, \ldots$

[3] where $X(\tau, t_1)$ is the index of the control system (1) below and

[4] $\Delta X^{(k)}(t_1 - jh, t_1) = X^{(k)}((t_1 - jh)^-, t_1) - X^{(k)}((t_1 - jh)^+, t_1)$. Cf. [1].

[5] showing that $Q_\infty(t_1)$ (t_1) is a linear combination of

$$Q_0(s), Q_1(s), \cdots, Q_{n-1}(s); s = 0, h, \cdots (n-1)h.$$

Our objective is to embellish and unify the subtasks in task (i) as well as investigate the cardinalities and computational complexity of the determining matrices. Tasks (ii) and (iii) will be prosecuted in other papers.

2.1 Identification of Work-Based Double-Delay Autonomous Control System:
We consider the double-delay autonomous control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + B\,u(t);\ t \geq 0 \tag{1}$$

$$x(t) = \phi(t),\ t \in [-2h, 0],\ h > 0 \tag{2}$$

where A_0, A_1, A_2 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constant matrix with real entries. The initial function ϕ is in $C([-2h, 0], \mathbf{R}^n)$, the space of continuous functions from $[-2h, 0]$ into the real n-dimension Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t \in [-2h, 0]} |\phi(t)|$, (the sup norm). The control u is in the space $L_\infty([0, t_1], \mathbf{R}^n)$, the space of essentially bounded measurable functions taking $[0, t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = ess \sup_{t \in [0, t_1]} |u(t)|$.

Any control $u \in L_\infty([0, t_1], \mathbf{R}^n)$ will be referred to as an admissible control. For full discussion on the spaces C^{p-1} and L_p (or L^p), $p \in \{1, ..., \infty\}$. Cf. Chidume [5, 6] and Royden [7].

2.2 Definition, Existence and Uniqueness of Determining Matrices for System (1)

Let $Q_k(s)$ be then $n \times n$ matrix function defined by:

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) + A_2 Q_{k-1}(s-2h) \tag{3}$$

for $k = 1, 2, \cdots;\ s > 0,$ with initial conditions:

$$Q_0(0) = I_n \tag{4}$$

$$Q_0(s) = 0;\ s \neq 0 \tag{5}$$

These initial conditions guarantee the unique solvability of (1.7). Cf. [2].

Let r_0, r_1, r_2 be nonnegative integers and let $P_{0(r_0),1(r_1),2(r_2)}$ denote the set of all permutations of $\underbrace{0,0,...0}_{r_0 \text{ times}}\ \underbrace{1,1,...1}_{r_1 \text{ times}}\ \underbrace{2,2,...2}_{r_2 \text{ times}}$: the permutations of the objects $0, 1,$ and 2 in which i appears r_i times, $i \in \{0,1,2\}$. Cf. [1].

The stage is now set for our results and discussions with the deployment of key facts about permutations.

III. RESULTS AND DISCUSSIONS

Ukwu [1] obtained the following expressions for the determining matrices of system (1)

3.1 Theorem on $Q_k(jh);\ 0 \leq j \leq k, k \neq 0$

For $0 \leq j \leq k,\ j,\ k$ integers, $k \neq 0,$

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k) \in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1} \cdots A_{v_k}$$

3.2 Theorem on $Q_k(jh);\ j \geq k \geq 1$

For $j \geq k \geq 1,\ j,\ k$ integers,

$$Q_k(jh) = \begin{cases} \displaystyle\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k) \in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1} \cdots A_{v_k}, & 1 \leq j \leq 2k \\ 0, & j \geq 2k+1 \end{cases}$$

Remarks

The expressions for $Q_k(jh)$ in theorems 3.1 and 3.2 coincide when $j = k \neq 0$, as should be expected.

3.3 First Corollary to theorem 3.2

(i) If $j > k$ and $A_2 = 0$, in theorem 3.2, then $Q_k(jh) = 0$.

(ii) $A_2 = 0 \Rightarrow Q_k(jh) = \left[\sum_{(v_1,\cdots,v_k)\in P_{0(k-j),1(j)}} A_{v_1} \cdots A_{v_k} \right] \mathrm{sgn}(\max\{0, k+1-j\})$

Proof

(i) Observe that in the expression for $Q_k(jh)$, A_2 appears

$r + j - k$ times for $r \in \left\{0, \cdots, \left[\left[\frac{2k-j}{2}\right]\right]\right\}$, in all feasible $\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \frac{k!}{r!(2k-j-2r)!(r+j-k)!}$ permutations of

A_0, A_1 and A_2. Since r is a non-negative integer, if $j > k$, then $j - k > 0$ and $r + j - k$ is a positive integer. Therefore, all the permutations must contain A_2. Hence, $j > k$, $A_2 = 0 \Rightarrow$ the sum of all permutation products is zero, and so $Q_k(jh) = 0$, for $j > k$ and $A_2 = 0$.

This result is in agreement with [3], p.10.

(ii) $A_2 = 0 \Rightarrow$ the only surviving terms in the expression for $Q_k(jh)$ are those with $r + j - k = 0$, in which case $r = k - j$ and $2k - j - 2r = 2k - j - 2(k - j) = j$; these, together with (i) reduce the expression for $Q_k(jh)$ to $Q_k(jh) = \left[\sum_{(v_1,\cdots,v_k)\in P_{0(k-j),1(j)}} A_{v_1} \cdots A_{v_k} \right] \mathrm{sgn}(\max\{0, k+1-j\})$, as desired.

3.4 Lemma on Consistency of theorem 3.2 for $j \geq 2k-2$.

Lemma 3.4 is consistent with lemma 2.6 of [1] (the results coincide) if $j \geq 2k-2$.

Proof

If $j \geq 2k+1$, then $2k - j < 0$, and hence $Q_k(jh) = 0$, for $j > 2k+1$, in theorem 3.2. We are now left with $j \in \{2k-2,\ 2k-1,\ 2k\}$.

$j = 2k-2 \Rightarrow r \in \{0, 1\}$, lemma 2.6 of [1] $\Rightarrow Q_k(jh) = Q_k([2k-2]h); r \in \{0, 1\}$

$$\Rightarrow Q_k([2k-2]h) = \sum_{(v_1,\cdots,v_k)\in P_{0(0),1(2),2(k-2)}} A_{v_1} \cdots A_{v_k} + \sum_{(v_1,\cdots,v_k)\in P_{0(1),1(0),2(k-1)}} A_{v_1} \cdots A_{v_k}$$

$$= \sum_{(v_1,\cdots,v_k)\in P_{1(2),2(k-2)}} A_{v_1} \cdots A_{v_k} + \sum_{(v_1,\cdots,v_k)\in P_{0(1),\,2(k-1)}} A_{v_1} \cdots A_{v_k}$$

$$= \sum_{(v_1,\cdots,v_k)\in P_{1(2k-j),2(j-k)}} A_{v_1} \cdots A_{v_k} + \sum_{(v_1,\cdots,v_k)\in P_{0(1),\,2(k-1)}} A_{v_1} \cdots A_{v_k};\ j = 2k-2$$

$j = 2k-1 \Rightarrow r = 0$, in theorem 3.2 $\Rightarrow Q_k(jh) = Q_k([2k-1]h)$

$$\Rightarrow Q_k([2k-1]h) = \sum_{(v_1,\cdots,v_k)\in P_{0(0),1(1),2(k-1)}} A_{v_1} \cdots A_{v_k} = \sum_{(v_1,\cdots,v_k)\in P_{1(1),2(k-1)}} A_{v_1} \cdots A_{v_k}$$

$j = 2k \Rightarrow r = 0$, in theorem 3.2 $\Rightarrow Q_k(jh) = Q_k(2kh)$

$$\Rightarrow Q_k(2kh) = \sum_{(v_1,\cdots,v_k)\in P_{0(0),1(0),2(k-1)}} A_{v_1}\cdots A_{v_k} = \sum_{(v_1,\cdots,v_k)\in P_{1(0),2(k)}} A_{v_1}\cdots A_{v_k} = A_2^k.$$

This concludes the proof of lemma 3.4 and establishes its validity.

3.5 A Theorem unifying theorems 3.1 and 3.2

For all nonnegative integers, $j,\ k;\ j$ and k not simultaneously equal to zero ($j+k\neq 0$),

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}$$

$$= \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k},\ \text{if}\ 0\le j\le k.$$

Proof

For $0\le j\le k, k\neq 0,$

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k}, \text{ by theorem 3.1}$$

Case: j even. Then: $\left[\left[\frac{j}{2}\right]\right]=\frac{j}{2}$, yielding $Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k}$

$$= \sum_{r=k-j}^{\frac{j}{2}+k-j} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k} = \sum_{r=k-j}^{\frac{2k-j}{2}} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k},$$

(using the change of variables $\tilde{r}=r+k-j$)

$$= \sum_{r=k-j}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k} = \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}. \quad (6)$$

The last equality in (6) holds because if $r<k-j,$ then $r+j-k<0,$ rendering those permutations infeasible and hence the sum of the permutation products may be conventionally set equal to zero.

If j is odd, then $j-1$ is even; thus $\left[\left[\frac{j}{2}\right]\right]=\left[\left[\frac{j-1}{2}\right]\right]=\frac{j-1}{2}$ and

$$\left[\left[\frac{j}{2}\right]\right]+k-j=\frac{j-1}{2}+\frac{2(k-j)}{2}=\left[\left[\frac{2k-j-1}{2}\right]\right]=\left[\left[\frac{2k-j}{2}\right]\right]$$

So, theorem 3.5 is valid in all cases for which $0\le j\le k,\ j+k\neq 0$.

If $j=0$ and $k=0,$ then $r=0;$ so, the permutations are nonexistent. Hence, vacuously the sum of the permutation products is zero. By this conventional contraption, the restriction $j+k\neq 0$ cannot be discarded or waived, since $Q_0(0)=I_n$.

3.6 Second Corollary to theorem 3.2: obviates the need for piece-wise representation of $Q_k(jh)$

For all nonnegative integers $j,k: j+k \neq 0, j \geq k,$

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k} \tag{7}$$

Proof

This follows by noting that if $j \geq 2k+1,$ the upper limit becomes negative.

Hence $Q_k(jh)=0, \ \forall j \geq 2k+1;$ this is consistent with lemma 2.6 of [1] and theorem 3.2.

3.7 Third Corollary to thm. 3.2: using a composite function to express $Q_k(jh)$

For all nonnegative integers $j,k: j+k \neq 0$

$$Q_k(jh) = \left[\sum_{r=\max\{0,k-j\}}^{\left|\left[\left[\frac{2k-j}{2}\right]\right]\right|} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}\right] \operatorname{sgn}\left(\max\{0,2k+1-j\}\right) \tag{8}$$

Proof

The first part of the expression for $Q_k(jh)$ in theorem 3.2 holds for nonnegative integers j and $k.$ The absolute value in the upper limit for r, in (8) ensures that the upper limit is never negative. Note that $r \geq 0$ and $2k-j-2r \geq 0$ in the given range for r. We require that $r+j-k$ be nonnegative for permutation feasibility; $r+j-k \geq 0$ if and only if $r \geq k-j.$ Combine this with the condition $r \geq 0$ to deduce that $r \geq \max\{0,k-j\}$, to maintain feasibility. The upper limit for r is never less than the lower limit since $\frac{2k-j}{2} = k - \frac{j}{2} \geq k-j \ \ \forall j \geq 0.$ Finally, the composition of max and sgn ensures that

$$Q_k(jh)=0 \text{ if } j \geq 2k+1, \text{ since } \operatorname{sgn}\left[\max\{0,2k+1-j\}\right] = \begin{cases} 0 & \text{if } j \geq 2k+1 \\ 1 & \text{if } j \leq 2k \end{cases}$$

This completes the proof of the corollary.

The above corollary effectively resolves permutation infeasibilities and obviates the need for explicit piece-wise representations of $Q_k(jh)$.

3.8 Fourth Corollary to theorem. 3.2: using max functions in the range of r to express $Q_k(jh)$

For all nonnegative integers $j,k: j+k \neq 0,$

$$Q_k(jh) = \left[\sum_{r=\max\{0,k-j\}}^{\max\left\{0,\left[\left[\frac{2k-j}{2}\right]\right]\right\}} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}\right] \operatorname{sgn}\left(\max\{0,2k+1-j\}\right) \tag{9}$$

Proof

The proof follows by noting that

$$\max\{0,k-j\} \leq \max\left\{0,\left[\left[\frac{2k-j}{2}\right]\right]\right\} = \begin{cases} \left[\left[\frac{2k-j}{2}\right]\right] = \left|\left[\left[\frac{2k-j}{2}\right]\right]\right|; \ j \leq 2k \\ 0; \ j \geq 2k+1 \left(\Rightarrow \operatorname{sgn}\left[\max\{0,2k+1-j\}\right]=0\right) \end{cases}$$

This completes the proof.

IV. Computational Complexity of $Q_k(jh)$

, even for moderately- sized j and k could be quite tedious. We appeal to multinomial distributions to obtain the following measures of computing complexity with respect to the terms of $Q_k(jh)$.

By lemmas 2.5 and 2.6 of [1],

$|Q_k(jh)| = 0, \forall j \in \left\{ \tilde{j} : \{\min\{\tilde{j}, k\} < 0\} \bigcup \{\tilde{j} \ge 2k+1\} \bigcup \{\tilde{j} \ne 0, \tilde{j}k = 0\right\}, |Q_k(0)| = 1, k \ge 0.$ Note that for $0 \le j \le k, k \ne 0,$ or for $j \ge k \ge 1, j, k$ integers, $|Q_k(jh)|$ is the number of nonzero terms (products) in $Q_k(jh),$ by theorems 3.1 and 3.2.

TABLE 1: COMPUTING COMPLEXITY TABLE FOR $Q_k(jh)$.

	Number of nonzero terms = Number of nonzero products = Cardinality of $Q_k(jh)$.	Number of additions	Size of permutation = sum of powers of the A_is
$Q_k(jh),$ $0 \le j \le k \ne 0$	$\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \frac{k!}{(r+k-j)!(j-2r)!r!}$	$\left[\left[\frac{j}{2}\right]\right]$	k
$Q_k(jh),$ $1 \le k \le j \le 2k$	$\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \frac{k!}{r!(2k-j-2r)!(r+j-k)!}$	$\left[\left[\frac{2k-j}{2}\right]\right]$	k

The complexity table for $Q_k(jh),\ k \ge j$ cannot be obtained by swapping j and $k,$ in $Q_k(jh), j \ge k.$ $Q_k([k+p]h)$ and $Q_{k+p}(kh)$ do not have the same complexity, for any positive integer, $p.$

TABLE 2: Electronic Implementation of Cardinalities of $Q_k(jh), j \ge k$ for Selected Inputs

EXCEL Computations for the number of terms in $Q_k(jh),\ 1 \le k \le j \le 2k, k \in \{2, \cdots, 6\}.$

k	j	$r=0$	1	2	3	Total
		Cardinality components				
2	2	1	2			3
2	3	2				2
2	4	1				1
3	3	1	6			7
3	4	3	3			6
3	5	3				3
3	6	1				1
4	4	1	12	6		19
4	5	4	12			16
4	6	6	4			10
4	7	4				4
4	8	1				1
5	5	1	20	30		51
5	6	5	30	10		45
5	7	10	20			30
5	8	10	5			15
5	9	5				5
5	10	1				1
6	6	1	30	90	20	141
6	7	6	60	60		126
6	8	15	60	15		90
6	9	20	30			50
6	10	15	6			21
6	11	6				6
6	12	1				1

EXCEL Computations for the number of terms in $Q_k(jh),\ 2 \le j \le k, k \in \{2, \cdots, 8\}.$

k	j	$r=0$	1	2	3	4	Total
		Cardinality components					
2	2	1	2				3
3	2	3	3				6
3	3	1	6				7
4	2	6	4				10
4	3	4	12				16
4	4	1	12	6			19
5	2	10	5				15
5	3	10	20				30
5	4	5	30	10			45
5	5	1	20	30			51
6	2	15	6				21
6	3	20	30				50
6	4	15	60	15			90
6	5	6	60	60			126
6	6	1	30	90	20		141
7	2	21	7				28
7	3	35	42				77
7	4	35	105	21			161
7	5	21	140	105			266
7	6	7	105	210	35		357
7	7	1	42	210	140		393
8	2	28	8				36
8	3	56	56				112
8	4	70	168	28			266
8	5	56	280	168			504
8	6	28	280	420	56		784
8	7	8	168	560	280		1016
8	8	1	56	420	560	70	1107

Table 2 was generated using the above table 1 and an embedded Microsoft Excel worksheet.

TABLE 3: Summary Table for Electronic Implementation of Cardinalities of $Q_k(jh), j \geq k$

k	j	No. of terms in $Q_k(jh),\ 2 \leq k \leq j \leq 2k, k \in \{2,\cdots,6\}.$	No. of terms in $Q_k(kh)$	$Q_k(kh)$ Cardinality Ratio
2	2	3	3	
2	3	2		
2	4	1		
3	3	7	7	2.33
3	4	6		
3	5	3		
3	6	1		
4	4	19	19	2.71
4	5	16		
4	6	10		
4	7	4		
4	8	1		
5	5	51	51	2.68
5	6	45		
5	7	30		
5	8	15		
5	9	5		
5	10	1		
6	6	141	141	2.76
6	7	126		
6	8	90		
6	9	50		
6	10	21		
6	11	6		
6	12	1		
7	7	393	393	2.78

k	j	No. of terms in $Q_k(jh),\ 2 \leq j \leq k, k \in \{2,\cdots,8\}.$	No. of terms in $Q_k(kh)$	$Q_k(kh)$ Cardinality ratio
2	2	3	3	
3	2	6		
3	3	7	7	2.33
4	2	10		
4	3	16		
4	4	19	19	2.71
5	2	15		
5	3	30		
5	4	45		
5	5	51	51	2.68
6	2	21		
6	3	50		
6	4	90		
6	5	126		
6	6	141	141	2.76
7	2	28		
7	3	77		
7	4	161		
7	5	266		
7	6	357		

7	7	393	393	2.78
8	2	36		
8	3	112		
8	4	266		
8	5	504		
8	6	784		
8	7	1016		
8	8	1107	1107	2.82

A glance at Table 3 is quite revealing. Observe that $\left|Q_j(kh)\right| > \left|Q_k(jh)\right|$, if $j < k$.
Notice how quickly the cardinalities of $Q_k(jh)$ grow astronomically from 3, for $j+k=4$, to a maximum of 784, for $j+k=14$. In particular, observe how the cardinalities of $Q_k(kh)$ leap from 3, for $k=2$, to 1107, for $k=8$. How, in the world could one manage 1107 permutations for just $Q_8(8h)$, not to bother about $Q_k(kh)$, for larger *k*. Also, the cardinality ratios for the cases $j=k$ reveal that on the average $\left|Q_{k+1}([k+1]h)\right| > 2.68\left|Q_k(kh)\right|$; obviously, $Q_k(kh)$ exhibits rapidly divergent geometric growth. It is clear that long-hand computations for $Q_k(jh)$, even for $j+k=16$, are definitely out of the question.

Practical realities/exigencies dictate that these computations should be implemented electronically for arbitrarily *j* and *k*. These challenges have been tackled head-on; the computations for $Q_k(jh)$ and their cardinalities have been achieved elsewhere on the C^{++} platform, for any appropriate input matrices, A_0, A_1, A_2 and positive integers $j,k : \min\{j,k\} \ge 1$, using the second corollary to theorem 3.2; needless to say that the cases $j,k : jk = 0$ have also been incorporated in the code, using the results in lemma 2.5 of [1].

IV. Mathematical computations of determining matrices and their cardinalities

5.1 Illustrations of Mathematical Computations of $Q_k(jh)$, with respect to system (1)

We will evaluate the expressions for $Q_k(jh)$, for $j,k \in \{3,4\}$. By theorem 3.1:

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k}, \text{for } 0 \le j \le k,\ j,\ k \text{ integers, } k \ne 0.$$

$$Q_3(3h) = \sum_{r=0}^{\left[\left[\frac{3}{2}\right]\right]} \sum_{(v_1,\cdots,v_3)\in P_{0(r),1(3-2r),2(r)}} A_{v_1}\cdots A_{v_3} = \sum_{r=0}^{1} \sum_{(v_1,\cdots,v_3)\in P_{0(r),1(3-2r),2(r)}} A_{v_1}\cdots A_{v_3}$$

$$= \sum_{(v_1,\cdots,v_3)\in P_{0(0),1(3-0),2(0)}} A_{v_1}\cdots A_{v_3} + \sum_{(v_1,\cdots,v_3)\in P_{0(1),1(3-2),2(1)}} A_{v_1}\cdots A_{v_3}$$

$$= A_1^3 + A_0A_1A_2 + A_0A_2A_1 + A_1A_2A_0 + A_1A_0A_2 + A_2A_0A_1 + A_2A_1A_0 \qquad (10)$$

$$Q_4(3h) \;=\; \sum_{r=0}^{1} \sum_{(v_1,\cdots,v_4)\in P_{0(r+1),1(3-2r),2(r)}} A_{v_1}\cdots A_{v_4}$$

$$= \sum_{(v_1,\cdots,v_4)\in P_{0(1),1(3)}} A_{v_1}\cdots A_{v_4} + \sum_{(v_1,\cdots,v_4)\in P_{0(2),1(1),2(1)}} A_{v_1}\cdots A_{v_4}$$

$$\begin{aligned} &= A_0A_1^3 + A_1A_0A_1^2 + A_1^2A_0A_1 + A_1^3A_0 + A_0^2A_1A_2 + A_0^2A_2A_1 + A_0A_1A_0A_2 + A_0A_2A_0A_1 \\ &\quad + A_0A_2A_1A_0 + A_1A_2A_0^2 + A_2A_1A_0^2 + A_0A_1A_2A_0 + A_1A_0A_2A_0 \\ &\quad + A_1A_0^2A_2 + A_2A_0^2A_1 + A_2A_0A_1A_0 \end{aligned} \tag{11}$$

By theorem 3.2:

$$Q_k(jh) \;=\; \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}, \text{for } \; k \le j \le 2k,\, j,\, k \text{ integers}, k \ge 1.$$

$$Q_3(4h) = \sum_{r=0}^{1} \sum_{(v_1,\cdots,v_4)\in P_{0(r),1(2-2r),2(r+1)}} A_{v_1}\cdots A_{v_4} = \sum_{(v_1,\cdots,v_4)\in P_{1(2),2(1)}} A_{v_1}\cdots A_{v_4} + \sum_{(v_1,\cdots,v_4)\in P_{0(1),2(2)}} A_{v_1}\cdots A_{v_4}$$

$$= A_2A_1^2 + A_1A_2A_1 + A_1^2A_2 + A_2^2A_0 + A_0A_0^2 + A_0^2A_0 + A_2A_0A_2 \tag{12}$$

It is clear that $Q_3(4h) \ne Q_4(3h)$; in general $Q_k(jh) \ne Q_j(kh)$, and $Q_k(jh)$ is independent of h. Also $\left| Q_k(jh) \right| \ne \left| Q_j(kh) \right|$, for $j \ne k$, where denotes the cardinality of $Q_k(jh)$.

$$\begin{aligned} Q_4(4h) \;&=\; \sum_{r=0}^{2} \sum_{(v_1,\cdots,v_4)\in P_{0(r),1(4-2r),2(r)}} A_{v_1}\cdots A_{v_4} \\ &= \sum_{(v_1,\cdots,v_4)\in P_{1(4)}} A_{v_1}\cdots A_{v_4} + \sum_{(v_1,\cdots,v_4)\in P_{0(1),1(2),2(1)}} A_{v_1}\cdots A_{v_4} + \sum_{(v_1,\cdots,v_4)\in P_{0(2),2(2)}} A_{v_1}\cdots A_{v_4} \\ &= A_1^4 + A_0A_1^2A_2 + A_1^2A_2A_0 + A_1A_0A_1A_2 + A_1A_2A_1A_0 \\ &\quad + A_1A_2A_0A_1 + A_0A_2A_1^2 + A_2A_0A_1^2 + A_1A_0A_2A_1 + A_0A_1A_2A_1 \\ &\quad + A_0A_1^2A_2 + A_2A_1^2A_0 + A_2A_1A_0A_1 + A_0^2A_2^2 + A_2^2A_0^2 + A_0A_2^2A_0 \\ &\quad + A_2A_0^2A_2 + A_0A_2A_0A_2 + A_2A_0A_2A_0 \end{aligned} \tag{13}$$

5.2 Illustrations of Mathematical Computations of Cardinalities of Determining matrices, with respect to system (1)

Recall that the cardinality, $|Q_k(jh)|$, of $Q_k(jh)$ is the number of nonzero summands in $Q_k(jh)$ and that the cardinality of an identically zero matrix is zero.

From corollary 3.8:

$$|Q_k(jh)| = \sum_{r=\max\{0,k-j\}}^{\left|\left[\left[\frac{2k-j}{2}\right]\right]\right|} \frac{k!}{r!\,|(2k-j-2r)|!\,(r+j-k)!} \operatorname{sgn}(\max\{0, 2k+1-j\})$$

$$|Q_3(2h)| = \sum_{r=\max\{0,1\}}^{\left|\left[\left[\frac{6-2}{2}\right]\right]\right|} \frac{3!}{r!(6-2-2r)!(r-1)!} \operatorname{sgn}(\max\{0, 7-2\}) = \sum_{r=1}^{2} \frac{3!}{r!(4-2r)!(r-1)!} = 3+6 = 9. \tag{14}$$

$$\left|Q_2(3h)\right| = \sum_{r=\max\{0,-1\}}^{\left|\left[\left[\frac{4-3}{2}\right]\right]\right|} \frac{2!}{r!(4-3-2r)!(r+1)!}\operatorname{sgn}(\max\{0,5-3\}) = \sum_{r=0}^{0} \frac{2!}{r!(1-2r)!(r+1)!} = 2. \quad (15)$$

$$j \geq 2k+1 \Rightarrow \left|Q_k(jh)\right| = \sum_{r=0}^{\left|\left[\left[\frac{2k-j}{2}\right]\right]\right|} \frac{k!}{r!\left|(2k-j-2r)\right|(2k-j-2r)!(r+j-k)!}\operatorname{sgn}(0) = 0. \quad (16)$$

This is consistent with the cardinality of a null set.

$$\left|Q_3(3h)\right| = \sum_{r=\max\{0,0\}}^{\left|\left[\left[\frac{6-3}{2}\right]\right]\right|} \frac{3!}{r!\left|(6-3-2r)\right|!(r)!}\operatorname{sgn}(4) = \sum_{r=0}^{1} \frac{3!}{r!\left|(3-2r)\right|!(r)!} = 1+6=7. \quad (17)$$

V. CONCLUSION

The results in this article attest to the fact that we have embellished the results in [1] by deft application of the max and sgn functions and their composite function sgn (max {.,.}) in the expressions for determining matrices. Such applications are optimal, in the sense that they obviate the need for explicit piece–wise representations of those and many other discrete mathematical objects and some others in the continuum.We have also examined the issue of computational feasibility and mathematical tractability of our results, as never been done before through indepth analyses of structures and cardinalities of determining matrices.

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