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Alternative Optimal Expressions For The Structure And Cardinalities Of Determining Matrices Of Single-Delay Autonomous Neutral Control Systems

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ABSTRACT :*This paper exploited the results in Ukwu [1] to obtain the cardinalities, computing complexity and alternative optimal expressions for the determining matrices of single – delay autonomous linear neutral differential systems through a sequence of theorems and corollaries and the invocation of key facts about permutations. The paper also derived a unifying theorem for the major results in [1].The proofs were achieved using ingenious combinations of summation notations, the multinomial distribution, change of variables techniques and compositions of signum and max functions. The computations were mathematically illustrated and implemented on Microsoft Excel platform for some problem instances.*

KEYWORDS: *Cardinalities, Determining, Neutral, Platform, Structure.*

I. INTRODUCTION

The importance of determining matrices stems from the fact that they constitute the optimal instrumentality for the determination of Euclidean controllability and compactness of cores of Euclidean targets. See Gabasov and Kirillova [2] and [3 and 4]. In sharp contrast to determining matrices, the use of indices of control systems on the one hand and the application of controllability Grammians on the other, for the investigation of the Euclidean controllability of systems can at the very best be quite computationally challenging and at the worst mathematically intractable. Thus, determining matrices are beautiful brides for the interrogation of the controllability disposition of delay control systems. See [1].However up-to-date review of literature on this subject reveals that there was no correct result on the structure of determining matrices single – delay autonomous linear neutral differential systems prior to [1]. This could be attributed to the severe difficulty in identifying recognizable mathematical patterns needed for inductive proof of any claimed result. This paper extends and embellishes the main results in [1] by effectively resolving ambiguities in permutation infeasibilities and obviating the need for explicit piece-wise representations of $Q_k(jh)$, as well as conducting careful analyses of the computational complexity and cardinalities of the determining matrices, thus filling the yawning gaps in [1] and much more.

II. On determining matrices and controllability of single-delay autonomous neutral control systems

We consider the class of neutral systems:

$$\frac{d}{dt}\left[x(t) - A_{-1}x(t-h)\right] = A_0x(t) + A_1x\left(t-h\right) + Bu\left(t\right),\ \ t \ge 0 \qquad (1)$$

where A_{-1}, A_0, A_1 are $n \times n$ constant matrices with real entries and B is an $n \times m$ constant matrix with the real entries. The initial function ϕ is in $C\left(\left[-h, 0\right], \mathbf{R}^n\right)$ equipped with sup norm. The control u is in $L_\infty\left(\left[0, t_1\right], \mathbf{R}^n\right)$. Such controls will be called admissible controls. $x\left(t\right), x\left(t-h\right) \in \mathbf{R}^n$ for $t \in \left[0, t_1\right]$. If $x \in C\left(\left[-h, t_1\right], \mathbf{R}^n\right)$, then for $t \in \left[0, t_1\right]$ we define $x_t \in C\left(\left[-h, 0\right], \mathbf{R}^n\right)$ by $x_t(s) = x(t+s),\ \ s \in \left[-h, 0\right]$.

2.1 Existence, uniqueness and representation of solutions

If $A_{-1} \neq 0$ and ϕ is continuously differentiable on $[-h, 0]$, then there exists a unique function $x:[-h, \infty)$ which coincides with ϕ on $[-h, 0]$, is continuously differentiable and satisfies system (1) except possibly at the points $jh;\ j = 0, 1, 2, \ldots$. This solution *x* can have no more derivatives than ϕ and continuously differentiable if and only if the relation:

$$\dot{\phi}(0) = A_{-1}\,\dot{\phi}(-h) + A_0\phi(0) + A_1\,\phi(-h) + Bu(0) \tag{2}$$

is satisfied. See Bellman and Cooke (1963) and theorem 7.1 in Dauer and Gahl (1977) for complete discussion on existence, uniqueness and representations of solutions of system (1).

The process of obtaining necessary and sufficient conditions for the Euclidean controllability of (1) will be initiated in the rest of the work as follows:

[1] Obtaining a workable expression for the determining matrices of system (1):

$$Q_k(jh) \quad \text{for} \quad j: t_1 - jh > 0,\ k = 1, 2, .. \tag{3}$$

[2] Showing that:

$$\Delta X^{(k)}(t_1 - jh, t_1) = (-1)^k Q_k(jh) \tag{4}$$

for $j: t_1 - jh > 0,\ k = 0, 1, 2, \ldots,$

[3] Showing that $Q_\infty(t_1)$ is a linear combination of:

$$Q_0(s), Q_1(s), \ldots, Q_{n-1}(s),\ s = 0, h, \ldots, (n-1)h \tag{5}$$

Sequel to [1], our objective is to embellish and unify the subtasks in task (i) as well as investigate the cardinalities and computational complexity of the determining matrices. Tasks (ii) and (iii) will be prosecuted in other papers.

We now define the determining equation of the $n \times n$ matrices, $Q_k(s)$.

For every integer *k* and real number *s*, define $Q_k(s)$ by:

$$Q_k(s) = A_{-1}Q_k(s-h) + A_0\,Q_{k-1}(s) + A_1\,Q_{k-1}(s-h) \tag{6}$$

for $k = 0, 1, \ldots;\ s = 0, h, 2h, \ldots.$ subject to $Q_0(0) = I_n$, the $n \times n$ identity matrix and $Q_k(s) = 0$ for $k < 0$ or $s < 0$.

Ukwu [1] obtained the following expressions for the determining matrices of system (1)

2.2 Theorem on explicit computable expression for determining matrices of system (1)

Let j and k be nonnegative integers.

If $j \geq k \geq 1$, then

$$Q_k(jh) = \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_j) \in P_{-1(j-k),1(k)}} A_{v_1}\cdots A_{v_j} + \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+r}}$$

If $k \geq j \geq 1$, then

$$Q_k(jh) = \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{(v_1,\cdots,v_k) \in P_{0(k-j),1(j)}} A_{v_1}\cdots A_{v_k} + \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1}\cdots A_{v_{k+r}}$$

The cases $j \geq k$ and $k \geq j$, in the preceding theorem can be unified by using a composition of the max and the signum functions as follows:

2.3 Theorem on computations of $Q_k(jh)$ of system (1) using a composite function

Let j and k be positive integers.

Then

$$\mathrm{Q}_k(jh) = \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1} \cdots A_{v_{j+k}}$$

$$+ \left[\sum_{(v_1,\cdots,v_j) \in P_{-1(j-k),1(k)}} A_{v_1} \cdots A_{v_j} + \sum_{r=1}^{k-1} \sum_{(v_1,\cdots,v_{j+r}) \in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1} \cdots A_{v_{j+r}} \right] \mathrm{sgn}(\max\{0, j+1-k\})$$

$$+ \left[\sum_{(v_1,\cdots,v_k) \in P_{0(k-j),1(j)}} A_{v_1} \cdots A_{v_k} + \sum_{r=1}^{j-1} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1} \cdots A_{v_{k+r}} \right] \mathrm{sgn}(\max\{0, k-j\})$$

<u>Proof</u>

If $j \geq k$, $\mathrm{sgn}(\max\{0, k-j\})$ annihilates the accompanying summations, and the summations accompanying $\mathrm{sgn}(\max\{0, j+1-k\})$ are preserved, in view of the fact that $\mathrm{sgn}(\max\{0, j+1-k\}) = 1$. This coincides with 2.2 for $j \geq k$.

If $k > j$, $\mathrm{sgn}(\max\{0, j+1-k\})$ annihilates the accompanying summations, and the summations accompanying $\mathrm{sgn}(\max\{0, k-j\})$ are preserved, since $\mathrm{sgn}(\max\{0, k-j\}) = 1$.

This coincides with theorem 2.2 for $k > j$. The case $j = k$ is embedded in ‘ $j \geq k$. ’ This completes the proof.

2.4 Theorem on Computations of $Q_k(jh)$ of system (1) using min and max functions

Let j and k be positive integers. Then

$$\mathrm{Q}_k(jh) = \sum_{(v_1,\cdots,v_{j+k}) \in P_{-1(j),0(k)}} A_{v_1} \cdots A_{v_{j+k}}$$

$$+ \sum_{(v_1,\cdots,v_{\max\{j,k\}}) \in P_{-1(\max\{j-k,0\}),\ 0(\max\{k-j,0\}),\ 1(\min\{j,k\})}} A_{v_1} \cdots A_{v_{\max\{j,k\}}}$$

$$+ \sum_{r=1}^{\min\{j,k\}-1} \sum_{(v_1,\cdots,v_{k+r}) \in P_{-1(r+\max\{j-k,0\}),0(r+\max\{k-j,0\}),1(\min\{j,k\})}} A_{v_1} \cdots A_{v_{\max\{j,k\}+r}}$$

<u>Proof</u>

If $j \geq k$, $\mathrm{sgn}(\max\{0, k-j\})$ annihilates the accompanying summations, and the summations accompanying $\mathrm{sgn}(\max\{0, j+1-k\})$ are preserved, in view of the fact that $\mathrm{sgn}(\max\{0, j+1-k\}) = 1$. This coincides with theorem 2.2 for $j \geq k$.

If $k > j$, $\mathrm{sgn}(\max\{0, j+1-k\})$ annihilates the accompanying summations, and the summations accompanying $\mathrm{sgn}(\max\{0, k-j\})$ are preserved, since $\mathrm{sgn}(\max\{0, k-j\}) = 1$.

This coincides with theorem 2.2 for $k > j$. The case $j = k$ is embedded in ‘ $j \geq k$. ’ This completes the proof.

2.5 First corollary to theorem 2.3

(i) If $A_{-1} = 0$, then, $Q_k(jh) = \left[A_1^k + \sum_{(v_1,\cdots,v_k) \in P_{0(k-j),1(j)}} A_{v_1} \cdots A_{v_k} \right] \mathrm{sgn}(\max\{0, k+1-j\})$

(ii) If $A_{-1} = A_1 = 0$, then, $Q_k(jh) = \begin{cases} 0, \text{ if } \min\{j,k\} \geq 1 \\ 0, \text{ if } k = 0,\ j \neq 0 \\ A_0^k, \text{ if } j = 0,\ k \neq 0 \end{cases}$

Proof of (i)

$A_{-1}=0 \Rightarrow$ no term survives in the expression for $Q_k(jh)$, for $j>k$, since that condition forces A_{-1} to appear in every feasible permutation; infeasible permutation products are equated to zero. We are left with the case $j \le k$, for which only $A_1^k + \sum_{(v_1,\cdots,v_k)\in P_{0(k-j),1(j)}} A_{v_1}\cdots A_{v_k}$ survives.

Notice that $\operatorname{sgn}(\max\{0,k+1-j\})=0$ if $j>k$, and 1 otherwise. This completes the proof of (i)

Proof of (ii)

$A_{-1}=A_1=0 \Rightarrow Q_k(jh)=0$, for $\min\{j,k\}\ge 1$. Then by an appeal to lemma 2.4 of [1], it is clear that only $Q_k(0)=A_0^k$ survives. This completes the proof of (ii).

2.6 Second corollary to theorem 2.3

For all nonnegative integers j, k and real $h>0$,

$$Q_k([j-1]h)A_{-1}+Q_{k-1}(jh)A_0+Q_{k-1}([j-1]h)A_1 = A_{-1}Q_k([j-1]h)+A_0Q_{k-1}(jh)+A_1Q_{k-1}([j-1]h)$$

Proof

We note from the determining equation (6) that $Q_k(jh)=A_{-1}Q_k([j-1]h)+A_0Q_{k-1}(jh)+A_1Q_{k-1}([j-1]h)$.

From the proof of theorem 2.2, we deduce that

$$\begin{aligned}
&Q_k([j-1]h)A_{-1}+Q_{k-1}(jh)A_0+Q_{k-1}([j-1]h)A_1\\
&=\sum_{i=-1}^{0}\sum_{(v_1,\cdots,v_{j+k})\in P^{iT}_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{i\in\{-1,1\}}\sum_{(v_1,\cdots,v_{j+k})\in P^{iT}_{-1(j-k),0(k)}} A_{v_1}\cdots A_{v_j}\\
&\quad+\sum_{i=-1}^{1}\sum_{r=1}^{k-1}\sum_{(v_1,\cdots,v_{j+r})\in P^{iT}_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+r}}\\
&=\sum_{i=-1}^{0}\sum_{(v_1,\cdots,v_{j+k})\in P^{iL}_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}} + \sum_{i\in\{-1,1\}}\sum_{(v_1,\cdots,v_{j+k})\in P^{iL}_{-1(j-k),0(k)}} A_{v_1}\cdots A_{v_j}\\
&\quad+\sum_{i=-1}^{1}\sum_{r=1}^{k-1}\sum_{(v_1,\cdots,v_{j+r})\in P^{iL}_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+r}}\\
&=Q_k(jh)=A_{-1}Q_k([j-1]h)+A_0Q_{k-1}(jh)+A_1Q_{k-1}([j-1]h),
\end{aligned}$$

as desired. The proof for the case $k \ge j$ is similar, using the expression for $Q_k(jh)$.

3. Computational complexity of $Q_k(jh)$

$|Q_k(jh)|=0$, for $\min\{\tilde{j},k\}<0, |Q_k(0)|=1, \forall k\ge 1, |Q_0(jh)|=1, j\ge 0$. By theorem 2.2, for $\min\{j,k\}\ge 1$, integers, $|Q_k(jh)|$ is the number of nonzero terms (products) in $Q_k(jh)$.

Let C_i denote the number of terms in the i^{th} component summations in $Q_k(jh)$; let $Q_k^{(i)}(jh)$ denote the i^{th} component summations, for $i\in\{1,\ 2,\ 3\}$; let $C=C_1+C_2+C_3$. Then, we have the following complexity table:

TABLE 1: Computing Complexity Table for $Q_k(jh)$ with respect to system (1)

	Number of nonzero terms = Number of nonzero products	Number of additions	Size of permutation = sum of powers of the A_I,s
$Q_k^{(1)}(jh)$	$C_1 = \frac{(j+k)!}{j!k!} = \binom{j+k}{j} = \binom{j+k}{k}$	$C_1 - 1$	$j+k$
$Q_k^{(2)}(jh)$	$C_2 = \frac{(j)!}{(j-k)!k!} = \binom{j}{k} = \binom{j}{j-k}$	$C_2 - 1$	j
$Q_k^{(3)}(jh)$	$C_3 = \sum_{r=1}^{k-1} \frac{(j+r)!}{(r+j-k)!r!(k-r)!}$	$C_3 - 1$	$j+r \in \{1,\cdots,k-1\}$ min size $= j+1$, max $= j+k-1$
$Q_k(jh)$	$C = \binom{j+k}{k} + \binom{j}{k} + \sum_{r=1}^{k-1} \frac{(j+r)!}{(r+j-k)!r!(k-r)!} = \binom{j+k}{k} + \binom{j}{k} + \sum_{r=1}^{k-1} \binom{j+r}{k}\binom{k}{r}$	$C-1$	

The complexity table for $Q_k(jh),\ k \ge j$ is obtained by swapping j and k. $Q_k([k+p]h)$ and $Q_{k+p}(kh)$ have the same complexity, for every nonnegative integer, p.

TABLE 2: Electronic Implementation of Computations of $Q_k(jh),\ j \ge k$ for selected inputs

EXCEL Computations for the number of terms in $Q_k(jh),\ j \ge k \in \{2,\cdots,8\}, k \le j \le k+2.$

		r =	1	2	3	4	5	6	7		Cardinality
k	*j*	C1 + C2	C3 components							T=C	ratios
2	2	7	6							13	
2	3	13	12							25	1.92
2	4	21	20							41	1.64
3	3	21	12	30						63	1.54
3	4	39	30	60						129	2.05
3	5	66	60	105						231	1.79
4	4	71	20	90	140					321	1.39
4	5	131	60	210	280					681	2.12
4	6	225	140	420	504					1289	1.89
5	5	253	30	210	560	630				1683	1.31
5	6	468	105	560	1260	1260				3653	2.17
5	7	813	280	1260	2520	2310				7183	1.97
6	6	925	42	420	1680	3150	2772			8989	1.25
6	7	1723	168	1260	4200	6930	5544			19825	2.21
6	8	3031	504	3150	9240	13860	10296			40081	2.02
7	7	3433	56	756	4200	11550	16632	12012		48639	1.21
7	8	6443	252	2520	11550	27720	36036	24024		108545	2.23
7	9	11476	840	6930	27720	60060	72072	45045		224143	2.06
8	8	12871	72	1260	9240	34650	72072	84084	51480	265729	1.19
8	9	24319	360	4620	27720	90090	168168	180180	102960	598417	2.25
8	10	43803	1320	13860	72072	210210	360360	360360	194480	1256465	2.10

Table 2 was generated using table 1 and an embedded Microsoft Excel sheet.

TABLE 3: $Q_k(jh)$ Cardinality Summary Table with respect to system (1)

k	j	No. of terms in $Q_k(jh);\ k\in\{2,\cdots,8\},\ j\in\{k,k+1,k+2\}.$	$Q_k(jh)$ Cardinality ratios	No. of terms in $Q_k(kh)$
2	2	13		13
2	3	25	1.92	
2	4	41	1.64	
3	3	63	1.54	63
3	4	129	2.05	
3	5	231	1.79	
4	4	321	1.39	321
4	5	681	2.12	
4	6	1289	1.89	
5	5	1683	1.31	1683
5	6	3653	2.17	
5	7	7183	1.97	
6	6	8989	1.25	8989
6	7	19825	2.21	
6	8	40081	2.02	
7	7	48639	1.21	48639
7	8	108545	2.23	
7	9	224143	2.06	
8	8	265729	1.19	265729
8	9	598417	2.25	
8	10	1256465	2.10	

A glance at Table 3 is quite revealing. Notice how quickly the cardinalities of $Q_k(jh)$ grow astronomically from 13, for $j+k=4$, to 1,256,465, for $j+k=18$. In particular, observe how the cardinalities of $Q_k(kh)$ leap from 13, for $k=2$, to 1683, for $k=5$. How, in the world could one manage 1683 permutations for just $Q_5(5h)$, not to bother about $Q_k(kh)$, for larger *k*. it is clear that long-hand computations for $Q_k(jh)$, even for $j+k=10$, are definitely out of the question.

Practical realities/exigencies dictate that these computations should be implemented electronically. These challenges have been tackled headlong; the computations for $Q_k(jh)$ and their cardinalities have been achieved on the C^{++} platform, for any appropriate input matrices, A_{-1}, A_0, A_1 and positive integers $j,k:\min\{j,k\}\geq 1$, using theorem 2.2; needless to say that the cases $j,k: jk=0$ have also been incorporated in the code, using lemma 2.4 of [1].

Now we have adequate tools to establish necessary and sufficient conditions for the Euclidean controllability of system (1) on $\left[0,t_1\right]$.

IV .Illustrations of mathematical computations of $Q_k(jh)$ with respect to system (1):

$$\mathrm{Q}_k(jh)=\sum_{(v_1,\cdots,v_{j+k})\in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}}+\sum_{(v_1,\cdots,v_j)\in P_{-1(j-k),1(k)}} A_{v_1}\cdots A_{v_j}$$

$$\mathrm{Q}_k(jh)=\sum_{(v_1,\cdots,v_{j+k})\in P_{-1(j),0(k)}} A_{v_1}\cdots A_{v_{j+k}}+\sum_{r=1}^{k-1}\sum_{(v_1,\cdots,v_{j+r})\in P_{-1(r+j-k),0(r),1(k-r)}} A_{v_1}\cdots A_{v_{j+r}}+\sum_{(v_1,\cdots,v_k)\in P_{0(k-j),1(j)}} A_{v_1}\cdots A_{v_k}\text{, for } j\geq k\geq 1$$

$$+\sum_{r=1}^{j-1}\sum_{(v_1,\cdots,v_{k+r})\in P_{-1(r),0(r+k-j),1(j-r)}} A_{v_1}\cdots A_{v_{k+r}}\text{, for } k\geq j\geq 1$$

$$\mathrm{Q}_2(2h)=\sum_{(v_1,\cdots,v_4)\in P_{-1(2),0(2)}}A_{v_1}\cdots A_{v_4}+\sum_{(v_1,v_2)\in P_{1(2)}}A_{v_1}A_{v_2}+\sum_{(v_1,\cdots,v_3)\in P_{-1(1),0(1),1(1)}}A_{v_1}\cdots A_{v_3}$$
$$=A_{-1}^2A_0^2+A_0^2A_{-1}^2+A_{-1}A_0A_{-1}A_0+A_0A_{-1}A_0A_{-1}+A_{-1}A_0^2A_{-1}+A_0A_{-1}^2A_0+A_1^2$$
$$+A_{-1}A_0A_1+A_{-1}A_1A_0+A_0A_{-1}A_1+A_0A_1A_{-1}+A_1A_0A_{-1}+A_1A_{-1}A_0 \tag{7}$$

$$=A_{-1}^3A_0^2+A_0^2A_{-1}^3+A_{-1}^2A_0A_{-1}A_0+A_{-1}^2A_0^2A_{-1}+A_{-1}A_0^2A_{-1}^2+A_0A_{-1}A_0A_{-1}^2+A_0A_{-1}^3A_0+A_0A_{-1}^2A_0A_{-1}$$
$$+A_{-1}A_0A_{-1}A_0A_{-1}+A_{-1}A_0A_{-1}^2A_0+A_{-1}A_1^2+A_1^2A_{-1}+A_1A_{-1}A_1+A_{-1}^2A_0A_1+A_{-1}^2A_1A_0+A_1A_{-1}^2A_0$$
$$+A_{-1}^2A_1A_0+A_0A_{-1}^2A_1+A_0A_1A_{-1}^2+A_1A_0A_{-1}^2+A_0A_{-1}A_1A_{-1}+A_1A_{-1}A_0A_{-1}+A_{-1}A_1A_{-1}A_0$$
$$+A_{-1}A_0A_{-1}A_1+A_{-1}A_1A_0A_{-1} \tag{8}$$

and replace 1 by 0 in those permutations involving only the indices 1 and 1. Therefore:

$$=A_{-1}^3A_0^3+A_0^3A_{-1}^3+A_{-1}^2A_0^3A_{-1}+A_0^2A_{-1}^3A_0+A_{-1}A_0^3A_{-1}^2+A_0A_{-1}^3A_0^2+A_{-1}A_0^2A_{-1}^2A_0+A_0A_{-1}^2A_0^2A_{-1}$$
$$+A_{-1}A_0^2A_{-1}A_0A_{-1}+A_0A_{-1}^2A_0A_{-1}A_0+A_{-1}^2A_0^2A_{-1}A_0+A_0^2A_{-1}^2A_0A_1+A_{-1}A_0A_{-1}^2A_0^2+A_0A_{-1}A_0^2A_{-1}^2$$
$$+A_{-1}^2A_0A_{-1}A_0^2+A_0^2A_{-1}A_0A_{-1}^2+A_{-1}A_0A_{-1}A_0A_{-1}A_0+A_0A_{-1}A_0A_{-1}A_0A_{-1}+A_{-1}A_0^2A_{-1}A_0A_{-1}$$
$$+A_0A_{-1}^2A_0A_{-1}A_0+A_1^3$$

To obtain $Q_3(2h)$, simply swap the indices 0 and -1 in the expanded expression for $Q_2(3h)$

$$Q_3(2h)=A_0^3A_{-1}^2+A_{-1}^2A_0^3+A_0^2A_{-1}A_0A_{-1}+A_0^2A_{-1}^2A_0+A_0A_{-1}^2A_0^2+A_{-1}A_0A_{-1}A_0^2+A_{-1}A_0^3A_{-1}$$
$$+A_{-1}A_0^2A_{-1}A_0+A_0A_{-1}A_0A_{-1}A_0+A_0A_{-1}A_0^2A_{-1}+A_0A_1^2+A_1^2A_0+A_1A_0A_1+A_0^2A_{-1}A_1$$
$$+A_0^2A_1A_{-1}+A_1A_0^2A_{-1}+A_0^2A_1A_{-1}+A_{-1}A_0^2A_1+A_{-1}A_1A_0^2+A_1A_{-1}A_0^2+A_{-1}A_0A_1A_0$$
$$+A_1A_0A_{-1}A_0+A_0A_1A_0A_{-1}+A_0A_{-1}A_0A_1+A_0A_1A_{-1}A_0 \tag{9}$$

$$\mathrm{Q}_3(3h)=\sum_{(v_1,\cdots,v_6)\in P_{-1(3),0(3)}}A_{v_1}\cdots A_{v_6}+\sum_{(v_1,\cdots,v_3)\in P_{1(3)}}A_{v_1}\cdots A_{v_3}+\sum_{r=1}^{2}\sum_{(v_1,\cdots,v_{3+r})\in P_{-1(r),0(r),1(3-r)}}A_{v_1}\cdots A_{v_{3+r}}$$
$$+A_1^2A_0A_{-1}+A_1^2A_{-1}A_0+A_1A_{-1}A_1A_0+A_1A_0A_1A_{-1}+A_1A_{-1}A_0A_1+A_0A_{-1}A_1^2+A_{-1}A_0A_1^2$$
$$+A_1A_0A_{-1}A_1+A_0A_1A_{-1}A_1+A_0A_1^2A_{-1}+A_{-1}A_1^2A_0+A_{-1}A_1A_0A_1$$

4.2 : Illustrations of Mathematical Computations of $|Q_k(jh)|$ with respect to system (1)

$$+A_{-1}^2A_0^2A_1+A_{-1}^2A_1A_0^2+A_0^2A_{-1}^2A_1+A_0^2A_1A_{-1}^2+A_{-1}A_0^2A_{-1}A_1+A_{-1}A_0^2A_1A_{-1}+A_{-1}A_1A_0^2A_{-1}$$
$$+A_0A_{-1}^2A_0A_1+A_0^2A_{-1}^2A_1A_0+A_0A_1A_{-1}^2A_0+A_{-1}^2A_0A_1A_0+A_{-1}A_0A_1A_0A_{-1}+A_{-1}A_0A_{-1}A_0A_1$$
$$+A_{-1}A_0A_1A_{-1}A_0+A_{-1}A_1A_0A_{-1}A_0+A_1A_{-1}^2A_0^2+A_1A_0^2A_{-1}^2+A_{-1}A_0A_{-1}A_1A_0+A_0A_1A_0A_{-1}^2$$

From theorem 2.2:

$$+A_{-1}A_1A_{-1}A_0^2+A_1A_{-1}A_0^2A_{-1}+A_{-1}A_0^2A_{-1}A_1+A_1A_0A_{-1}^2A_0+A_0^2A_{-1}A_1A_{-1}+A_1A_0A_{-1}A_0A_{-1}$$
$$+A_1A_{-1}A_0A_{-1}A_0+A_0A_{-1}A_1A_{-1}A_0+A_0A_1A_{-1}A_0A_{-1}+A_0A_{-1}A_0A_1A_{-1}+A_0A_{-1}A_0A_{-1}A_1 \tag{10}$$

$$|Q_2(2h)|==\binom{4}{2}+\binom{2}{2}+\sum_{r=1}^{1}\frac{(r+2)!}{r!(r)!(2-r)!}=6+1+6=13. \tag{11}$$

$$\left|Q_3(3h)\right| = \binom{6}{3} + \binom{3}{3} + \sum_{r=1}^{2} \frac{(r+3\})!}{r!(r)!(3-r)!} = 20 + 1 + \frac{4!}{2!} + \frac{5!}{2!2!} = 33 + 30 = 63. \qquad (13)$$

$$\left|Q_k(jh)\right| = \left|Q_j(kh)\right| = \binom{j+k}{k} + \binom{\max\{j,k\}}{k} + \sum_{r=1}^{\min\{j,k\}-1} \frac{(r+\max\{j,k\})!}{r!(r+\left|j-k\right|)!(\min\{j,k\}-r)!}$$

$$\left|Q_2(3h)\right| = \binom{5}{2} + \binom{3}{2} + \sum_{r=1}^{1} \frac{(r+3)!}{r!(r+1)!(2-r)!} = 10 + 3 + \frac{4!}{2!} = 25. \qquad (12)$$

These are consistent with the number of permutation products obtained in the computations of $Q_k(jh)$, for $j,k \in \{2,3\}$ in subsection 4.1:

$$\left|Q_3(4h)\right| = \binom{7}{3} + \binom{4}{3} + \sum_{r=1}^{2} \frac{(r+4)!}{r!(r+1)!(3-r)!} = 35 + 4 + \frac{5!}{2!2!} + \frac{6!}{2!3!} = 129. \qquad (14)$$

Above computations could be effected using the following established results:

$$\text{If } j \ge k \ge 2, \left| Q_k(jh) \right| = \binom{j+k}{k} + \binom{j}{k} + \sum_{r=1}^{k-1} \frac{(j+r)!}{(j+r-k)!r!(k-r)!}$$

$$= \binom{j+k}{k} + \binom{j}{k} + \sum_{r=1}^{k-1} \binom{j+r}{k}\binom{k}{r} \qquad (15)$$

$$= \binom{j+k}{j} + \binom{k}{j} + \sum_{r=1}^{j-1} \binom{k+r}{j}\binom{j}{r} \qquad (16)$$

It is clear that the mathematical implementation of $Q_k(jh)$ is computationally prohibitive for $\min\{j,k\} \ge 4$.

$$\text{If } k \ge j \ge 2, \left| Q_k(jh) \right| = \binom{j+k}{j} + \binom{k}{j} + \sum_{r=1}^{j-1} \frac{(j+r)!}{r!(k+r-j)!(j-r)!}$$

III. CONCLUSION

The results in this article attest to the fact that we have embellished the results in [1] by deft application of the max and sgn functions and their composite function sgn (max {.,.}) in the expressions for determining matrices. Such applications are optimal, in the sense that they obviate the need for explicit piece–wise representations of those and many other discrete mathematical objects and some others in the continuum.

We have also examined the issue of computational feasibility and mathematical tractability of our results, as never been done before through indepth analyses of structures and cardinalities of determining matrices.

REFERENCES

[1] Ukwu, C. (2014h). The structure of determining matrices for single-delay autonomous linear neutral control systems. International Journal of Mathematics and Statistics Invention, (IJMSI), 2014, Volume 2, Issue 3, pp 14-30.

[2] Gabasov, R. and Kirillova, F. The qualitative theory of optimal processes. Marcel Dekker Inc., New York, 1976.

[3] Ukwu, C. Euclidean Controllability and Cores of Euclidean Targets for Differential difference systems Master of Science Thesis in Applied Math. with O.R. (Unpublished), North Carolina State University, Raleigh, N. C. U.S.A., 1992.

[4] Ukwu, C. An exposition on Cores and Controllability of differential difference systems, ABACUS, 1996, Vol. 24, No. 2, pp. 276-285.

[5] Bellman, R. and Cooke, K.(1963). Differential-difference equations. Academic Press, New York.

[6] Dauer, J. P. and Gahl, R. D. (1977). Controllability of nonlinear delay systems. J.O.T.A. Vol. 21, No. 1, January.