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Norlund Index Summability of a Factored Fourier series

Chitaranjan Khadanga[1] Shubhra Sharma[2]
Lituranjan Khadanga[3]
Professor Dept .of Mathematics R C E T Bhilai(c.g)
Chitakhadanga@gmail.com
Shubhrasharma13@gmail.com
Asst.Professor Dept. of Mathematics RCET Bhilai(c.g)
Asst.professor Dept of Mathematics CEC Bhilai(c.g)

ABSTRACT :- *In this paper we have been proved an analogue theorem on* $\left|N,\ p_n\right|_k$ *-summability,* $k \geq 1$.

KEY WORDS:- *Summability Theory., Infinite Series , Fourier series* **.**

I. INTRODUCTION:-

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive numbers such that

(1.1) $$P_n = \sum_{r=0}^{n} p_v \to \infty,\ as\ n \to \infty\ \left(P_{-i} = p_{-i} = 0,\ i \geq 1\right)$$

The sequence to sequence transformation is defined by

(1.2) $$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v}\ s_v$$

and which defines the sequence of the $\left(N,\ p_n\right)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be summable $\left|N,\ p_n\right|_k,\ k \geq 1,$

if (1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty$$

In this case, when $p_n = 1$, for all $n\ and\ k = 1, \left|N, p_n\right|_k$ summability is same as $\left|C,1\right|$ summability.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without loss of generality, we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

(1.4) $$f(t) \approx \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) = \sum_{n=1}^{\infty} A_n(t)$$

II. Dealing with $\left|\overline{N},\ p_n\right|_k,\ k \geq 1$, summability factors of Fourier series, Bor has been proved the following theorem:

Theorem-2.1

If $\{\lambda_n\}$ is a non-negative and non-increasing sequence such that $\sum p_n \ \lambda_n < \infty$, where $\{p_n\}$ is a sequence of positive numbers such that $P_n \to \infty \ as \ n \to \infty$ and $\sum_{v=1}^{n} P_v \ A_v(t) = 0(P_n)$. Then the factored Fourier series $\sum A_n(t) \ P_n \ \lambda_n$ is summable $|N, p_n|_k, \ k \geq 1$.

III. MAIN THEOREM

Let $\{p_n\}$ is a sequence of positive numbers such that $P_n = p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$ and $\{\lambda_n\}$ is a non-negative, non-increasing sequence such that $\sum p_n \ \lambda_n \ < \ \infty$. If

(2.1) (i). $\sum_{v=1}^{n} P_v \ A_v(t) = 0(P_n)$

(2.2) (ii). $\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{p_{n-v-1}}{P_{n-1}}\right) = 0\left(\frac{p_v}{P_v}\right), \ as \ m \to \infty$ and

(2.3) (iii). $P_{n-v-1} \ \Delta\lambda_v = 0\left(p_{n-v} \ \lambda_v\right)$, then the series $\sum A_n(t) P_n \ \lambda_n$ is summable $|N, p_n|_k, \ k \geq 1$.

3. We need the following Lemma for the proof of our theorem.

Lemma-3.1

If $\{\lambda_n\}$ is a non-negative and non-increasing sequence such that $\sum p_n \ \lambda_n < \infty$, where $\{p_n\}$ is a sequence of positive numbers such that $P_n \to \infty \ as \ n \to \infty$ then $P_n \ \lambda_n = 0(1) \ as \ n \to \infty$ and $\sum P_n \ \Delta\lambda_n < \infty$. **PROOF OF MAIN THEOREM**

Let $t_n(x)$ be the *n*-th (N, p_n) mean of the series $\sum_{n=1}^{\infty} A_n(x) \ P_n \ \lambda_n$, then by definition we have

$$t_n(x) = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} \sum_{r=0}^{v} A_r(x) \ P_r \ \lambda_r$$

$$= \frac{1}{P_n} \sum_{r=0}^{n} A_r(x) \ P_r \lambda_r \sum_{v=r}^{n} p_{n-v}$$

$$= \frac{1}{P_n} \sum_{r=0}^{n} A_r(x) \ P_r \ P_{n-r} \lambda_r \ .$$

Then

$$t_n(x) - t_{n-1}(x) = \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r} \ P_r \ \lambda_r \ A_r(x) - \frac{1}{P_{n-1}} \sum_{r=0}^{n-1} P_{n-r-1} \ P_r \lambda_r A_r(x)$$

$$= \sum_{r=1}^{n} \left(\frac{P_{n-r}}{P_n} - \frac{P_{n-r-1}}{P_{n-1}}\right) P_r \ \lambda_r \ A_r(x)$$

$$= \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n-1} \left(P_{n-v} \; P_{n-1} - P_{n-r-1} \; P_n\right) P_r \, \lambda_r A_r(x)$$

$$= \frac{1}{P_n P_{n-1}} \left[\sum_{r=1}^{n-1} \Delta\left\{\left(P_{n-r} \; P_{n-1} - P_{n-r-1} \; P_n\right) \lambda_r\right\} \left(\sum_{v=1}^{r} P_v \; A_v(x) \right) \right]$$, using partial summation formula with $p_o = 0$

$$= \frac{1}{P_n P_{n-1}} \left[\sum_{r=1}^{n-1} \left(p_{n-r} \; P_{n-1} - p_{n-r-1} \; P_n\right) \lambda_r P_r \right.$$

$$\left. + \sum_{r=1}^{n-1} \left(p_{n-r-1} \; P_{n-1} - P_{n-r-2} \; p_n\right) P_r \Delta\lambda_r \right]$$, using (2.1)

$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}$, say.

In order to complete the proof of the theorem, using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left|T_{n,i}\right|^k < \infty, \; for \; i = 1,2,3,4.$$

Now, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left|T_{n,1}\right|^k = \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \frac{1}{P_n^k} \left(\sum_{v=1}^{n-1} p_{n-v} \, P_v \, \lambda_v \right)^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \frac{1}{P_n} \left(\sum_{v=1}^{n-1} p_{n-v} \, P_v^k \, \lambda_v^k \right) \left(\frac{1}{P_n} \sum_{v=1}^{n-1} p_{n-v} \right)^{k-1},$$

using Holder's inequality

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \frac{1}{P_n} \sum_{v=1}^{n-1} p_{n-v} \; P_v \, \lambda_v \left(P_v \; \lambda_v\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_v \, \lambda_v \sum_{n = v+1}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \frac{p_{n-v}}{P_n}$$

$$= O(1) \sum_{v=1}^{m} p_v \, \lambda_v \quad \text{(using 2.2)}$$

$$= O(1), \; as \; m \to \infty.$$

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left|T_{n.2}\right|^k = \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_{n-v-1} \; P_v \; \lambda_v \right)^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} p_{n-v-1} \; P_v^k \; \lambda_v^k \right) \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_{n-v-1} \right)^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-n}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_{n-v-1}\;P_v\;\lambda_v(P_v\,\lambda_v)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}P_v\,\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{p_{n-v-1}}{P_{n-1}}\right),$$

$$=0(1)\sum_{v=1}^{m}p_v\,\lambda_v,\ \text{using (2.2)}$$

$$=0(1)\,,\ as\ m\to\infty.$$

Now,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,3}\right|^k=\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n^k}\left(\sum_{v=1}^{n-1}p_{n-v-1}\;P_v\;\Delta\lambda_v\right)^k$$

$$\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\left(\sum_{v=1}^{n-1}p_{n-v-1}\,(P_v\;\Delta\lambda_v)^k\right)\left(\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v-1}\right)^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v-1}\;P_v\;\Delta\lambda_v(P_v\,\Delta\lambda_v)^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_n}\sum_{v=1}^{n-1}p_{n-v-1}\;P_v\;\Delta\lambda_v\;,\ \text{by the lemma (3.1)}$$

$$=O(1)\sum_{v=1}^{m}P_v\;\Delta\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{n-1}\left(\frac{p_{n-v-1}}{P_n}\right)$$

$$=0(1)\sum_{v=1}^{m}p_v\;\Delta\lambda_v\,,\ \text{using (2.2)}$$

$$=0(1)\,,\ m\to\infty.$$

Finally,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,4}\right|^k=\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v\;P_{n-v-2}\;\Delta\lambda_v\right)^k$$

$$\leq 0(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}p_{n-v-1}\;P_v\;\lambda_v\right)^k,\ \text{using (2.3)}$$

$$\leq 0(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_{n-v-1}\;(P_v\;\lambda_v)^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_{n-v-1}\right)^{k-1}$$

$$=0(1)\sum_{n=2}^{n+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_{n-v-1}\;P_v\,\lambda_v$$

$= 0(1),\ as\ m \to \infty$, as above

This completes the proof of the theorem.

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