International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 5 || May. 2014 || PP-44-52

On EPr Bimatrices

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ABSTRACT: *The aim of this article is to introduce the notion of EPr bimatrices. This paper is concerned solely with developing the properties of EPr bimatrices. Real and complex EPr bimatrices are studied for their own inherent properties and a number of generalizations of the results analogous to EPr matrices have been obtained.*

KEYWORDS: *EP bimatrix, EPr bimatrix, Permutation bimatrix* ***AMS classification:*** *15A09, 15A15, 15A57.*

I. INTRODUCTION

The concept of a normal matrix with entries from the complex field was introduced in 1918 by O.Toeplitz [6] who gave a necessary and sufficient condition that a complex matrix to be normal. Since, then many researchers have developed the concept and many generalizations of normality were studied. First, results about EPr complex matrices, a concept introduced by H.Schwerdtfeger in [5] as a generalization of normality were obtained and then in [3,4] the notion of EPr was extended to matrices over arbitrary fields and applied to obtain results about normal matrices.

In this paper we introduce the notion of EPr bimatrices as an extension of bimatrices. A complex matrix A of order n is called EP if the range spaces of A and A^* are equal. Greville in [6] termed an EP Matrix as a range hermitian matrix.

II. ON EP_r BIMATRICES

Definition 2.1

Let $A_B = A_1 \cup A_2$ be a bimatrix. Then the null space of A_B is defined by,

$$N(A_B) = \{x \in \mathbb{C}_n / A_B\, x = 0 \text{ that is } A_1 x \cup A_2 x = 0\}.$$

Example 2.2

Let $A_B = \begin{pmatrix} -3 & 6 & -1 \\ 1 & -2 & 2 \\ 2 & -4 & 5 \end{pmatrix} \cup \begin{pmatrix} 2 & -4 & 1 \\ -6 & 12 & 0 \\ -4 & 8 & 3 \end{pmatrix} = A_1 \cup A_2 \ (say)$

Here $x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ such that $A_B x = 0$. Hence $x \in N(A_B)$.

Definition 2.3

A bimatrix $A_B \in \mathbb{C}_{nxn}$ is said to be EP if it satisfies the condition $A_B x = 0 \Leftrightarrow A_B^* x = 0$ (or) equivalently $N(A_B) = N(A_B^*)$.

Definition 2.4

A bimatrix $A_B = A_1 \cup A_2$ is said to be of rank r if both the components of A_B ,that is A_1 and A_2 are of same rank r.

Example 2.5

Let $A_B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix} = A_1 \cup A_2 \ (say)$

Since, both A_1 and A_2 are of rank 3 then rank of A_B is 3.

Remark 2.6

Throughout this paper we consider bimatrices whose components are of same rank.

Definition 2.7

A nxn bimatrix A_B with entries from the complex field $\mathbb{C}_{nxn}$ is called EPr if it satisfies the following conditions:

(i) A_B has rank r.

(ii) $\sum_{i=1}^{n} \alpha_i \ A_{B_i} = 0$ if and only if $\sum_{i=1}^{n} \bar{\alpha}_i \ A_B^i = 0$

That is $\sum_{i=1}^{n} \alpha_i \left(A_{1_i} \cup A_{2_i} \right) = 0$ if and only if $\sum_{i=1}^{n} \bar{\alpha}_i \ \left(A_1^i \cup A_2^i \right) = 0$ where $\alpha_i \in \mathbb{C}, i = 1, 2, \ldots, n$

Remark 2.8

Here A_{B_i} denote the i[th] row of the bimatrix A_B *and* A_B^i denote the i[th] column of A_B (*that is both* A_1 *and* A_2).

Example 2.9

Let $A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} = A_1 \cup A_2 (say)$

$$N(A_B) = \left\{ \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}, \begin{pmatrix} -a \\ 0 \\ a \end{pmatrix} \right\}, \forall a \in R$$

Here rank$\left(A_B\right)$ is 2, Since A_1 and A_2 are of rank 2. Also the condition $\sum_{i=1}^{n} \alpha_i A_{B_i} = 0 \Leftrightarrow \sum_{i=1}^{n} \bar{\alpha}_{\mathrm{i}} A_B^i$ is satisfied.

Therefore A_B is an EP_2 bimatrix.

Theorem 2.10

If A_B is a bimatrix then the following statements are equivalent :

(i) A_B is an nxn EPr bimatrix.

(ii) A_B has rank r and there is an nxn bimatrix N_B such that $A_B^* = N_B A_B$.

(iii) A_B has rank r and there is a non singular nxn bimatrix N_B such that $A_B^* = N_B \ A_B$.

(iv) A_B can be represented as,

$$A_B = P_B \left[\begin{array}{c|c} D_B & D_B X_B^* \\ \hline X_B D_B & X_B D_B X_B^* \end{array} \right] P_B^*$$

$$= P_1 \left[\begin{array}{c|c} D_1 & D_1 X_1^* \\ \hline X_1 D_1 & X_1 D_1 X_1^* \end{array} \right] P_1^* \cup P_2 \left[\begin{array}{c|c} D_2 & D_2 X_2^* \\ \hline X_2 D_2 & X_2 D_2 X_2^* \end{array} \right] P_2^*$$

$$= P_1 \left[\begin{array}{c|c} I_{1_r} & 0 \\ \hline X_1 & I_1 \end{array}\right] \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I_{1_r} & X_1^* \\ \hline 0 & I_1 \end{array}\right] P_1^* \cup$$

$$P_2 \left[\begin{array}{c|c} I_{2_r} & 0 \\ \hline X_2 & I_2 \end{array}\right] \left[\begin{array}{c|c} D_2 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I_{2_r} & X_2^* \\ \hline 0 & I_2 \end{array}\right] P_2^*$$

where P_B is a permutation bimatrix and D_B is an rxr non singular bimatrix.

(v) $A_B\, \xi = 0$ if and only if $A_B^*\, \xi = 0$ where $\zeta \in \mathbb{C}_n$.

To Prove: $(v) \Rightarrow (i)$

If part:

Suppose $\sum_{i=1}^{n} \alpha_i A_{B_i} = 0$ where $\alpha_i \in \mathbb{C}, i = 1, 2..., n$

That is $\sum_{i=1}^{n} \alpha_i A_{1_i} = 0$ and $\sum_{i=1}^{n} \alpha_i A_{2_i} = 0 \longrightarrow$ (1)

Let $\xi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{C}_n$

Then $\xi^t\, A_B = (\alpha_1\, \alpha_2 \ldots \alpha_n) A_B$

$= (\alpha_1\, \alpha_2 \ldots \alpha_n) A_1 \cup (\alpha_1\, \alpha_2 \ldots \alpha_n) A_2$

$= \sum_{i=1}^{n} \alpha_i A_{1_i} \cup \sum_{i=1}^{n} \alpha_i A_{2_i}$

$\xi^t A_B = 0 \quad (\sin ce\ by\ (1))$

Taking conjugate transpose on both sides,

$(\xi^t A_B)^* = A_B^*\, \bar{\xi}$

Applying (v) we have

$A_B\, \bar{\xi} = 0$

Thus, $\sum_{i=1}^{n} \bar{\alpha}_i A_B^i = 0$.

Only if part:

Suppose $\sum_{i=1}^{n} \bar{\alpha}_i A_B^i = 0$ where $\alpha_i \in \mathbb{C}, i = 1, 2, ..., n$

$\Rightarrow \sum_{i=1}^{n} \bar{\alpha}_i A_1^i = 0$ and $\sum_{i=1}^{n} \bar{\alpha}_i A_2^i = 0$ $(\sin ce\, A_B = A_1 \cup A_2)$

Let $\bar{\xi} = \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix} \in \mathbb{C}_n$

Then $(\bar{\xi})^t\, A_B = (\bar{\alpha}_1\, \bar{\alpha}_2 \ldots \bar{\alpha}_n) A_B$

$= (\bar{\alpha}_1\, \bar{\alpha}_2 \ldots \bar{\alpha}_n) A_1 \cup (\bar{\alpha}_1\, \bar{\alpha}_2 \ldots \bar{\alpha}_n) A_2$

$$= \sum_{i=1}^{n} \bar{\alpha}_i A_1^i \cup \sum_{i=1}^{n} \bar{\alpha}_i A_2^i$$

$$(\bar{\xi})^t \; A_B = 0$$

Taking conjugate transpose on both sides,

$$\left(\bar{\xi}^t A_B\right)^* = A_B^* \, \xi = 0$$

Applying (v) we have

$$A_B \, \bar{\xi} = 0$$

Thus, $\sum_{i=1}^{n} \alpha_i A_{B_i} = 0$

Hence $\sum_{i=1}^{n} \alpha_i A_{B_i} = 0$ if and only if $\sum_{i=1}^{n} \bar{\alpha}_i A_B^i = 0$

$\Rightarrow A_B$ is an $nxn \; EP_r$ bimatrix.

To prove $(i) \Rightarrow (iv)$

Let A_B be EP_r and let the rows $A_{1_1}, A_{1_2}, \ldots, A_{1_r}$ and $A_{2_1}, A_{2_2}, \ldots, A_{2_r}$ be linearly independent.

If $\sum_{i=1}^{r} \beta_i \; A_B^i = 0$ then $\sum_{i=1}^{r} \bar{\beta}_i \; A_{B_i} = 0$

Hence $\bar{\beta}_1 = \bar{\beta}_2 = \cdots = \bar{\beta}_r = 0$

Thus, $\beta_1 = \beta_2 = \cdots = \beta_r = 0$ and so the columns $A_1^1, A_1^2, \ldots, A_1^r$ and $A_2^1, A_2^2, \ldots, A_2^r$ are linearly independent. Since the rank of A_1 and A_2 are r, the sub bimatrices D_1 and D_2 formed by the elements in the intersection of rows $A_{1_1}, A_{1_2}, \ldots A_{1_r}$ and the columns $A_1^1, A_1^2, \ldots, A_1^r$ of A_1 and also the rows $A_{2_1}, A_{2_2}, \ldots A_{2_r}$ and the columns $A_2^1, A_2^2, \ldots, A_2^r$ of A_2 respectively, are an rxr non singular bimatrix [5,P-52].

Now, Premultiply and postmultiply $A_1 \; and \; A_2$ by the Permutation matrix P_1, P_2 and P_1^*, P_2^* respectively such that $A_1 \; and \; A_2$ can be written in the form,

$$B_1 = P_1 \; A_1 \; P_1^* = \begin{bmatrix} D_1 & E_1 \\ F_1 & G_1 \end{bmatrix} \text{ and } B_2 = P_2 \; A_2 \; P_2^* = \begin{bmatrix} D_2 & E_2 \\ F_2 & G_2 \end{bmatrix}$$

$$\Rightarrow B_B = P_B \; A_B \; P_B^* = \begin{bmatrix} D_1 & E_1 \\ F_1 & G_1 \end{bmatrix} \cup \begin{bmatrix} D_2 & E_2 \\ F_2 & G_2 \end{bmatrix} = \begin{bmatrix} D_B & E_B \\ F_B & G_B \end{bmatrix}$$

Since the first block row of B_1 and B_2 are of the same rank r as A_1 and A_2 and thus B_B there is an $(n-r) \times r$ bimatrix $H_B = H_1 \cup H_2$ such that,

$$\begin{bmatrix} F_1 & G_1 \end{bmatrix} = H_1 \begin{bmatrix} D_1 & E_1 \end{bmatrix} \text{ and } \begin{bmatrix} F_2 & G_2 \end{bmatrix} = H_2 \begin{bmatrix} D_2 & E_2 \end{bmatrix}$$

and hence by (1), such that

$$\begin{bmatrix} E_1 \\ G_1 \end{bmatrix} = \begin{bmatrix} D_1 \\ F_1 \end{bmatrix} H_1^* \text{ and } \begin{bmatrix} E_2 \\ G_2 \end{bmatrix} = \begin{bmatrix} D_2 \\ F_2 \end{bmatrix} H_2^*$$

If D_1 and D_2 have rank $< r$, then $D_1 \xi = 0$ and $D_2 \xi = 0$

So that $D_B \xi = 0 \;\; \left(\sin ce \; D_B = D_1 \cup D_2\right)$ hold for some $\xi \in \mathbb{C}_r$

$$\Rightarrow\begin{bmatrix}D_1\\F_1\end{bmatrix}\xi=\begin{bmatrix}D_1\,\xi\\H_1D_1\,\xi\end{bmatrix}=0 \text{ and } \begin{bmatrix}D_2\\F_2\end{bmatrix}\xi=\begin{bmatrix}D_2\,\xi\\H_2D_2\,\xi\end{bmatrix}=0\Rightarrow\begin{bmatrix}D_B\\F_B\end{bmatrix}\xi=\begin{bmatrix}D_B\,\xi\\H_BD_B\,\xi\end{bmatrix}=0$$

Which is a contradiction to the assumption that the first block column of B_B has rank r.
With H_B as above, let

$$Q_B=\left[\begin{array}{c|c}I_{B_r}&0\\\hline -H_B&I_{B_{n-r}}\end{array}\right]=\left[\begin{array}{c|c}I_{1_r}&0\\\hline -H_1&I_{1_{n-r}}\end{array}\right]\cup\left[\begin{array}{c|c}I_{2_r}&0\\\hline -H_2&I_{2_{n-r}}\end{array}\right]$$

It is verified that

$$\left[\begin{array}{c|c}D_1&0\\\hline 0&0\end{array}\right]=Q_1\,B_1\,Q_1^*=Q_1\,P_1\,A_1\,P_1^*\,Q_1^* \text{ and } \left[\begin{array}{c|c}D_2&0\\\hline 0&0\end{array}\right]=Q_2\,B_2\,Q_2^*=Q_2\,P_2\,A_2\,P_2^*\,Q_2^*$$

$$\Rightarrow A_1=P_1^{-1}\,Q_1^{-1}\left[\begin{array}{c|c}D_1&0\\\hline 0&0\end{array}\right]Q_1^{-1^*}P_1^{-1^*} \text{ and } A_2=P_2^{-1}\,Q_2^{-1}\left[\begin{array}{c|c}D_2&0\\\hline 0&0\end{array}\right]Q_2^{-1^*}P_2^{-1^*}$$

$$\Rightarrow A_1=\;P_1^{-1}\left[\begin{array}{c|c}I_{1_r}&0\\\hline -X_1&I_1\end{array}\right]\left[\begin{array}{c|c}D_1&0\\\hline 0&0\end{array}\right]\left[\begin{array}{c|c}I_{1_r}&-X_B^*\\\hline 0&I_1\end{array}\right]P_1^{-1^*}$$

$$\text{and } A_2=P_2^{-1}\left[\begin{array}{c|c}I_{2_r}&0\\\hline -X_2&I_2\end{array}\right]\left[\begin{array}{c|c}D_2&0\\\hline 0&0\end{array}\right]\left[\begin{array}{c|c}I_{2_r}&-X_2^*\\\hline 0&I_2\end{array}\right]P_2^{-1^*}$$

$$\Rightarrow A_B=A_1\cup A_2$$

$$\Rightarrow A_B=P_B^{-1}\left[\begin{array}{c|c}I_{B_r}&0\\\hline -X_B&I_B\end{array}\right]\left[\begin{array}{c|c}D_B&0\\\hline 0&0\end{array}\right]\left[\begin{array}{c|c}I_{B_r}&-X_B^*\\\hline 0&I_B\end{array}\right]P_B^{-1^*}$$

Set $P_B^{-1}=P_B,\ X_B=-X_B$ and $Q_B=Q_B^{-1}$ we have,

$$A_B=P_B\left[\begin{array}{c|c}I_{B_r}&0\\\hline X_B&I_B\end{array}\right]\left[\begin{array}{c|c}D_B&0\\\hline 0&0\end{array}\right]\left[\begin{array}{c|c}I_{B_r}&X_B^*\\\hline 0&I_B\end{array}\right]P_B^*\quad\rightarrow(1)$$

To prove $(iv)\Rightarrow(iii)$

Assume that (1) is true.

Set $Q_B=Q_1\cup Q_2=\left[\begin{array}{c|c}I_{1_r}&0\\\hline X_1&I_1\end{array}\right]\cup\left[\begin{array}{c|c}I_{2_r}&0\\\hline X_2&I_2\end{array}\right]=\left[\begin{array}{c|c}I_{B_r}&0\\\hline X_B&I_B\end{array}\right]$

If $Q_BA_BQ_B^*=\left[\begin{array}{c|c}D_B&0\\\hline 0&0\end{array}\right]$, where $Q_B=Q_1\cup Q_2$ is non singular and D_B is rxr non singular bimatrix then,

$$\begin{aligned}\left(Q_B\,A_B\,Q_B^*\right)^*&=\left[\left(Q_1\cup Q_2\right)\left(A_1\cup A_2\right)\left(Q_1\cup Q_2\right)^*\right]^*\\&=\left[\left(Q_1A_1\cup Q_2A_2\right)\left(Q_1^*\cup Q_2^*\right)\right]^*\\&=\left(Q_1\,A_1\,Q_1^*\cup Q_2\,A_2\,Q_2^*\right)^*\\&=\left(Q_1\,A_1\,Q_1^*\right)^*\cup\left(Q_2\,A_2\,Q_2^*\right)^*\end{aligned}$$

$$\left(Q_B\,A_B\,Q_B^*\right)^*=Q_1\,A_1^*\,Q_1^*\cup Q_2\,A_2^*\,Q_2^*$$

Pre multiply and post multiply by Q_B^{-1} and $Q_B^{-1^*}$

$$Q_B^{-1}\left(Q_B\,A_B\,Q_B^*\right)^*Q_B^{-1^*}=Q_B^{-1}\left(Q_1\,A_1^*\,Q_1^*\cup Q_2\,A_2^*\,Q_2^*\right)Q_B^{-1^*}$$

$$=\left(Q_1 \cup Q_2\right)^{-1}\left(Q_1\, A_1^*\, Q_1^* \cup Q_2\, A_2^*\, Q_2^*\right)\left(Q_1 \cup Q_2\right)^{-1^*}$$

$$=\left(Q_1^{-1} \cup Q_2^{-1}\right)\left(Q_1\, A_1^*\, Q_1^* \cup Q_2\, A_2^*\, Q_2^*\right)\left(Q_1^{-1^*} \cup Q_2^{-1^*}\right)$$

$$=Q_1^{-1}\, Q_1\, A_1^* Q_1^* \cup Q_2^{-1}\, Q_2\, A_2^*\, Q_2^* \quad \left(Q_1^{-1^*} \cup Q_2^{-1^*}\right)$$

$$=Q_1^{-1}\, Q_1\, A_1^* Q_1^* Q_1^{*-1} \cup Q_2^{-1}\, Q_2\, A_2^*\, Q_2^* Q_2^{*-1}$$

$$\left(\sin ce\, A^{-1^*}=A^{*-1}\right)$$

$$=A_1^* \cup A_2^*$$

$$=\left(A_1 \cup A_2\right)^*$$

$$=A_B^*$$

$$\Rightarrow A_B^*=Q_B^{-1}\left[\begin{array}{c|c}(D_1 \cup D_2)^* & 0 \\ \hline 0 & 0\end{array}\right] Q_B^{-1^*}$$

$$=Q_1^{-1}\left[\begin{array}{cc}D_1^* & 0 \\ \hline 0 & 0\end{array}\right] Q_1^{-1^*} \cup Q_2^{-1}\left[\begin{array}{cc}D_2^* & 0 \\ \hline 0 & 0\end{array}\right] Q_2^{-1^*}$$

$$=\left\{Q_1^{-1}\left[\begin{array}{c|c}D_1^* D_1^{-1} & 0 \\ \hline 0 & I_1\end{array}\right] Q_1 Q_1^{-1}\left[\begin{array}{c|c}D_1 & 0 \\ \hline 0 & 0\end{array}\right] Q_1^{-1^*}\right\} \cup$$

$$\left\{Q_2^{-1}\left[\begin{array}{c|c}D_2^* D_2^{-1} & 0 \\ \hline 0 & I_2\end{array}\right] Q_2 Q_2^{-1}\left[\begin{array}{c|c}D_2 & 0 \\ \hline 0 & 0\end{array}\right] Q_2^{-1^*}\right\}$$

$$=N_1\, A_1 \cup N_2\, A_2$$

$$=\left(N_1 \cup N_2\right)\left(A_1 \cup A_2\right)$$

$$=N_B\, A_B$$

where $N_1=Q_1^{-1}\left[\begin{array}{cc}D_1^* D_1^{-1} & 0 \\ \hline 0 & I_1\end{array}\right] Q_1$ and $N_2=Q_2^{-1}\left[\begin{array}{cc}D_2^* D_2^{-1} & 0 \\ \hline 0 & I_2\end{array}\right] Q_2$ and N_1, N_2 are nonsingular.

Proof of $(iii) \Rightarrow (ii)$ is evident.

To prove $(ii) \Rightarrow (v)$

Let $A_B^*=N_B\, A_B$

Let $\mathrm{N}\left(A_B\right)$ denote the null space of A_B and let $\xi \in N\left(A_B\right)$

$$\Rightarrow A_B\, \xi=0$$

That is $A_1\, \xi=0$ and $A_2\, \xi=0$

Then $A_B^*\, \xi=\left(N_B\, A_B\right)\xi$

$$=\left[\left(N_1 \cup N_2\right)\left(A_1 \cup A_2\right)\right]\xi$$

$$=\left(N_1 A_1 \cup N_2\, A_2\right)\xi$$

$$=\left(N_1 A_1\right)\xi \cup \left(N_2\, A_2\right)\xi$$

$$=N_1\left(A_1 \xi\right) \cup N_2\left(A_2 \xi\right)$$

$$= N_1(0) \cup N_2(0)$$

$$\Rightarrow A_B^* \, \xi = 0$$

$$\Rightarrow N(A_B) \subseteq N(A_B^*)$$

But, since rank (A_B) = rank (A_B^*)

$$\Rightarrow \dim N(A_B) = n - rank\,(A_B)$$

$$= n - rank\,(A_B^*)$$

$$\dim\ N(A_B) = \dim N(A_B^*)$$

$$\Rightarrow N(A_B) = N(A_B^*)$$

Thus, if $A_B^* \, \xi = 0$ then $A_B \, \xi = 0$

Lemma 2.11

Let A_B and B_B be nxn EPr bimatrices. Then $R(A_B) = R(B_B)$ if and only if $N(A_B) = N(B_B)$.

More generally, let A_B and B_B be nxn bimatrices of rank r then $R(A_B) = R(B_B)$ if and only if $N(A_B^*) = N(B_B^*)$.

Proof

Let $R(A_B) = R(B_B)$

By known result, there is a bimatrix C_B such that

$$B_B = A_B \, C_B$$

Then, $B_B^* = (A_B \, C_B)^* = C_B^* \, A_B^*$

$$\Rightarrow N(A_B^*) \subseteq N(B_B^*) \qquad (\sin ce\, N(A_1^*) \subseteq N(B_1^*) \ \ if \ B_1 = C_1 A_1)$$

Since rank $(A_B^*) = rank\,(B_B^*)$, we have

$$N(A_B^*) = N(B_B^*)$$

Conversely,

$$N(A_B^*) = N(B_B^*)$$

By known result, there is a bimatrix C_B such that

$$A_B^* = C_B^* \, B_B^*$$

$$(A_B^*)^* = (C_B^* \, B_B^*)^* = B_B \, C_B$$

$$A_B = B_B \, C_B$$

$$\Rightarrow R(B_B) \subseteq R(A_B)$$

But rank $(A_B) = rank\,(B_B)$

We have $R(A_B) = R(B_B)$

Lemma 2.12

Let A_B and B_B be complex $n \times n$ bimatrices satisfying $A_B B_B = B_B A_B$. Then

$$A_B^* B_B^* \square_n \subseteq \left[N(A_B) + N(B_B) \right]^{\perp}$$

Proof

Let $v \in \mathbb{C}_n, x \in N\left(A_B\right)$ and $y \in N\left(B_B\right)$

$\Rightarrow A_B x = 0$ and $B_B y = 0$ ⟶ (1)

$$\left(A_B^* B_B^* v, x+y\right) = \left(\left(B_B A_B\right)^* v, x+y\right)$$

$$= \left(v, B_B A_B\left(x+y\right)\right)$$

$$= \left(v, B_B A_B x + B_B A_B y\right)$$

$$= \left(v, B_B A_B x + A_B B_B y\right) \left(\sin ce A_B B_B = B_B A_B\right)$$

$$= \left(v, B_B\left(A_B x\right) + A_B\left(B_B y\right)\right)$$

$$= \left(v, 0\right) \quad \left(\sin ce\, by\,(1)\right)$$

$$\Rightarrow A_B^* B_B^* v \in \left[N\left(A_B\right) + N\left(B_B\right)\right]^{\perp}$$

$$\Rightarrow A_B^* B_B^* \mathbb{C}_n \subseteq \left[N\left(A_B\right) + N\left(B_B\right)\right]^{\perp}$$

Lemma 2.13

Let A_B and B_B be complex $n \times n$ bimatrices, satisfying $A_B B_B = B_B A_B$. Then $A_B^* B_B^* \mathbb{C}_n \subseteq N\left(A_B B_B\right)^{\perp}$

Proof

Let $v \in \mathbb{C}_n$ and $z \in N\left(A_B B_B\right)$

$\Rightarrow \left(A_B B_B\right) z = 0$

$$\left(A_B^* B_B^* v, z\right) = \left(\left(B_B A_B\right)^* v, z\right)$$

$$= \left(v, B_B A_B z\right)$$

$$= \left(v, A_B B_B z\right) \quad \left(\sin ce\, A_B B_B = B_B A_B\right)$$

$$= \left(v, 0\right)$$

$$\left(A_B^* B_B^* v, z\right) = 0$$

$$\Rightarrow A_B^* B_B^* v \in N\left(A_B B_B\right)^{\perp}$$

$$A_B^* B_B^* \mathbb{C}_n \subseteq N\left(A_B B_B\right)$$

Lemma 2.14

Let A_B and B_B be complex $n \times n$ bimatrices satisfying $A_B B_B = B_B A_B$. Then $B^*\left(N\left(A_B\right)^{\perp}\right) \subseteq N\left(A_B\right)^{\perp}$

Proof

Let $x \in N\left(A_B\right)^{\perp}$ and $y \in N\left(A_B\right)$

Now, $A_B\left(B_B y\right) = B_B\left(A_B y\right)$

$$= 0 \quad (\sin ce\, y \in N\left(A_B\right))$$

$$\Rightarrow B_B y \in N\left(A_B\right)$$

Hence $\left(B_B^* x, y\right) = \left(x, B_B y\right)$

$$= \left(x, 0\right) \quad (\sin ce B_B y \in N\left(A_B\right))$$

$$\Rightarrow B_B^* x \in N\left(A_B\right)^{\perp}$$

$$\Rightarrow B_B^*\left(N\left(A_B\right)^{\perp}\right)\subseteq N\left(A_B\right)^{\perp}$$

Theorem 2.15

Let $\text{rank}\left(A_B B_B\right)=\text{rank}\ (B_B)=r_1$ and $\text{rank}(B_B A_B)=\text{rank}\ (A_B)=r_2$. If $A_B B_B$ and B_B are $E\,\text{Pr}_1$ and A_B is $E\,\text{Pr}_2$ then $B_B A_B$ is $E\,\text{Pr}_2$.

Proof

If $A_B\xi=0$ then $B_B A_B\xi=0$

Therefore, $N\left(A_B\right)\subseteq N\left(B_B A_B\right)$

Since $rank(A_B)=rank\ (B_B A_B)$,

We have $N\left(A_B\right)=N\left(B_B A_B\right)$

Similarly $N\left(B_B\right)=N\left(A_B B_B\right)$

Then $B_B A_B\xi=0$

$\Leftrightarrow A_B\xi=0$

$\Leftrightarrow A_B^*\xi=0$ Since $A_B\ is\ E\,\text{Pr}_2$

$\Leftrightarrow B_B^* A_B^*\xi=0$

$\Leftrightarrow A_B B_B\xi=0$ Since $A_B\ is\ E\,\text{Pr}_2$

$\Leftrightarrow B_B\xi=0$

$\Leftrightarrow B_B^*\xi=0$

$\Leftrightarrow A_B^*\ B_B^*\ \xi=0$

Hence $N\left(B_B\ A_B\right)\subseteq N\left(A_B^*\ B_B^*\right)=N\left(B_B\ A_B\right)^*$

But $rank(B_B\ A_B)=rank\left(B_B\ A_B\right)^*$

There fore,

$$N\left(B_B\ A_B\right)=N\left(A_B^*\ B_B^*\right)$$

$$\Rightarrow B_B\ A_B\ is\ EP_{r_2}$$

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