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α-Ψ Contractive Type Mapping in Complex Valued G-Metric Spaces

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ABSTRACT: *In this paper, we introduce the notion of α-ψ contractive type mappings in complex valued G-metric spaces and establish fixed point theorems for these mappings.*

KEY WORDS: Complex valued G-metric space, α-ψ contractive mappings.

I. INTRODUCTION

In the last few year ,fixed point theory has been one of the most interesting research fields in nonlinear functional analysis.In2012 Samet et.al.[3] Introduced the notion of α-ψ contractive mappings and α-admissible mapping in metric spaces.In2013,Alghamdi and Karapinar[4] introduced the notion of α-ψ contractive mappings and α-admissible mapping in G-metric spaces Recently, Mustafa and Sims [1,2] have shown that most of the results concerning Dhage's D-metric spaces are invalid , therefore they introduced an improved version of the generalized metric space structure which they called G-metric spaces. In 2006, Mustafa and Sims [2] introduced the concept of G- metric spaces as follows:

Definition 1.1.[2] Let X be a non-empty set, and let G: $X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) G(x, y, z) =0 if x = y = z,

(G2) 0 < G(x, y, z) for all x, y ∈ X with x ≠ y,

(G3) G(x, x, y) ≤ G(x, y, z) for all x, y, z ∈ X with y ≠ z,

(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) =... (Symmetry in all three variables),

(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z) for all x, y, z, a ∈ X (rectangle inequality).

Then, the function G is called a generalized metric or, more specially, a G-metric on X, and the pair (X, G) is called a G- metric space. The idea of complex metric space was initiated by Azam et.al.[5] to exploit the idea of complex valued normed spaces and complex valued Hilbert spaces.

Definition 1.2.[5] Let ℂ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on ℂ as follows:

$z_1 \precsim z_2$ if and only if Re $(z_1) \le$ Re (z_2) and Im $(z_1) \le$ Im (z_2)

That is $z_1 \precsim z_2$ if one of the following holds

(C1): Re $(z_1) =$ Re (z_2) and Im $(z_1) =$ Im (z_2)

(C2): Re $(z_1) <$ Re (z_2) and Im $(z_1) =$ Im (z_2)

(C3): Re $(z_1) =$ Re (z_2) and Im $(z_1) <$ Im (z_2)

(C4): Re $(z_1) <$ Re (z_2) and Im $(z_1) <$ Im (z_2)

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Remark 1. We obtained that the following statements hold:

(i) a, b ∈ R and a ≤ b ⟹ az $\precsim$ bz for all z ∈ ℂ

(ii) $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

In 2013, Kang et.al. introduce the notion of complex valued G-metric space[6] akin to the notion of complex valued metric spaces [1] as follows:

Definition 1.4.[6] Let X be a non-empty set. Let G: $X \times X \times X \to \mathbb{C}$ be a function satisfying the following properties:

(CG1) G(x, y, z) =0 if x = y = z,

(CG2) $0 \prec G(x, y, z)$ for all x, y ∈ X with x ≠ y,

(CG3) $G(x, x, y) \precsim G(x, y, z)$ for all x, y, z ∈ X with y ≠ z,

(CG4) G(x, y, z) = G(x, z, y) = G(y, z, x) =... (Symmetry in all three variables)

(CG5) $G(x, y, z) \precsim G(x, a, a) + G(a, y, z)$ for all x, y, z, a ∈ X.

Then, the function G is called a complex valued generalized metric or more specially, a complex valued G-metric on X, and the pair (X, G) is called a complex valued G- metric space.

II. THE COMPLEX VALUED G-METRIC TOPOLOGY

A point x ∈ X is called *interior point* of a set A⊆ X, whenever there exists $0 \prec r \in \mathbb{C}$ such that

$B_G(x, r) = \{ y \in X: G(x, y, y) \prec r\} \subseteq A.$

A point x ∈ X is called *limit point* of a set A whenever there exists $0 \prec r \in \mathbb{C}$,

$B_G(x, r) \cap (A/X) \neq \emptyset.$

A is called *open* whenever each element of A is an interior point of A. A subset B⊆ X is called *closed* whenever each limit point of B belongs to B.

Proposition 2.1.[6] Let (X, G) be complex valued G-metric space, then for any $x_0 \in X$ and $r \succ 0$, we have

(1) If $G(x_0, x, y) \prec r$ *then* $x, y \in B_G(x_0, r)$,

(2) If $y \in B_G(x_0, r)$ *then there exists a* $\delta > 0$ *such that* $B_G(y, \delta) \subseteq B_G(x_0, r)$.

Proposition 2.2.[6] Let (X, G) be complex valued G-metric space, then for all $x_0 \in X$ and $r \succ 0$, we have,

$$B_G\left(x_0, \tfrac{1}{3}r\right) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r).$$

where, $d_G(x, y) = G(x, y, y) + G(x, x, y)$.

III. CONVERGENCE, CONTINUITY AND COMPLETENESS IN COMPLEX VALUED G-METRIC SPACES

Definition 3.1.[6] Let (X, G) be a complex valued G-metric space, let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is complex valued G-convergent to x if for any $\epsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) \prec \epsilon$, for all n, m ≥ k. We refer to x as the limit of the sequence $\{x_n\}$ and we write $x_n \xrightarrow{(G)} x$.

Proposition 3.1.[6] Let (X, G) be complex valued G-metric space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$, the following are equivalent:

(1) $\{x_n\}$ *is complex valued* $G -$ *convergent to* x

(2) $|G(x_n, x_n, x)| \to 0$ as $n \to \infty$

(3) $|G(x_n, x, x)| \to 0$ as $n \to \infty$

(4) $|G(x_m, x_n, x)| \to 0$ as $n, m \to \infty$

Definition 3.2.[6] Let (X, G) and (X', G') be two complex valued G-metric spaces. Then a function f : $X \to X'$ is complex valued G-continuous at a point $x_0 \in X$ if $f^{-1}(B_{G'}(f(x_0), r)) \in \tau(G)$, for all $r \succ 0$. We say f is complex valued G-continuous if it complex valued G-continuous at all points of X; that is, continuous as a function from X with the $\tau(G)$- topology to X' with $\tau(G')$- topology.

Since complex valued G-metric topologies are metric topologies we have :

Proposition 3.2.[6] Let (X, G) and (X', G') be two complex valued G-metric spaces. Then a function f : $X \to X'$ is complex valued G-continuous at a point $x \in X$ if and only if it is complex valued G-sequentially continuous at x: that is whenever $\{x_n\}$ is complex valued G-convergent to x we have $(f\{x_n\})$ *is* complex valued G-convergent to f(x).

Proposition 3.3.[6] Let (X, G) be a complex valued G-metric spaces, then the function G(x,y,z) is jointly continuous in all three of its variables.

Definition 3.2.[6] Let (X, G) be a complex valued G-metric space, a sequence $\{x_n\}$ is complex valued G-Cauchy if given $\epsilon > 0$, there exists k ∈ N such that $G(x_n, x_m, x_l) \prec \epsilon$ for all n, m, $l \geq$ k.

Definition 3.3.[6] A complex valued G-metric space (X, G) is said to be complex valued G-complete if every complex valued G-Cauchy sequence is complex valued G-converg[ent in (X, G).

Proposition 3.4.[6] Let (X, G) be a complex valued G-metric space. Then the following are equivalent:

(1) The sequence $\{x_n\}$ is a complex valued G-Cauchy in X.For every $\epsilon > 0$, there exists k ∈ N such that $G(x_n, x_m, x_m) \prec \epsilon$, for all n, m ≥ k .

(2) $\{x_n\}$ is a Cauchy sequence in the complex valued metric space (X, d_G).

Proposition 3.5.[6] Let (X, G) be a complex valued G-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G- convergent to x if and only if $|G(x, x_n, x_m)| \to 0$ as n, m $\to \infty$.

Proposition 3.6.[6] Let (X, G) be a complex valued G-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G- Cauchy sequence if and only if $|G(x_n, x_m, x_l)| \to 0$ as n, m $\to \infty$.

IV. PROPERTIES OF COMPLEX VALUED G-METRIC SPACES.

Proposition 4.1.[6] Let (X, G) be a complex valued G-metric space. Then, for any x, y, z, a in X it follows that:

(i) If G(x, y, z) =0 if x = y = z

(ii) G(x, y, z) $\precsim G(x, x, y) + G(x, x, z)$

(iii) G(x, y, y) $\precsim 2G(y, x, x)$

(iv) G(x, y, z) $\precsim G(x, a, z) + G(a, y, z)$

(v) G(x, y, z) $\precsim 2/3(G(x, y, a) + G(x, a, z) + G(a, y, z))$

(vi) G(x, y, z) $\precsim (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Proposition 4.2.[6] Let (X, G) be a complex valued G-metric space. Then, the following are equivalent:

(i) (X, G) is symmetric.

(ii) G(x, y, y) $\precsim G(x, y, a)$, for all x, y, a ∈ X.

(iii) G(x, y, z) $\precsim G(x, y, a) + G(z, y, b)$ for all x, y, a, b ∈ X.

Denote with χ the family of non decreasing functions of ψ:[0,∞)→[0,∞) such that $\sum_{n=1}^{\infty} \psi^n < +\infty$ for each t>0,where ψ^n is the nth iterate of ψ.

Lemma1. For every function ψ:[0,∞)→[0,∞) the following holds, if ψ is non decreasing, then each t>0, $\lim_{n\to\infty} \psi^n$(t)=0 implies ψ(t)<t.

Definition4.1. Let (X,G) be a G-metric space and T:X→X be given mapping. We say that T is G-α-ψ contractive mapping of type I if there exists two function α:X×X×X→[0,∞) and ψ∈χ such that

α(x,y,z)G(Tx,Ty,Tz)≾ψ(G(x,y,z)) for all x,y,z∈X. (1)

Definition4.2. Let (X,G) be a G-metric space and T:X→X be given mapping. We say that T is G-α-ψ contractive mapping of type A if there exists two function α:X×X×X→[0,∞) and ψ∈χ such that

α(x,y,Tx)G(Tx,Ty,T^2x)≾ψ(G(x,y,T^2x)) for all x ,y, z ∈X. (2)

Definition4.3.[4] Let T:X→X and α:X×X×X→[0,∞). We say that T is α-admissible if x,y∈X, α(x,y,z)≥1 implies α(Tx,Ty,Tz)≥1. (3)

Example 2[4]Let X=[0,∞) define T:X→X and α:X×X×X→[0,∞) by

$$Tx=\begin{cases} 2\ln x & \text{if } x\neq 0 \\ e & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha(x, y, z)=\begin{cases} e & \text{if } x\succsim y\succsim z \\ o & \text{otherwise} \end{cases}$$

V. MAIN RESULT

Now we prove our main results for α-ψcontractive type mapping in complex valued G-metric space

Theorem 5.1 Let (X,G) be a complete G-metric space and T:X→X be an α-ψ contractive mapping of type A and satisfying following condition:

(i) T is α-admissible,

(ii) There exists, x_0 ∈X such that α(x_0,Tx_0,Tx_1)≥1;
(iii) T is G-continous. Then T has a fixed point, that is, there exists x^*∈X such that T $x^* = x^*$.

Proof Let x_0∈X such that α(x_0,Tx_0,Tx_0)≥1 Define a sequence {x_n} in X as x_{n+1}=Tx_n for all n∈N. If x_n=x_{n+1} for some n∈N, then x^*= x_n is a fixed point for T. Now we assume that $x_n \neq x_{n+1}$ for all n∈N, since T is α-admissible, therefore we have α(x_0, x_1, x_1) =α(x_0, Tx_0,Tx_0)≥1 implies α(Tx_0,Tx_1,Tx_1)= α(x_1, x_2, x_2)≥1.

By induction we get,

α(x_n, x_{n+1}, x_{n+1})≥1 for all n=0,1,2.... (4)

Now G(x_n,x_{n+1},x_{n+1})=G(Tx_{n-1},Tx_n,Tx_n)

=G(Tx_{n-1},Tx_n,T^2x_{n-1})

≾α(x_{n-1},x_n,x_n)G(Tx_{n-1},Tx_n,T^2x_{n-1})

G(x_n, x_{n+1}, x_{n+1}) ≾ψ(G(x_{n-1}, x_n, x_n)).

Since ψ non decreasing,by induction,we have

G(x_n, x_{n+1}, x_{n+1}) ≾ ψ^n(G(x_0,x_1,x_1)) for all n≥1. (5)

Using (G5) and (5),we have

$$G(x_n, x_m, x_m) \precsim G(x_n, x_{n+1}, x_{n+1})+G(x_{n+1}, x_{n+2}, x_{n+2})+ \dots\dots\dots +G(x_{m-1}, x_m, x_m)$$

$$G(x_n, x_m, x_m) \precsim \sum_{k=n}^{m-1} G(x_k, x_{k+1}, x_{k+1})$$

$$\precsim \sum_{k=n}^{m-1} \psi^k(G(x_k, x_{k+1}, x_{k+1})).$$

Since, ψϵψ and0≾G(x_0, x_1, x_1), by Lemma 1, we get

$\sum_{k=n}^{m-1} \psi^k|(G(x_0, x_1, x_1))| < \infty$. Thus, we have

$$\lim_{n,m\to\infty} |G(x_n, x_m, x_m)| = 0.$$

By Proposition 3.4 , this implies that {x_n} is a G-Cauchy sequence in G-metric space (X,G).

Since (X, G) is complete , there exists, x^*∈X such that $x_n \to x^*$ as n→∞. From continuity of T, it follows that x_{n+1}=Tx_n→Tx^* as n→∞. By uniqueness of limit we get x^*=Tx^*, that is, x^* is a fixed point of T.

In next theorem we omit the continuity hypothesis of T.

Theorem 5.2 Let (X, G) be a complete G-metric space and T:X→X ba an α-ψ contractive mapping of type A satisfying the following conditions:

(i) T is α-admissible,
(ii) There exists x_0ϵX such that α(x_0, Tx_0, Tx_0)≥1;
(iii) if {x_n} is a sequence in X such that α(x_n, x_{n+1}, x_{n+1})≥1 for all n and x_n→x∈X as n→∞. Then α(x_n, x,x_{n+1})≥1 for all n.
Then T has a fixed point.

Proof . Following the proof of 5.2 we know that {x_n} is G-cauchy sequence in G-metric space (X,G). Then there exists x^*∈X such that $x_n \to x^*$ as n→∞. On the other hand (4) and hypothesis (iii)

α(x_n, x^*,x^*) ≥1 for all n≥0 (6)

Using basic properties of G-metric together with (2) and (6), we have

G(x_{n+1},Tx^*,x_{n+2}) ≾G(Tx_n,Tx^*,T^2x_n)

≾ α(x_n, x^*,x_{n+1}) G(Tx_n,Tx^*,T^2x_n)

≾ψ(G(x_n, x^*,x_{n+1}))

Letting n→∞, using Proposition 3.1 and since ψ is continuous at t=0,we get

$\lim_{n\to\infty}|G(x^*, Tx^*, x^*)| = 0$. By Proposition 4.1, we obtain x^*=Tx^*.

Example 5.1 Let X=[0,∞) be the G- metric space , where G(x, y, z)=|x-y|+|y-z|+|z-x| for all x, y, z∈X

Define the mapping T:X→X by

$$Tx=\begin{cases} 5x-\frac{5}{3} & if\ x>1 \\ \frac{x}{3} & if\ 0\leq x\leq 1 \\ 0 & if\ x<0 \end{cases}$$

At first we obtain that Banach contraction principal can not be applied,
G(T1,T2,T2)=16>2=G(1,2,2)

Now we define the mapping $\alpha:X\times X\times X\to[0,\infty)$

$$\alpha(x, y, z)=\begin{cases}1 & if\ x,y,z\in[0,1]\\ 0 & otherwise\end{cases}$$

Clearly, T is an α-ψ contraction mapping with $\psi(t)=t/2$ for all t≥0. Infact, x, y, z ∈ X, we have, $\alpha(x, y, z)G(Tx ,Ty, Tz)\le\frac{1}{2}G(x, y, z)$

Moreover, there exists x_0, ∈X such that $\alpha(x_0,Tx_0,Tx_0)\ge1$ infact $x_0=1$, we have α(1,T1,T1)=1.
Obviously, T is continuous and so it remains to show that T is α-admissible.
Let x, y∈ X such that α(x, y, y)≥1 implies x, y ∈[0,1], by definition of T and α, we have $Tx=\frac{x}{3}\in[0,1]$, $Ty=\frac{y}{3}\in[0,1]$.Hence α(Tx,Ty,Tz)≥1.
Then T is α-admissible. Now all hypothesis of theorem 5.1 are satisfied, consequently T has a fixed point but not uniqueness. Here, 0 and $\frac{5}{12}$ are two fixed point of T.
Now, we give example involving a function T that is not continuous.

Example 5.2 Let X=[0,∞) be the G- metric space , where G(x,y,z)=|x-y|+|y-z|+|z-x| for all x,y,z∈X
Define the mapping T:X→X by

$$Tx=\begin{cases}5x-\frac{5}{3} & if\ x>1\\ \frac{x}{3} & if\ 0\le x\le 1\\ 0 & if\ x<0\end{cases}$$

It is clear that T is not continous at 1. Then Banach contraction principle and also theorem 5.1 are not applicable in this case.
Define the mapping $\alpha:X\times X\times X\to[0,\infty)$ by $\alpha(x,y,z)=\begin{cases}1 & if\ x,y,z\in[0,1]\\ 0 & otherwise\end{cases}$

Clearly, T is an α-ψ contractive mapping with $\psi(t)=\frac{t}{2}$ for all t≥0. Infact, for all x, y ∈ X such that $\alpha(x, y, y)G(Tx,Ty,Ty)\le\frac{1}{2}G(x, y, y)$.

Moreover there exists $x_0\in X$ such that $\alpha(x_0,Tx_0,Tx_0)\ge1$and so for $x_0=1$, we have α(1,T1,T1)=1. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1},x_{n+1})\ge1$ for all n and $x_n\to x\in X$ as n→∞. Since $\alpha(x_n, x_{n+1}, x_{n+1})\ge1$ for all n, by definition of α, we have $x_n\in[0,1]$.Thus $\alpha(x_n,x,x)\ge1$.To show T satisfies all conditions of Theorem5.2,it is sufficient to show T is α-admissible.
For this , let x, y∈X such that α(x ,y, y)≥1 implies x ,y,∈[0,1] and by definition of T and α we have $Tx=\frac{x}{3}\in[0,1]$, $Ty=\frac{y}{3}\in[0,1]$and α(T x, Ty, T z)=1 i.e T is α-admissible. Here 0 and $\frac{5}{12}$ are two fixed points of T.
To ensure the uniqueness of the fixed point, we will consider the following hypothesis
(H): For all x, y∈X, there exists z ϵ X such that α(x, z, z)≥1 and α(y, z, z)≥1.

Theorem 5.3 Adding condition (H) to the hypothesis of theorem 5.1 and theorem 5.2 we obtain uniqueness of the fixed point of T.

Proof Suppose x^*, y^*, are two fixed point of T. From (H) there exists z ∈ X such that $\alpha(x^*, x^*, z)\ge1$ $\alpha(y^*,y^*,z)\ge1$.
Since T is α-admissible, we get by induction that
$\alpha(x^*,x^*,T^nz)\ge1$and $\alpha(y^*,y^*,T^nz)\ge1$ for all n=1,2,.. (7)
Using (7) and (2), we have

$$G(x^*,T^nz, x^*)= G(Tx^*,T(T^{n-1}z),T^2x^*)$$
$$\precsim\alpha(x^*,T^{n-1}z,Tx^*)\, G(Tx^*,T(T^{n-1}z),T^2x^*)$$
$$\precsim\psi(G(x^*,T^{n-1}z,Tx^*))= \psi(G(x^*,T^{n-1}z,x^*)).$$

Thus , we get by induction that
$G(x^*,T^nz, x^*)\precsim \psi^n(G(x^*,z,x^*))$ for all n=1,2,3 ...
By (CG4), we get
$G(x^*, x^*,T^nz)\precsim \psi^n(G(x^*,x^*,z))$
Letting n→∞,we get

$|G(x^*, x^*, T^n z)|=0$. This implies that $\{T^n z\}$ is G-convergent to x^*. Similarly, we get $\{T^n z\}$ is G − convergent to y^*. By uniqueness of limit we get,we get $x^* = y^*$,that is ,T has a unique fixed point.

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