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Fixed Points for ζ-α Expansive Mapping in 2-metric Spaces

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ABSTRACT: *In this paper, first we introduce the notion of (ζ-α) expansive mappings in the setting of 2-metric spaces and then prove some fixed point theorems for these maps. Also, we provide some examples in support of our results.*

MSC: *47H10, 54H25*

KEY-WORDS: *2-metric spaces, (ζ-α) expansive mappings, α-admissible map*

I. INTRODUCTION

The concept of 2-metric space has been investigated by Gahler [1] to generalize the concept of metric i.e., distance function and has been developed broadly by Gähler [2, 3] and more. After this the cocept of compatible maps, weakly compatible etc was introduced in 2-metric space and fixed point results was obtained for such type of maps. Various authors [13, 14, 15] used the concepts of weakly commuting mappings, compatible mappings of type (A) and (P) and weakly compatible mappings of type(A) to prove fixed point theorems in 2-metric space. Commutability of two mappings was weakened by Sessa [14] with weakly commuting mappings. Jungck [15] extended the class of non-commuting mappings by compatible mappings. Iseki [4] set out the tradition of proving fixed point theorems for various contractive conditions in 2-metric spaces. The study was further enhanced by Rhoades [8], Iseki [4], Sharma [9, 10, 11], Khan [5] and Ashraf [7].

Definition 1.1[1] Let X denotes a set of nonempty set and d : X×X×X→ R be a map satisfying the following conditions:
(1.1) For every pair of distinct points a, b ∈ X, there exists a point c ∈X such that d(a, b, c) ≠ 0.
(1.2) d(a, b, c) = 0, only if at least two of three points are same.
(1.3) The symmetry: d(a, b, c) = d(a, c, b) = d(b, c, a) = d(b, a, c) = d(c, a, b) = d(c, b, a) for all a, b, c ϵ X.
(1.4) The rectangular inequality: d(a, b, c) ≤ d(a, b, d) + d(b, c, d) + d(c, a, d) for all a, b, c, d ϵ X.
Then d is called 2-metric on X and (X, d) is called a 2-metric.

Definition 1.2 A sequence { x_n} is said to be Cauchy sequence in 2-metric space, if for each a ϵ X, $\lim_{m,n\to\infty} d(x_n, x, a) = 0.$

Definition 1.3 A sequence { x_n} in 2-metric space is said to be convergent to an element x ϵX , if for each a ϵ X,

Definition 1.4 A complete 2-metric space is one in which every cachy sequence in X is convergent to an element of X.
$\lim_{n\to\infty} d(x_n, x, a) = 0$.

In 2012, Shahi et.al. [12] gave a notion of (ζ-α) expansive mapping in metric spaces as follows:
Let χ denote all the functions ζ:[0,∞)→[0,∞), that satisfies the following properties:
(i) ζ is non decreasing;
(ii) $\sum_{n=1}^{+\infty} \zeta^n$(a) < +∞ for each a > 0,where ζ^n is the nth iterate of ζ;
(iii) $\zeta(a+b)$ = ζ(a) + ζ(b) for all a, b ∈ [0,∞).

Definition 1.5 [12] Let (*X*, *d*) be a metric space and *T* : *X* →*X* be a given mapping. We say that *T* is an (*ξ*, *α*)-expansive mapping if there exist two functions *ξ* ∈ *χ* and *α* : *X* × *X* →[0,+∞) such that

$$\xi(d(Tx, Ty)) \geq \alpha(x, y)\, d(x, y) \text{ for all } x, y \in X.$$

Definition 1.6 [6] Let T : X→X and α : X ×X→[0,+∞). T is said to be α-admissible, if x, y ∈ X, α(x, y) ≥ 1 implies α(Tx, Ty) ≥ 1.

Now we introduce the notion of (ζ-α) expansive mappings in the setting of 2-metric spaces akin to the notion of (ζ-α) expansive mappings in the setting of 2-metric spaces .

Definition 1.7 Let T: X→X and α: X×X×X →[0,∞). T is said to be α-admissible for 2-metric space X if x, y ∈ X, α(x, y, a) ≥ 1 implies α(Tx, Ty, a) ≥ 1 for all a∈X.

Example 1.1. Let X be the set of all non-negative real numbers.
Let α:X × X × X →[0,∞) be defined as

$$\alpha(x, y, z) = \begin{cases} 2, & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Define a map T : X→X by Tx = 2x+1 for all x ∈ X. Then T is α-admissible.

Definition 1.6 Let (X, d) be a 2-Metric space and T be a self map on G-metric space X. We say T is (ζ, α) expansive mapping if there exists two functions ζ∈χ and α: X×X×X→[0,∞) such that

$$\zeta(d(Tx, Ty, a)) \geq \alpha(x, y, a)\, d(x, y, a) \text{ for all } x, y, a \in X. \qquad (1.5)$$

II. MAIN RESULTS

Before proving our main results we need the following Lemma.
Lemma 2.1. [12] If ζ: [0,∞)→[0,∞) is a non decreasing function,then for each a > 0, $\lim_{n\to+\infty}\zeta^n(a)$ = 0 implies ζ(a) < a.
Now we present our main theorem as follow:
Theorem 2.2. Let (X,d) be a complete 2-metric space and T: X→X be a bijective (ζ,α) expansive mapping satisfying the following conditions:
(2.1) T^{-1} is α-admissible;
(2.2) There exists x_0∈X such that α(x_0, $T^{-1}x_0$, a) ≥ 1, for all a ∈ X;
(2.3) T is continous.
Then T has a fixed point, that is, there exists u ∈ X such that Tu = u.
Proof. Let us define a sequence{x_n} in X by x_n = Tx_{n+1}, for all n ∈ N.
Since T is bijective, so $x_{n+1} = T^{-1}x_n$.
For x_0 ∈ X,we have α(x_0, $T^{-1}x_0$, a) ≥ 1, i.e., α(x_0, x_1, a) ≥ 1.
Now if $x_n = x_{n+1}$ for any n ∈ N, then x_n is a fixed point of T by definition of x_n.
Without loss of generality, we can suppose $x_n \neq x_{n+1}$ for each n ∈ N.
Now α(x_0, x_1, a) ≥ 1 and T^{-1} is α-admissible.
Therefore, we have α($T^{-1}x_0$, $T^{-1}x_1$, a) ≥ 1 implies α(x_1, x_2, a) ≥ 1.
By induction we have, α(x_n, x_{n+1},a) ≥ 1 for all n ∈ N. (2.4)
Consider

$$\begin{aligned} d(x_n, x_{n+1}, a) &\leq \alpha(x_n, x_{n+1}, a)\, d(x_n, x_{n+1}, a) \\ &\leq \zeta(d(Tx_n, Tx_{n+1}, a)) \\ &= \zeta(d(x_{n-1}, x_n, a)). \end{aligned}$$

In the similar manner,we have,
$d(x_n, x_{n+1}, a) \leq \zeta^n d((x_0, x_1, a))$ for all n ∈ N.
For any n > m, we have,

$$\begin{aligned} d(x_m, x_n, u) &\leq d(x_m, x_{m+1}, x_n) + d(x_m, x_{m+1}, a) + d(x_{m+1}, x_n, a) \\ &\leq d(x_m, x_{m+1}, x_n) + d(x_m, x_{m+1}, a) + d(x_{m+1}, x_{m+2}, x_n) + d(x_{m+1}, x_{m+2}, a) \\ &\quad + d(x_{m+2}, x_n, a) \\ &\leq 2\zeta^m d((x_0, x_1, a) + 2\zeta^m d((x_0, x_1, a) + \cdots + 2\zeta^{n-1} d((x_0, x_1, a), \text{ for all } a \in X. \end{aligned}$$

Now from Lemma 2.1, It follows that {x_n} is a Cauchy sequence. Since (X, d) is complete 2-metric space, so there exists u ∈ X such that x_n→u as n→∞. From the continuity of T, it follows that $x_n = Tx_{n+1}$→ Tu as n→∞.

So by uniqueness of limit we have u = Tu, that is, u is fixed point of T.
We now relax the continuity of map T by using the following condition:
Condition [P] If {x_n} is a sequence in X such that α(x_n, x_{n+1}, a) ≥ 1 for all n ∈ N and {x_n}→x as n→∞, then α($T^{-1}x_n$, T^{-1}x, a) ≥ 1 for all n ∈ N. (2.5)
Theorem 2.3 In Theorem 2.2 ,we replace the continuity of T by condition [P] , then result still holds true.
Proof We know that {x_n} is a sequence in X such that α(x_n, x_{n+1}, a) ≥ 1,
for all n ∈ N and {x_n}→u as n→∞.

From (2.5) we have

$$\alpha(T^{-1}x_n, T^{-1}x_n, u) \geq 1 \quad \text{for all } n \in N \text{ and } a \in X. \tag{2.6}$$

Now $d(T^{-1}u, u, a) \leq d(T^{-1}u, u, x_{n+1}) + d(T^{-1}u, x_{n+1}, a) + d(x_{n+1}, u, a)$

$= d(T^{-1}u, T^{-1}x_n, u) + d(T^{-1}u, T^{-1}x_n, a) + d(x_{n+1}, u, a)$

$\leq \alpha(T^{-1}x_n, T^{-1}u, u) d(T^{-1}x_n, T^{-1}u, u) + \alpha(T^{-1}x_n, T^{-1}u, a)\, d(T^{-1}x_n, T^{-1}u, a) + d(x_{n+1}, u, a)$

$\leq \zeta(d(x_n, u, u)) + \zeta(d(x_n, u, u)) + d(T^{-1}x_n, u, a).$

Now continuity of ζ at t = 0 implies that $d(T^{-1}u, u, u) = 0$ as $n\to\infty$, that is, $T^{-1}u = u$

Now $Tu = T(T^{-1}u) = u$ implies that u is fixed point of T.

Example 2.1. Let $X = [0,\infty)$ with 2-metric $d(x,y,z) = \min(|x-y|, |y-z|, |z-x|)$ for all x, y, z ∈ X.
$T : X\to X$ and $\alpha: X\times X\times X \to [0,+\infty)$ defined as

$$Tx = \begin{cases} x^2, & x \geq 1, \\ x, & 0 \leq x < 1. \end{cases} \quad \text{and} \quad \alpha(x,y,z) = \begin{cases} \frac{1}{5} & \text{if } x, y \epsilon [0,1), \\ 1 & \text{if } x = y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If any one of x,y >1 then α(x, y, a) = 0, for all a ϵ X then
ζ(d(Tx,Ty,a)) ≥ α(x,y,a) d(x,y,a) holds.

Now if x = y = 1 then α(x, y, a) = 1, for all a ϵ X and we have d(x, y, a) = 0 for all a ϵ X.

Now for x, y ϵ [0,1), we have $\alpha(x,y,z) = \frac{1}{5}$.

Here T is $(\zeta - \alpha)$expansive mapping with $\zeta(a) = \frac{a}{2}$ for all $a \geq 0$, since

$\frac{1}{2} d(Tx, Ty, a) \geq \alpha(x, y, a)\, d(x, y, a)$holds for all x, y, aϵ X.

Also there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0, a) \geq 1$ for all aϵX.

Infact for $x_0 = 1$,we have $\alpha(1, T^{-1}1, a) = 1$. Also T is continuous.

Now it remain to show T^{-1} is α-admissible.

Let x, yϵ X such that α(x, y, a) ≥ 1 for all aϵX implies x = y = 1.

$\alpha(T^{-1}1, T^{-1}1, a) \geq 1$ for all aϵ X implies that T^{-1} is α − admissible.

We note that all hypothesis of theorem 2.2 are satisfied. So T has a fixed point. Infact in thi example all x ϵ [0,1) are fixed point of T.

Example 2.2. Let $X = [0,\infty)$ with 2-metric $d(x,y,z) = \min(|x-y|, |y-z|, |z-x|)$ for all x, y, z ∈ X.
$T : X\to X$ and $\alpha: X\times X\times X \to [0,+\infty)$ defined as

$$Tx = \begin{cases} x & \text{if } x \epsilon [0,1], \\ 5 - x^2 & \text{if } x \epsilon (1,2), \\ x^2 & \text{if } x \geq 2. \end{cases} \quad \text{and} \quad \alpha(x,y,z) = \begin{cases} \frac{1}{5} & \text{if } x, y \epsilon [0,1), \\ 1 & \text{if } x = y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If any one of x,y >1 then α(x, y, a) = 0, for all a ϵ X then
ζ(d(Tx,Ty,a)) ≥ α(x,y,a) d(x,y,a) holds.

Now if x = y = 1 then α(x, y, a) = 1, for all a ϵ X and we have d(x, y, a) = 0 for all a ϵ X.

Now for x, y ϵ [0,1), we have $\alpha(x,y,z) = \frac{1}{5}$.

Here T is $(\zeta - \alpha)$expansive mapping with $\zeta(a) = \frac{a}{2}$ for all $a \geq 0$, since

$\frac{1}{2} d(Tx, Ty, a) \geq \alpha(x, y, a)\, d(x, y, a)$holds for all x, y, aϵ X.

Also there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0, a) \geq 1$ for all aϵX.

Infact for $x_0 = 1$,we have $\alpha(1, T^{-1}1, a) = 1$. Also T is discontinuous at x=1 and at x=2.

Now it remain to show T^{-1} is α-admissible.

Let x, yϵ X such that α(x, y, a) ≥ 1 for all aϵX implies x = y = 1.

$\alpha(T^{-1}1, T^{-1}1, a) \geq 1$ for all aϵ X implies that T^{-1} is α − admissible.

Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all n ∈ N and $\{x_n\}\to x$, as $n\to\infty$ then
$\alpha(T^{-1}x_n, T^{-1}x, a) \geq 1$.

We note that all the conditions of theorem 2.3 are satisfied. So T has a fixed point. Infact in this example all x ϵ [0,1] are fixed points of T.

Remark 2.1. Now to ensure uniqueness of the fixed point in Theorems 2.2 and 2.3, by considering the following condition.
Condition [U] For all u, v $\in$X, there exists t $\in$ X such that $\alpha(u, t, a) \geq 1$ and $\alpha(v, t, a) \geq 1$ for all aϵX.
Theorem 2.4. Adding condition [U] to Theorems 2.2 and 2.3, we obtain the uniqueness of fixed point of T.
Proof Let u and v be two fixed points of T, that is, Tu = u and Tv = v.
From condition [U], there exists, w $\in$ X such that

$$\alpha(u, w, a) \geq 1 \text{ and } \alpha(v, w, a) \geq 1 \text{ for all } a \epsilon X. \quad (2.7)$$

As T^{-1} is α-admissible. Therefore (2.7) implies ,

$$\alpha(u, T^{-1}w, a) \geq 1 \text{ and } \alpha(u, T^{-1}w, a) \geq 1 \text{ for all } a \epsilon X. \quad (2.8)$$

Repeating α-admissible property of T^{-1},we get ,

$$\alpha(u, T^{-n}w, a) \geq 1 \text{ and } \alpha(v, T^{-n}w, a) \geq 1 \text{ for all } n\epsilon N \text{ and for all } a \epsilon X. \quad (2.9)$$

Using (1.1) and (2.9) we get,

$$d(u, T^{-n}w, a) \leq \alpha(u, T^{-n}w, a)\, d(u, T^{-n}w, a)$$
$$\leq \zeta(d(u, T^{-(n-1)}w, a) \quad \text{for all } a \epsilon X.$$

In a similar way, we get,

$$d(u, T^{-n}w, a) \leq \zeta^{n}(d(u, w, a)) \quad \text{for all } n \epsilon N \text{ and for all } a \epsilon X.$$

Thus we have ,T^{-n}w→u as n→∞.
Similarly T^{-n}w→v as n→∞, as uniqueness of limit of T^{-n}w gives u = v.

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