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# A Comparative Study of Multiset Orderings

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***ABSTRACT:*** *All efforts made by various researchers, particularly in the wake of resolving difficulties arising while addressing the termination issues in computer science, seem to be oriented toward generalizing the notion of multiset orderings on a class of finite multisets by taking the generic set finite and partially ordered. These orderings are maps from a partially ordered base set to partial orders on the class of finite multisets. We present in this paper a comparative study of such multiset orderings described in the literature during the last two decades or so and the efficiency of implementations.*

***KEY WORDS:*** *Generic set, Multiset, Partial ordering, Multiset Orderings.*

## 1 INTRODUCTION

Multiset orderings have been studied for many years. In recent years, it has been exploited in demonstrating termination of programs and that of term rewriting systems including production systems where programs are written in terms of rewrite rules (see (Knuth (1973),Dershowitz and Manna(1979), Dershowitz (1982), Jouannaud and Lescanne (1982) and Martin(1982)). In the sequel, a great variety of multiset path orderings have been extensively exploited in termination, AC termination and dependency pairs of term rewriting systems (see Dershowitz (1982), Bachmair and Plaisted (1985), Martin(87/88) Thistlewaite et al (1988), Kapur et al (1990), Baclawski (1981), Leclerc (1995), Kapur and Sivakumar (1997), Baader and Nipkow (1998), Kusakari (2000) and Jouannaud (2003)).

Consequent to the seminal idea advanced in Floyd (1967) that well-founded sets could be exploited for proving that programs terminate, the task of program verification has culminated into finding a termination function that maps the values of the program variables into some well-founded set such that the value of the termination function gets continually reduced throughout the process of computation. Dershowitz and Manna(1979) is the first place to profoundly demonstrate how the multiset ordering permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs. Infact, the multiset ordering proposed by Dershowitz and Manna (1979) has formed a basis for constructing a variety of multiset orderings .

In our presentation, it is observed that all efforts made by various researchers, particularly in the wake of resolving difficulties arising while addressing the termination issues in computer science, seem to be oriented toward generalizing the notion of multiset orderings on a class of finite multisets over a finite partially ordered generic set. These orderings are maps from a partially ordered base set to partial orders on the class of finite multisets. The section 2 of this paper contains some preliminaries required to make it self-contained. In section 3, a comprehensive study of the various formulations of multiset orderings is carried out. A comparative study of such multiset orderings is presented in section 4. We deduce the implementation efficiency of these definitions from the comparative study in section 5.

## 2 PRELIMINARIES

### Definition 2.1 Multiset and Multiset Operations

A multiset (mset, for short) is an unordered collection of objects (called its elements) in which unlike standard (Cantorian) sets, elements are allowed to repeat. In other words, an mset on a set $\mathcal{S}$ is an unordered sequence of elements of .

Formally, an mset $M$ over a set $\mathcal{S}$ is a cardinal-valued function. That is, $M$ on $\mathcal{S}$ is a map from $\mathcal{S}$ to the set $\mathbb{N}$ of natural numbers including zero. For elements $x \in \mathcal{S}, M(x)$ $or$ $M_x$ is called the multiplicity of $x$ in $M$. It follows by definition that $M(x) > 0 \quad \forall x \in M$ and $M(x) = 0$ for all $x \notin M$. An empty mset will be denoted by $\{ \quad \}$ or simply by $\emptyset$.

For any msets $M, N \in \mathfrak{M}(\mathcal{S})$, the *additive union* denoted $\uplus$ of $M$ and $N$ is the mset $M \uplus N$ such that $M \uplus N(x) = M(x) + N(x)$. The *difference* of $M$ and $N$ is the mset $M - N$ such that $M - N(x) = max(M(x) - N(x), 0)$. The mset $M$ is a submultiset (submset for short ) of an mset $N$ denoted $M \subseteq N$ if $M(x) \leq N(x)$ for all $x \in \mathcal{S}$.

We denote the class of all finite msets containing objects from the ground (generic) set $\mathcal{S}$ by $\mathfrak{M}(\mathcal{S})$ where a finite mset over a set $\mathcal{S}$ is a set formed with finitely many elements from $\mathcal{S}$ such that each element has a finite multiplicity. Hence, the class of all finite msets over $\mathcal{S}$ can be represented by

$\mathfrak{M}(\mathcal{S}) = \{M | M: \mathcal{S} \longrightarrow \mathbb{N} \; and \; M(x) = 0 \; for \; all \; but \; finitely \; many \; x \in \mathcal{S}\}$

### Definition 2.2 Nested Msets

These are msets whose elements may be members of some generic set $\mathcal{S}$ or may be msets of elements of $\mathcal{S}$ or both elements of $\mathcal{S}$ and msets of elements of $\mathcal{S}$, and so on. For example $\left\{\{1,1\}, \{\{0\}, 1,2\}, 0\right\}$ is a nested mset over the generic set $\mathcal{S} = \mathbb{Z}_+$, the set of positive integers.

We denote the set of nested msets over $\mathcal{S}$ by $\mathfrak{M}^*(\mathcal{S})$ and define $\mathfrak{M}^0(\mathcal{S}) = \mathcal{S}$ and $\mathfrak{M}^{i+1}(\mathcal{S})$ as containing the msets whose elements are taken from each of $\mathfrak{M}^0(\mathcal{S}), \mathfrak{M}^1(\mathcal{S}), \dots, \mathfrak{M}^i(\mathcal{S})$. A depth of a nested mset $M$ is a nonnegative integer $i$ for which the elements of $M$ are taken from each of $\mathfrak{M}^0(\mathcal{S}), \mathfrak{M}^1(\mathcal{S}), \dots, \mathfrak{M}^{i-1}(\mathcal{S})$. We denote the set of nested msets of depth $i$ by $\mathfrak{M}^i(\mathcal{S})$.

### Definition 2.3 List Msets

Let $\gtrdot$ be a total ordering on $\mathcal{S}$. Then the ordered list, $Lis(M)$ for any mset $M \in \mathfrak{M}(\mathcal{S})$ is defined by $Lis(M) = (x_1, \dots, x_n)$ with $j > i \Longrightarrow x_j \leq x_i$ where $\gtrdot$ is the usual ordering on the set of real numbers

### Definition 2.4 Well-foundedness

A well-founded relation consists of a set $\mathcal{S}$ and a transitive and irreflexive ordering $\succ$ defined on the elements of $\mathcal{S}$ such that there can be no sequence of elements $\{x_1, x_2, x_3, \dots\}$ such that $x_1 \succ x_2 \succ x_3 \succ \cdots$ i.e., there does not exist an infinite descending sequence of elements of $\mathcal{S}$. In other words, for a relation being well-founded, every nonempty subset of its domain must have minimal elements under the relation.

## 3. MULTISET ORDERINGS ON $\mathfrak{M}(\mathcal{S})$ WHEN $\mathcal{S}$ IS PARTIALLY ORDERED.

### 3.1 Some definitions

Mset ordering is an ordering defined on the class $\mathfrak{M}(\mathcal{S})$ of finite msets built from a ground set $\mathcal{S}$. We use $\succ$ or $\prec$ and $\gtrdot$ or $\lessdot$ on $\mathcal{S}$ to represent strict partial ordering and strict total ordering if it is irreflexive and transitive respectively.

Let $\succ\succ$ or $\prec\prec$ and $\ggg$ or $\lll$ be the associated ordering on $\mathfrak{M}(\mathcal{S})$ induced by $\succ$ or $\prec$ and $\gtrdot$ or $\lessdot$ respectively. We describe below various construction of orderings on $\mathfrak{M}(\mathcal{S})$ taking $\mathcal{S}$ equipped with a partial ordering $\succ$ or $\prec$. The symbol $\succ\succ$ or $\prec\prec$ shall be accompanied with a subscript for identification of the various formulations during the last two decades or so. We denote the incomparable elements $M, N \in \mathfrak{M}(\mathcal{S})$ by $M \# N$.

Essentially, an mset ordering $\succ\succ$ on $\mathfrak{M}(\mathcal{S})$ is a partial ordering induced by the partial ordering $\succ$ defined on $\mathcal{S}$. We regard the set of all partial orders on $\mathcal{S}$ denoted by $O(\mathcal{S})$ as a set partially ordered by the set inclusion relation $\subset$, where $\succ_1 \subset \succ_2$ if and only if $x \succ_1 y \rightarrow x \succ_2 y$ and $\succ_3 = \succ_1 \cap \succ_2 \rightarrow x \succ_3 y$ if and only if $x \succ_1 y$ and $x \succ_2 y$. Similarly, the set of partial orders on $\mathfrak{M}(\mathcal{S})$ denoted by $O(\mathfrak{M}(\mathcal{S}))$ can be regarded partially ordered by $\subset$. The order $\succ\succ \in O(\mathfrak{M}(\mathcal{S}))$ is said to inherit the order $\succ \in O(\mathcal{S})$ if $\{x\} \succ\succ \{y\}$ whenever $x \succ y$. The inheritance property ensures that the order on $\mathfrak{M}(\mathcal{S})$ is related in a way to the order on $\mathcal{S}$.

An mset ordering on $\mathfrak{M}(\mathcal{S})$ induced by a partial order on $\mathcal{S}$ is defined as a function $f: O(\mathcal{S}) \longrightarrow O(\mathfrak{M}(\mathcal{S}))$. This function $f$ is said to be *monotonic* if for any $\succ_1, \succ_2 \in O(\mathcal{S})$, $\succ_1 \subset \succ_2$ implies $f(\succ_1) \subset f(\succ_2)$. The function $f$ is said to be an *extension* function if it is monotonic and an *hereditary* extension function if it is monotonic and each order $\succ \in O(\mathcal{S})$ is inherited by $f(\succ) \in O(\mathfrak{M}(\mathcal{S}))$.

An extension function $g$ is said to be *maximal(minimal)* if for any other extension function $h$, $g(\succ) \subseteq h(\succ)(h(\succ) \subseteq g(\succ)) \rightarrow g = h \ \forall \succ \in O(\mathcal{S})$.

### 3.2 Dershowitz-Manna (1979) Ordering

In order to facilitate proving termination of programs and that of term rewriting systems, Dershowitz and Manna (1979) introduce an ordering on msets, usually called the *standard ordering* defined as follows:
Let $\succ$ be a partial ordering on $\mathcal{S}$. For $M, N \in \mathfrak{M}(\mathcal{S})$, if $M \neq N$ then $N \prec\prec_{DM} M$ if and only if for some finite msets $X, Y \in \mathfrak{M}(\mathcal{S})$,

(i) $\{ \ \} \neq X \subseteq M$ ,

(ii) $N = (M - X) \uplus Y$ and

(iii) $(\forall y \in Y)(\exists x \in X) x \succ y$.

Equivalently, $N \prec\prec_{DM} M$ if for some msets $X, Y, Z \in \mathfrak{M}(\mathcal{S})$ where $X$ is nonempty, $M = X \uplus Z$ , $N = Y \uplus Z$ and $(\forall y \in Y)(\exists x \in X) x \succ y$.

Note that $Y$ may be empty and hence $X \neq Y$. Also, $X \neq Y \neq Z$ to avoid triviality. For, if $X = Y$, then (iii) implies that $X$ is infinite, a contradiction to $X \in \mathfrak{M}(\mathcal{S})$. Moreover, $M \neq N$ which in turn means $X \neq M$. For, if $X = M$, then $N = Y$ and (iii) fails as $[3,3,4,0] \succ\succ [3,2,1,2,0,4]$ holds lexicographically, but (iii) does not hold.

This definition is difficult to use in order to prove that two multisets are not related by an inclusion. The definition only shows how to reduce a multiset. However, the efficient implementation of this definition is proposed in Dershowitz and Manna (1979).

The mset ordering $(\mathfrak{M}(\mathcal{S}), \prec\prec_{DM})$ over $(\mathcal{S}, \prec)$ is well founded, total and irreflexive if and only if $(\mathcal{S}, \prec)$ is well founded, total and irreflexive. Also, $\prec\prec_{DM}$ is monotonic and maximal (see Jouannaud and Lescanne (1982)).

While proving the transitivity property of the ordering $\prec\prec_{DM}$, Dershowitz and Manna (1979) define a *one step reduction order* as follows: Let $\succ$ be a partial ordering on $\mathcal{S}$, and $\mathfrak{M}(\mathcal{S})$ be the set of all finite msets built from $\mathcal{S}$. For $M, N \in \mathfrak{M}(\mathcal{S})$, the one step mset reduction order $\succ\succ^1$ is defined:

$M \succ\succ^1 N$ if and only if there exist $M_0, K \in \mathfrak{M}(\mathcal{S})$ and $x \in \mathcal{S}$ such that $M = M_0 \uplus \{x\}$ and $N = M_0 \uplus K$ and for all $y \in K, y \neq x$, we have $y \prec x$.
The ordering $\succ\succ^1$ is well-founded if and only if the order $\prec$ is well-founded on $\mathcal{S}$ (see Nipkow(1998) for details).

**Proposition 3.1**
The ordering $\gg^1$ on $\mathfrak{M}(\mathcal{S})$ is (i) total if $\succ$ is total on $\mathcal{S}$ and (ii) Monotonic.
Proof:

(i) Let $(\mathcal{S}, >)$ be a totally ordered set. Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \neq N$.

By definition, we have $M(z) \neq N(z)$ for some $z \in \mathcal{S}$.

In this case, either $M(z) < N(z)$ or $M(z) > N(z)$.

Now $M(z) < N(z) \rightarrow N(z) - M(z) > 0 \rightarrow N - M \neq \emptyset$

$\rightarrow \exists P(P \in \mathfrak{M}(\mathcal{S}) \wedge P \neq \emptyset \wedge N = M \uplus P)$.

Let $M_0 \in \mathfrak{M}(\mathcal{S})$ and $x \in \mathcal{S}$ such that $M = M_0 \uplus \{x\}$. Then

$N = M \uplus P = (M_0 \uplus \{x\}) \uplus P = M_0 \uplus (\{x\} \uplus P) = M_O \uplus K$ where $K = \{x\} \uplus P$.

Let $y \in K$ such that $y \neq x$. Since $(\mathcal{S}, >)$ is total, either $y > x$ or $y < x$.

If $y > x$, we have $N \gg^1 M$ and if $y < x$, we have $M \gg^1 N$.

Thus, for $M \neq N$, we have either $N \gg^1 M$ or $M \gg^1 N$.

(ii) Let $\prec_1 \subseteq \prec_2$. We show that ${}^1\prec\prec_1 \subseteq {}^1\prec\prec_2$.

Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $N\ {}^1\prec\prec_1 M$.

By definition, there exist $M_0, K \in \mathfrak{M}(\mathcal{S})$ and $x \in \mathcal{S}$ such that $M = M_0 \uplus \{x\}$ and $N = M_0 \uplus K$ and for all $y \in K$ we have $y \prec_1 x$. But for $y \prec_1 x$, we have $y \prec_2 x$(by hypothesis).

Thus, there exist $M_0, K \in \mathfrak{M}(\mathcal{S})$ and $x \in \mathcal{S}$ such that $M = M_0 \uplus \{x\}$ and $N = M_0 \uplus K$ and for all $y \in K$ we have $y \prec_2 x$. In this case, we have $N\ {}^1\prec\prec_2 M$ and the result follows.

Dershowitz-Manna (1979) also provides a generalization of mset orderings defined on $\mathfrak{M}(\mathcal{S})$ by defining, *nested mset* orderings $\succ\succ^*$ on the class $\mathfrak{M}^*(\mathcal{S})$ of nested msets as follows:

$M \succ\succ^* N$ if and only if (i)' $M, N \in \mathcal{S}$ and $M \succ N$ or (ii)' $M \notin \mathcal{S}$ and $N \in \mathcal{S}$ or

$M, N \notin \mathcal{S}$ and for some $X, Y \in \mathfrak{M}^*(\mathcal{S})$ where $\{\quad\} \neq X \subseteq M$, $N = (M - X) \uplus Y$ and

$(\forall y \in Y)(\exists x \in X) x \succ\succ^* y$.

This is a recursive version of the standard multiset ordering $\prec\prec_{DM}$.

It is startling to observe that the sequence $\mathfrak{M}^{*i}(\mathcal{S})$ typically forms a cumulative type of structure (see Drake and Singh (1996)). Hence, every $\mathfrak{M}^{*k}(\mathcal{S})$; $k = 1,2, \ldots$ is bounded in rank and belongs to von Neumann universe (see Singh and Singh (2009)), a necessary condition forbidding the occurrence of any infinite descending chain and granting well-foundedness in turn. The following theorem strengthens the aforesaid generalization:

The nested mset ordering $(\mathfrak{M}^*(\mathcal{S}), \succ\succ^*)$ over $(\mathcal{S}, \succ)$ is well-founded, irreflexive and total if and only if $(\mathcal{S}, \succ)$ is well-founded, irreflexive and total.

Also, $M \succ\succ^* N$ if and only if $N \in \mathfrak{M}^{*i}(\mathcal{S}) \wedge M \in \mathfrak{M}^{*j}(\mathcal{S}) \wedge i < j$. In other words, if the depth of the nested Multiset $N$ is less than the depth of the nested Multiset $M$ then $M \succ\succ^* N$ and conversely i.e the elements of $\mathfrak{M}^{*i}(\mathcal{S})$ are less than those of $\mathfrak{M}^{*j}(\mathcal{S})$ provided $i < j$.

**Proposition 3.2**
The class of nested multisets of distinct depths under the nested multiset ordering $\succ\succ^*$ is total and independent of the order on the underlying set $\mathcal{S}$.
**Proof:**
Let $M \in \mathfrak{M}^{*i}(\mathcal{S})$ and $N \in \mathfrak{M}^{*j}(\mathcal{S})$, where $i \neq j$.
Since for all distinct $, j$ ; either $i < j$ or $j < i$ and hence, either $N \succ\succ^* M$ or $M \succ\succ^* N$.

### 3.3 Huet-Oppen (1980) Ordering

Let $(\mathcal{S}, \prec)$ be a strictly ordered set. The Huet-Oppen mset ordering $\prec\prec_{HO}$ on $\mathfrak{M}(\mathcal{S})$ is defined as follows:

$M \prec\prec_{HO} N$ if and only if $M \neq N$ and $[M(y) > N(y) \to (\exists x \in \mathcal{S}) y \prec x$ and

$M(x) < N(x)]$. In words, if an object $y$ occurs more frequently in $M$, there exists another object $x$ greater than $y$ that occurs more frequently in $N$.

In *Martin (1989)*, it is observed that if $\prec$ is total, then $\prec\prec_{HO}$ is total and becomes lexicographic ordering on $\mathbb{R}^n$ with respect to $\prec$. That is, if
$\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ with $s_1 \succ s_2 \succ \ldots \succ s_n$, then $N \prec\prec_{HO} M$ if and only if $M \neq N$, and
$\forall i$ ; $1 \leq i \leq n, M(s_i) < N(s_i) \to \exists j < i$: $M(s_j) > N(s_j)$.

Note that the implementation of this definition can be derived. However, it is not efficient (see Jouannaud and Lescanne (1982), for details).

**Proposition 3.3**
The ordering $\prec\prec_{HO}$ on $\mathfrak{M}(\mathcal{S})$ is monotonic and well-founded if and only if $(\mathcal{S}, \prec)$ is well-founded.

**Proof:**
Let $(\mathcal{S}, \prec_1)$ and $(\mathcal{S}, \prec_2)$ be partially ordered sets where $\prec_1 \subset \prec_2$. Then we show that $\prec\prec_{1_{HO}} \subseteq \prec\prec_{2_{HO}}$. Now let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \prec\prec_{1_{HO}} N$.
By definition, $M \neq N$ and $[M(y) > N(y) \to (\exists x \in \mathcal{S}) y \prec x$ and

$M(x) < N(x)]$. In particular, $M \neq N$ and $[M(y) > N(y) \to (\exists x \in \mathcal{S}) y \prec x$ and

$M(x) < N(x)]$ since by hypothesis $\prec_1 \subset \prec_2$.

Hence, $M \prec\prec_{2_{HO}} N$ and $\prec\prec_{1_{HO}} \subseteq \prec\prec_{2_{HO}}$.

The ordering $\prec\prec_{HO}$ on $\mathfrak{M}(\mathcal{S})$ is maximal. It is also well-founded if and only if $(\mathcal{S}, \prec)$ is well-founded (see Jouannaud and Lescanne (1982)).

### 3.4 Jouannaud-Lescanne(1982) Ordering

Jouannaud and Lescanne(1982) define two *partition based orderings* $\prec\prec_{\mathfrak{R}}$ and $\prec\prec_{\mathcal{S}}$ on $\mathfrak{M}(\mathcal{S})$ as follows:

$M \prec\prec_{\mathfrak{R}} N$ if and only if $\bar{M} \prec_{\mathfrak{R}}^{lex} \bar{N}$ where

(i). $\bar{M} = \left\{ M_i \middle| \begin{array}{c} x \in M_i \to M_i(x) = M(x) \wedge x \in M_i, y \in M_i \\ \to x \# y \wedge \forall i \in [2..p]\ x \in M_i \to \exists y \in M_{i-1}\ x \prec y \end{array} \right\}$

(ii). $\bar{N} = \left\{ N_i \middle| \begin{array}{c} x \in N_i \to N_i(x) = N(x) \wedge x \in N_i, y \in N_i \\ \to x \# y \wedge \forall i \in [2..p]\ x \in N_i \to \exists y \in N_{i-1}\ x \prec y \end{array} \right\}$

(iii). $M_i \prec_{\mathfrak{R}} N_i$ if and only if $M_i \neq N_i$ and $\forall x \in M_i$, $M_i(x) \leq N_i(x)$ or

$\exists y \in N_i\ \ x \prec y$.

$M \prec\prec_{\mathcal{S}} N$ if and only if $\bar{M} \prec_{\mathcal{S}}^{lex} \bar{N}$ where

(i).$\bar{M} = \{S_i | S_i(x) \leq 1 \wedge x \in S_i, y \in S_i \to x \# y \wedge \forall i \in [2..p]\ x \in S_i \to \exists y \in S_{i-1}\ x \preccurlyeq y, i = 1,2, \ldots, p\}$

(ii).$\bar{N} = \{T_i | T_i(x) \leq 1 \wedge x \in T_i, y \in T_i \to x \# y \wedge \forall i \in [2..p]\ x \in T_i \to \exists y \in T_{i-1}\ x \preccurlyeq y, i = 1,2, \ldots, q\}$

(iii). $S_i \prec_{\mathcal{S}} T_i$ if and only if $S_i \neq T_i \wedge \forall x \in S_i\ \exists y \in T_i\ \ x \preccurlyeq y$.

Intuitively, the partition $\bar{M}$ is built by first computing the multiset $M_1$ or $S_1$ of all the maximal objects and then recursively computing the partition $M - M_1$ or $M - S_1$ respectively and similarly for $\bar{N}$ in each case.

The two partition based orderings $\prec\prec_{\mathfrak{R}}$ and $\prec\prec_{\mathcal{S}}$ as defined above are well-founded if the irreflexive transitive relation $\prec$ on $\mathcal{S}$ is well-founded.

However, the orderings $\prec_{\mathfrak{R}}, \prec_{\mathcal{S}}$ and the partition based orderings $\prec\prec_{\mathfrak{R}}, \prec\prec_{\mathcal{S}}$ are not monotonic (see Jouannaud and Lescanne(1982), pp.59-60 ).

Jouannaud and Lescanne (1982), using the concept of $List$, define an ordering $\prec\prec_*$ as follows:

Given a partial order $\prec$ on $\mathcal{S}$, an ordering $\prec\prec_*$ on $\mathfrak{M}(\mathcal{S})$ is defined:

$M \prec\prec_* N$ if and only if $List(M) <^{lex} List(N)$ for all total orderings $<$ containing $\prec$.

The poset $(\mathfrak{M}(\mathcal{S}), \prec\prec_*)$ is not total. Since if $\mathcal{S} = \{\mathrm{a,b,c|a} \prec \mathrm{b}\}$, then all the total orderings $<_i$ such that $\prec \subseteq <_i$ are:

$a <_1 b <_1 c, c <_2 a <_2 b, a <_3 c <_3 b$ . For the msets $M = [a,a,c,b]$ and $N = [b,a,a,c]$, we have for $a <_1 b <_1 c$,

$List(M) = (c,c,b,a), List(N) = (c,b,a,a)$ and $List(N) <_1^{lex} Lis(M)$.

But, for $c <_2 a <_2 b$, $List(M) = (b,a,c,c)$, $List(N) = (b,a,a,c)$ and $List(M) <_2^{lex} Lis(N)$. Therefore $M$ and $N$ are not comparable.

Note that under $\prec\prec_*$ on $\mathfrak{M}(\mathcal{S})$, it may not be easy to compare msets in $\mathfrak{M}(\mathcal{S})$ where $\mathcal{S}$ is a finite large size antichain. It is because of the fact that its linear extensions is of factorial order (see Rüetschi (2006)).

However, the poset $(\mathfrak{M}(\mathcal{S}), \prec\prec_*)$ is total and well-founded if and only if the ordering $\succ$ is total and well-founded on $\mathcal{S}$ (see Jouannaud and Lescanne (1982)).

**Proposition 3.4**
The ordering $\prec\prec_*$ on $\mathfrak{M}(\mathcal{S})$ is monotonic.

Proof:
Let $\prec_1$ and $\prec_2$ be two partial orders on $\mathcal{S}$ such that $\prec_1 \subset \prec_2$. Then we show that $\prec\prec*_1 \subseteq \prec\prec*_2$. Now let $M \prec\prec*_1 N$.

By definition, $List(M) <_1^{lex} List(N)$ for all total orderings $<_1$ such that $\prec_1 \subset <_1$.

Let $<_2$ be a total ordering such that $\prec_2 \subset <_2$.

Since $\prec_1 \subset \prec_2$ and $\prec_2 \subset <_2$, we have $\prec_1 \subset <_2$. In particular, $List(M) <_2^{lex} List(N)$.

Hence, for all total orderings $<_2$ such that $\prec_2 \subset <_2$, we have $List(M) <_2^{lex} List(N)$.

Thus, $M \prec\prec*_2 N$. In particular, $\prec\prec*_1 \subseteq \prec\prec*_2$.

### 3.5 Melven Krom (1985) Ordering
Melven Krom (1985) defines some binary relations on $\mathfrak{M}(\mathcal{S})$ taking strict transitive ordered set $(\mathcal{S}, \prec)$. To motivate these definitions, an intuitive description of the one denoted by $\prec^\infty$ is given by $M \prec^\infty N$ incase $M$ can be obtained from $N$ by a sequence of moves in which an occurrence of an object is removed and some finite number of occurrences of smaller objects are added. Formally, a binary relation $\prec\prec^n$ on $\mathfrak{M}(\mathcal{S})$ is defined for each $n \in \mathbb{N}$ as follows:

(1) $M \prec\prec^n N$ if and only if

(i) there exists exactly one element $y \in \mathcal{S}$ such that $M(y) < N(y)$, (ii) for this one element, $M(y) + 1 = N(y)$.

(iii) if $M(x) > N(x)$ then $x \prec y$ in $\mathcal{S}$ and (iv) $\sum\big(M(x) - N(x)\big) \leq n$

where the sum is over all $x \in \mathcal{S}$ such that $M(x) > N(x)$.Similarly the binary relation $\prec\prec^\infty$ is defined:

(2) $M \prec\prec^\infty N$ if and only if (iv) is omitted from the definition above

.

Taking $\prec^n$ as the transitive closure of $\prec\prec^n$ defined in (1) above, $\prec^n$ is defined:

(3) $M \prec^n N$ if and only if there is a finite $\prec\prec^n$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_k$ such that $M = M_0, N = M_k$ and $M_{i-1} \prec\prec^n M_i$ , $i = 1,2,\ldots,k$.

Also, denoting the transitive closure of $\prec\prec^{\infty}$ by $\prec^{\infty}$, the relation $\prec^{\infty}$ is defined:

(4) $M \prec^{\infty} N$ if and only if there is a finite $\prec\prec^{\infty}$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_k$ such that $M = M_0, N = M_k$ and $M_{i-1} \prec\prec^{\infty} M_i$ , $i = 1,2,\ldots,k$.

An extension of the given partial ordering $\prec$ of $\mathcal{S}$ to an ordering of $\mathfrak{M}(\mathcal{S})$ is the ordering $\prec_{\varepsilon}$ defined:

(5) $M \prec_{\varepsilon} N$ if and only if there existS $x \in \mathcal{S}$ such that $N(x) > 0$ and for any $y \in \mathcal{S}$, if $M(y) > 0$ then there is a $z \in \mathcal{S}$ such that $y \prec z$ and $N(z) > 0$ i.e $M \prec_{\varepsilon} N$

iff $N \neq \phi \wedge \forall y \in M \rightarrow \exists z \in N \wedge y \prec z$ .

If $(\mathfrak{M}(\mathcal{S}), \prec_{\varepsilon})$, $(\mathfrak{M}(\mathcal{S}), \prec^{\infty})$ $or$ $(\mathfrak{M}(\mathcal{S}), \prec^{n})$ is well-founded then $(\mathcal{S}, \prec)$ also is well-founded.

However, in each case, the converse is also true with the use of the axiom of choice . The relation $\prec^{0}$ is always well-founded but does not inherit $\prec$ (see Krom (1985)).

**Proposition 3.5**
The orderings $\prec\prec^{\infty}, \prec\prec^{\boldsymbol{n}}, \prec^{\infty}, \prec^{\boldsymbol{n}}$ and $\prec_{\boldsymbol{\varepsilon}}$ are monotonic extensions of the strict

order $\prec$ on $\mathcal{S}$.

Proof:

Let $\prec_1, \prec_2 \in O(\mathcal{S})$ such that $\prec_1 \subseteq \prec_2$. We show that $\prec\prec_1{}^{\infty} \subseteq \prec\prec_2{}^{\infty}$, $\prec\prec_1{}^{n} \subseteq \prec\prec_2{}^{n}$, $\prec_1{}^{\infty} \subseteq \prec_2{}^{\infty}$, $\prec_1{}^{n} \subseteq \prec_2{}^{n}$ and $\prec_{1_{\boldsymbol{\varepsilon}}} \subseteq \prec_{2_{\boldsymbol{\varepsilon}}}$.

Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \prec\prec_1{}^{\infty} N$. Then by definition, there exists exactly one element $y \in \mathcal{S}$ such that $M(y) < N(y)$. For this one element $y$, $M(y) + 1 = N(y)$. If $M(x) > N(x)$ then $x \prec_1 y$ in $\mathcal{S}$. Thus, If $M(x) > N(x)$ then $x \prec_2 y$ in $\mathcal{S}$ (by hypothesis). Hence, $M \prec\prec_2{}^{\infty} N$ and the result follows.
Suppose $M \prec\prec_1{}^{n} N$. Then by definition, there exists exactly one element $y \in \mathcal{S}$ such that $M(y) < N(y)$. For this one element $y$, $M(y) + 1 = N(y)$.
If $M(x) > N(x)$ then $x \prec_1 y$ in $\mathcal{S}$, and $\sum\big(M(x) - N(x)\big) \leq n$ where the sum is over all $x \in \mathcal{S}$ such that $M(x) > N(x)$. From the hypothesis, If $M(x) > N(x)$ then $x \prec_2 y$ in $\mathcal{S}$, and $\sum\big(M(x) - N(x)\big) \leq n$ where the sum is over all $x \in \mathcal{S}$ such that $M(x) > N(x)$. Thus $M \prec\prec_2{}^{n} N$ and the result follows.

Let $M \prec_1{}^{\infty} N$. Then by definition, there is a finite $\prec\prec_1{}^{\infty}$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_k$ such that $M = M_0, N = M_k$ and $M_{i-1} \prec\prec_1{}^{\infty} M_i$ , $i = 1,2,\ldots,k$. From monotonicity of $\prec\prec^{\infty}$, we have $M_{i-1} \prec\prec_2{}^{\infty} M_i$ , $i = 1,2,\ldots,k$. Thus, by definition, there is a finite $\prec\prec_2{}^{\infty}$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_k$ such that $M = M_0, N = M_k$ and $M_{i-1} \prec\prec_2{}^{\infty} M_i$ $i = 1,2,\ldots,k$. Hence $M \prec_2{}^{\infty} N$ and the result follows.

Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \prec_1{}^{n} N$. Then by definition, there is a finite $\prec\prec_1{}^{n}$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_k$ such that $M = M_0, N = M_k$ and $M_{i-1} \prec\prec_1{}^{n} M_i$ $i = 1,2,\ldots,k$. From the monotonicity of $\prec\prec^{n}$, it follows that $M_{i-1} \prec\prec_2{}^{n} M_i$ , $i = 1,2,\ldots,k$. Thus, there is a finite $\prec\prec_2{}^{n}$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_k$ such that $M = M_0, N = M_k$ and $M_{i-1} \prec\prec_2{}^{n} M_i$ , $i = 1,2,\ldots,k$. By definition, $M \prec_2{}^{n} N$. Hence, the result follows.

Let $M \prec_{1_{\boldsymbol{\varepsilon}}} N$. Then by definition, there exists $x \in \mathcal{S}$ such that $N(x) > 0$ and for any $y \in \mathcal{S}$, if $M(y) > 0$ then there is a $z \in \mathcal{S}$ such that $y \prec_1 z$ and $N(z) > 0$ i.e $M \prec_{1_{\boldsymbol{\varepsilon}}} N$ iff $N \neq \phi \wedge \forall y \in M \rightarrow \exists z \in N \wedge y \prec_1 z$ .

From the hypothesis, there exists $x \in \mathcal{S}$ such that $N(x) > 0$ and for any $y \in \mathcal{S}$, if $M(y) > 0$ then there is a $z \in \mathcal{S}$ such that $y \prec_2 z$ and $N(z) > 0$ . Thus $M \prec_{2_{\boldsymbol{\varepsilon}}} N$ and the result follows.

### 3.6 Dershowitz (1987) Ordering

Dershowitz (1987) recursively defined a quasi-ordering $_{Der}\succcurlyeq\succcurlyeq$ on $\mathfrak{M}(\mathcal{S})$ over a quasi-ordered base set $(\mathcal{S}, \succcurlyeq)$ as follows:

Let $X, Y \in \mathfrak{M}(\mathcal{S})$. Then,

$X = \{x_1, x_2, \dots, x_m\}\ _{Der}\succcurlyeq\succcurlyeq\ \{y_1, y_2, \dots, y_n\} = Y$ if and only if

$X = Y$ or if $x_i = y_j$ and $X - \{x_i\}\ _{Der}\succcurlyeq\succcurlyeq\ Y - \{y_j\}$ for some $i \in [1, m]$ and $j \in [1, n]$ or $x_i > y_{j_1}, y_{j_2}, \dots, y_{j_k}$ and $X - \{x_i\}\ _{Der}\succcurlyeq\succcurlyeq\ Y - \{y_{j_1}, y_{j_2}, \dots, y_{j_k}\}$ for $i \in [1, m]$

and $j_1 < j_2 < \cdots < j_k \le n$ ( $k \ge 1$).

Note that the linearity of $(\mathcal{S}, \succcurlyeq)$ does not necessarily imply the linearity of $\left(\mathfrak{M}(\mathcal{S}), _{Der}\succcurlyeq\succcurlyeq\right)$. For example, the msets [2,2,4,4] and [3,3,5,1,2] are not comparable. However a quasi-ordering $\succcurlyeq$ on a set $\mathcal{S}$ is well-founded if and only if the induced multiset ordering $_{Der}\succcurlyeq\succcurlyeq$ on $\mathfrak{M}(\mathcal{S})$ is well-founded (see Dershowitz (1987) for details).

**Proposition 3.6**

The ordering $_{Der}\succcurlyeq\succcurlyeq$ on $\mathfrak{M}(\mathcal{S})$ is Monotonic.

Proof:

Let $(\mathcal{S}, \preccurlyeq_1)$ and $(\mathcal{S}, \preccurlyeq_2)$ be two quasi-ordered sets such that $\preccurlyeq_1 \subset \preccurlyeq_2$ . We show that $\preccurlyeq\preccurlyeq_{1_{Der}} \subseteq \preccurlyeq\preccurlyeq_{2_{Der}}$.

Let $N = \{x_1, x_2, \dots, x_m\}, M = \{y_1, y_2, \dots, y_n\}$ and $M \preccurlyeq\preccurlyeq_{1_{Der}} N$. Then, by definition, we have

$M = N$ or if $x_i = y_j$ and $M - \{y_j\} \preccurlyeq\preccurlyeq_{1_{Der}} N - \{x_i\}$ for some $i \in [1, m]$ and $j \in [1, n]$ or $y_{j_1}, y_{j_2}, \dots, y_{j_k} \prec_1 x_i$ and

$M - \{y_{j_1}, y_{j_2}, \dots, y_{j_k}\}N - \{x_i\} \preccurlyeq\preccurlyeq_{1_{Der}} N - \{x_i\}$ for $i \in [1, m]$

and $j_1 < j_2 < \cdots < j_k \le n$ ( $k \ge 1$).

In particular, $M = N$ or if $x_i = y_j$ and $M - \{y_j\} \preccurlyeq\preccurlyeq_{2_{Der}} N - \{x_i\}$ for some $i \in [1, m]$ and $j \in [1, n]$ or $y_{j_1}, y_{j_2}, \dots, y_{j_k} \prec_2 x_i$ and $M - \{y_{j_1}, y_{j_2}, \dots, y_{j_k}\} \preccurlyeq\preccurlyeq_{2_{Der}} N - \{x_i\}$ for $i \in [1, m]$ and $j_1 < j_2 < \cdots < j_k \le n$ ( $k \ge 1$), since by hypothesis $\preccurlyeq_1 \subset \preccurlyeq_2$. Thus, $M \preccurlyeq\preccurlyeq_{2_{Der}} N$ and $\preccurlyeq\preccurlyeq_{1_{Der}} \subseteq \preccurlyeq\preccurlyeq_{2_{Der}}$.

### 3.7 Martin(1989) Ordering

Martin(1989) defines a multiset ordering $_{Mar}\succ\succ$ and $f_A$ on $\mathfrak{M}(\mathcal{S})$ as follows:

**Definition $_{Mar}\succ\succ$ on $\mathfrak{M}(\mathcal{S})$**

Let $\succ$ be any strict order defined on $\mathcal{S}$ and an $n\ by\ n$ matrix $A$ over $\mathbb{N}$ indexed by the elements of $\mathcal{S}$ whose $i, j$ entry is denoted by $a_{ij}$ .

(1) $M\ _{Mar}\succ\succ\ N$ if and only if $f_x(M) \ge f_x(N)\ \ \forall x \in \mathcal{S}$, where $f_x(M) = \sum_{y \succcurlyeq x} M(y)$.

Note that the linearity of $(\mathcal{S}, \succ)$ does not necessarily imply that of $\left(\mathfrak{M}(\mathcal{S}), _{Mar}\succ\succ\right)$.

For example, let $\mathcal{S} = \{1,2,3,4,8,12 | 1 < 2 < 3 < 4 < 8 < 12\}$.

Given $M = [4,3,3,8,12]$ and $N = [1,2,2,3,8,8]$, we have

$f_1(M) = 5, f_1(N) = 6$

$f_2(M) = 5, f_2(N) = 5$

$f_3(M) = 5, f_3(N) = 3$

$f_4(M) = 3, f_4(N) = 2$

$f_8(M) = 2, f_8(N) = 2$

$f_{12}(M) = 1, f_{12}(N) = 0$

Here, $f_3(M) = 5 > f_3(N) = 3$ and $f_1(N) = 6 > f_1(M) = 5$.

Hence, [4,3,3,8,12] and [1,2,2,3,8,8] are not comparable.

**Proposition 3.7**

The ordering ${}_{Mar}\succ\succ$ on $\mathfrak{M}(\mathcal{S})$ is (i) reflexive and transitive,
(ii) well-founded and (iii) Monotonic.

Proof:

(i) Let $\neg\left(M \ {}_{Mar}\succ\succ \ M\right)$. Then, there exists $x \in \mathcal{S}$ such that $\neg\left(f_x(M) \geq f_x(M)\right)$. Thus, $f_x(M) < f_x(M)$, a contradiction.
Thus, $M \ {}_{Mar}\succ\succ M$ (reflexivity).

Let $M \ {}_{Mar}\succ\succ N \ {}_{Mar}\succ\succ P$. Then by definition, we have $f_x(M) \geq f_x(N) \geq f_x(P)$ for all $x \in \mathcal{S}$. Thus, $f_x(M) \geq f_x(P)$ for all $x \in \mathcal{S}$.
Hence, $M \ {}_{Mar}\succ\succ P$ (transitivity).

(ii) Let the poset $\left(\mathfrak{M}(\mathcal{S}), {}_{Mar}\succ\succ\right)$ be not well-founded.
Let $M_1 \ {}_{Mar}\succ\succ M_2 \ {}_{Mar}\succ\succ \ldots$ be an infinite descending sequence. Then, by definition, $f_x(M_1) \geq f_x(M_2) \geq \cdots$ for all $x \in \mathcal{S}$ is infinite. In particular, $f_x(M_1) > f_x(M_2) > \cdots$ for all $x \in \mathcal{S}$. But $f_x(M_1) > f_x(M_2) > \cdots$ is an infinite descending sequence in $\mathbb{N}$. Since $f_x(M_i) \in \mathbb{N}$ for all $\in \mathcal{S}$ , $i = 1,2, \ldots$ , a contradiction. Thus, the poset $\left(\mathfrak{M}(\mathcal{S}), {}_{Mar}\succ\succ\right)$ must be well-founded.

(iii) Let $(\mathcal{S}, \prec_1)$ and $(\mathcal{S}, \prec_2)$ be such that $\prec_1 \subseteq \prec_2$.
We show that $\prec\prec_{1_{Mar}} \subseteq \prec\prec_{2_{Mar}}$.
Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \prec\prec_{1_{Mar}} N$.
By definition, we have $f_x(M) \geq f_x(N) \ \forall x \in \mathcal{S}$, where $f_x(M) = \sum_{x \preccurlyeq_1 y} M(y)$.
In particular, we have $f_x(M) \geq f_x(N) \ \forall x \in \mathcal{S}$, where $f_x(M) = \sum_{x \preccurlyeq_2 y} M(y)$
(by hypothesis). In this case, $M \prec\prec_{2_{Mar}} N$. Thus, $\prec\prec_{1_{Mar}} \subseteq \prec\prec_{2_{Mar}}$.

**Definition** $f_A(\succ)$ on $\mathfrak{M}(\mathcal{S})$

$Mf_A(\succ)N$ if and only if $[AM] \ {}_{HO}\succ\succ [AN]$ where $[AM]$ denotes the usual matrix product of $A$ and $M$ (equally a multiset) so that $[AM]_i = \sum_j a_{ij} M_j$.

The ordering $f_A(\succ)$ is well-founded provided $A$ is invertible. The ordering $f_A(\succ)$ inherits $\succ$ if and only if $A_x \ {}_{HO}\succ\succ A_y$ whenever $x \succ y$ where $A_x$ and $A_y$ are the columns indexed by $x$ and $y$ (elements of $\mathcal{S}$ ) respectively of $A$, an invertible matrix . $f_A$ is a hereditary maximal extension function whenever $A$ is an invertible $n$ by $n$ matrix over $\mathbb{N}$ and for each $i$ , $a_{i1} = a_{i2} = \cdots = a_{i,i-1} = a_{i,i+1} = \cdots a_{in} < a_{ii}$
(see Martin (1989) for details).

### 3.8 Zantema (1992) Ordering

Zantema (1992) defines mset ordering $\prec\prec_{Zan}$ on $\mathfrak{M}(\mathcal{S})$ as follows:
Let $\prec$ be an order on $\mathcal{S}$. Then, for $M, N \in \mathfrak{M}(\mathcal{S})$,

$M \prec\prec_{Zan} N$ if and only if $M \neq N \wedge \left(\forall a \in \mathcal{S}: N(a) \geq M(a) \vee \left(\exists a' \in \mathcal{S}: a \prec a' \wedge N(a') > M(a')\right)\right)$.

Note that $(\mathfrak{M}(\mathcal{S}), \prec\prec_{Zan})$ is total if and only if $(\mathcal{S}, \prec)$ is total, $(\mathfrak{M}(\mathcal{S}), \prec\prec_{Zan})$ is well-founded if and only if $(\mathcal{S}, \prec)$ is well-founded and if $(\mathcal{S}, \prec)$ corresponds to the ordinal $\alpha$, then $(\mathfrak{M}(\mathcal{S}), \prec\prec_{Zan})$ corresponds to the ordinal $\omega^{\alpha}$ (see Zantema (1992) for details).

**Proposition 3.8**

$(\mathfrak{M}(\mathcal{S}), \prec\prec_{Zan})$ is irreflexive and monotonic.

Proof:

$\prec\prec_{Zan}$ is irreflexive, since by definition, $M \prec\prec_{Zan} M$ implies $M \neq M$, a contradiction. Also, $\prec\prec_{Zan}$ is monotonic since for $(\mathcal{S}, \prec_1)$ and $(\mathcal{S}, \prec_2)$ such that $\prec_1 \subseteq \prec_2$, we have $M \prec\prec_{1_{Zan}} N$ if and only if

$M \neq N \wedge \left(\forall a \in \mathcal{S}: N(a) \geq M(a) \vee \left(\exists a' \in \mathcal{S}: a \prec_1 a' \wedge N(a') > M(a')\right)\right)$.

In particular, $M \prec\prec_{1_{Zan}} N$ if and only if

$M \neq N \wedge \left(\forall a \in \mathcal{S}: N(a) \geq M(a) \vee \left(\exists a' \in \mathcal{S}: a \prec_2 a' \wedge N(a') > M(a')\right)\right)$.
Thus, $M \prec\prec_{2_{Zan}} N$. In this case, we have $\prec\prec_{1_{Zan}} \subseteq \prec\prec_{2_{Zan}}$.

### 3.9 Baader and Nipkow(1998) Ordering

Baader and Nipkow(1998) define mset ordering $\prec\prec_{BN}$ on $\mathfrak{M}(\mathcal{S})$ as follows:
Let $(\mathcal{S}, \prec)$ be a strict ordered set. Then, for any $M, N \in \mathfrak{M}(\mathcal{S})$,
$M \prec\prec_{BN} N$ if and only if $N - M \neq \emptyset$ and $\forall x \in M - N, \exists y \in N - M$ such that $x \prec y$.

**Proposition 3.9**

The poset ( $\mathfrak{M}(\mathcal{S}) \prec\prec_{BN}$) is (i) irreflexive and monotonic,
(ii) ( $\mathfrak{M}(\mathcal{S}) \prec\prec_{BN}$) is total if and only if $(\mathcal{S}, \prec)$ is total.

Proof:
(i). Let $M \in \mathfrak{M}(\mathcal{S})$ such that $M \prec\prec_{BN} M$.
By definition, $M - M \neq \emptyset$ (a contradiction).
Thus, $\prec\prec_{BN}$ is irreflexive.
Let $(\mathcal{S}, \prec_1)$ and $(\mathcal{S}, \prec_2)$ be such that $\prec_1 \subseteq \prec_2$.We show that $\prec\prec_{1_{BN}} \subseteq \prec\prec_{2_{BN}}$.
Let $M, N \in \mathfrak{M}(\mathcal{S})$ be such that $M \prec\prec_{1_{BN}} N$.
We have $N - M \neq \emptyset$ and $\forall x \in M - N, \exists y \in N - M$ such that $x \prec_1 y$.
In particular, $N - M \neq \emptyset$ and $\forall x \in M - N, \exists y \in N - M$ such that $x \prec_2 y$.
Thus, $M \prec\prec_{2_{BN}} N$ and $\prec\prec_{1_{BN}} \subseteq \prec\prec_{2_{BN}}$.

(ii) Let ( $\mathfrak{M}(\mathcal{S}), \prec\prec_{BN}$) be total. Then we show that $(\mathcal{S}, \prec)$ is total.
Let $x, y \in \mathcal{S}$ such that $x \neq y$. Then $\{x\} \neq \{y\}$. Thus either $\{x\} \prec\prec_{BN} \{y\}$ or $\{y\} \prec\prec_{BN} \{x\}$. Let $\{y\} - \{x\} \neq \emptyset$. Then $\forall z \in \{x\} - \{y\} = \{x\}$, there exists $w \in \{y\} - \{x\} = \{y\}$ such that $z \prec w$. That is, $x \prec y$.
Let $\{x\} - \{y\} \neq \emptyset$. Then $\forall p \in \{y\} - \{x\} = \{y\}$, there exists $q \in \{x\} - \{y\} = \{x\}$ such that $p \prec q$; that is, $y \prec x$.Thus, for all $x, y \in \mathcal{S}$ with $x \neq y$, we have either $x \prec y$ or $y \prec x$. Hence $(\mathcal{S}, \prec)$ is total.

Conversely, let $(\mathcal{S}, \prec)$ be total and $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \neq N$. Then we show either $M \prec\prec_{BN} N$ or $N \prec\prec_{BN} M$. Now, $M \neq N$ implies $M(x) \neq N(x)$ for some $x \in \mathcal{S}$. In this case, either $M(x) < N(x)$ or $N(x) < M(x)$. That is, either $N - M \neq \emptyset$ or $M - N \neq \emptyset$.
Let $y \in M - N$ and $z \in N - M$ such that $y \neq z$.
Since $(\mathcal{S}, \prec)$ is total, either $y \succ z$ or $y \prec z$.
Now, for $y \succ z$, we have $N \prec\prec_{BN} M$ and for $y \prec z$, we have $M \prec\prec_{BN} N$.
Thus, for any $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \neq N$, we have either
$M \prec\prec_{BN} N$ or $N \prec\prec_{BN} M$ and ( $\mathfrak{M}(\mathcal{S}), \prec\prec_{BN}$) is total.

The poset ( $\mathfrak{M}(\mathcal{S}) \prec\prec_{BN}$) is well-founded if and only if $(\mathcal{S}, \prec)$ is well-founded
(see Ruiz-Reina et al. (2000 ) for details).

### 3.10 Wehrman (2006) Ordering

Wehrman (2006) defines mset ordering $\preccurlyeq\preccurlyeq_{Weh}$ on $\mathfrak{M}(\mathcal{S})$ as follows:

Let $(\mathcal{S}, \preccurlyeq)$ be a reflexive transitive ordering. Then, for $M, N \in \mathfrak{M}(\mathcal{S})$, $M \preccurlyeq\preccurlyeq_{Weh} N$ if and only if $M = (N - X) + Y$ for some $X, Y \in \mathfrak{M}(\mathcal{S})$ with $\emptyset \neq X \subseteq N$ and for all $y \in Y$ there exists $x \in X$ with $y \preccurlyeq x$, and $x \preccurlyeq y$ for $0,1, \ldots, |X|$ of the elements $y \in Y$.

Note that $(\mathfrak{M}(\mathcal{S}), \preccurlyeq\preccurlyeq_{Weh})$ is reflexive and transitive and well-founded if and only if $(\mathcal{S}, \preccurlyeq)$ is well-founded.
In particular, $(\mathfrak{M}(\mathbb{N}), \leq\leq_{Weh})$ over $(\mathbb{N}, \leq)$ is well-founded (see Wehrman (2006) for details).
However, $(\mathfrak{M}(\mathcal{S}), \preccurlyeq\preccurlyeq_{Weh})$ is not necessarily total whenever $(\mathcal{S}, \preccurlyeq)$ is total. For example,
$\neg([3,3,4,0] \ {}_{Weh}{\geq\geq} \ [4,3,2,1,1])$ and $\neg([4,3,2,1,1] \ {}_{Weh}{\geq\geq} \ [3,3,4,0])$.
In the first case, for $X = [3,0]$ and $Y = [2,1,1]$, $2,1,1 < 3$ but $3 \nleq 2,1,1$. In the second case, for $X = [2,1,1]$ and $Y = [3,0]$ there exists no $x \in X$ for which $x \geq 3$.

**Proposition 3.10**

The ordering $\preccurlyeq\preccurlyeq_{Weh}$ on $\mathfrak{M}(\mathcal{S})$ is monotonic.

Proof:
Let $(\mathcal{S}, \preccurlyeq_1)$and $(\mathcal{S}, \preccurlyeq_2)$ be such that $\preccurlyeq_1 \subseteq \preccurlyeq_2$.
We have for any $M, N \in \mathfrak{M}(\mathcal{S})$, $M \preccurlyeq\preccurlyeq_{1_{Weh}} N$ if and only if $M = (N - X) + Y$ for some $X, Y \in \mathfrak{M}(\mathcal{S})$ with $\emptyset \neq X \subseteq N$ and for all $y \in Y$ there exists $x \in X$ with $y \preccurlyeq_1 x$, and $x \preccurlyeq_1 y$ for atmost $|X|$ of the elements $y \in Y$.
In particular, $M \ll_{Weh} N$ if and only if $M = (N - X) + Y$ for some $X, Y \in \mathfrak{M}(\mathcal{S})$ with $\emptyset \neq X \subseteq N$ and for all $y \in Y$ there exists $x \in X$ with $y \preccurlyeq_2 x$, and $x \preccurlyeq_2 y$ for atmost $|X|$ of the elements $y \in Y$ (by hypothesis). Thus, $M \preccurlyeq\preccurlyeq_{2_{Weh}} N$. Hence, $\preccurlyeq\preccurlyeq_{1_{Weh}} \subseteq \preccurlyeq\preccurlyeq_{2_{Weh}}$.

## 4 COMPARATIVE ANALYSIS OF ORDERINGS ON $\mathfrak{M}(\mathcal{S})$ WHEN $\mathcal{S}$ IS PARTIALLY ORDERED

${}_{DM}{\succ\succ} \supseteq {\succ\succ}^1$ (see Dershowitz and Manna(1979)). (i)
$\prec\prec_{DM} = {}^*\prec\prec$ (see Dershowitz and Manna(1979)) (ii)
$\prec\prec_{DM} = \prec\prec_{HO}$ (iii)

$\prec\prec_{\mathfrak{R}} \#_{\subseteq} \prec\prec_{\mathcal{S}}$ (iv)

$\prec\prec_{DM} \subseteq \prec\prec_{\mathfrak{R}}$ (v)

$\prec\prec_{DM} \subseteq \prec\prec_{\mathcal{S}}$ (vi)

$\prec\prec_{*} = \prec\prec_{DM}$ (vii)

(see Jouannaud and Lescanne (1982) for details).

**Proposition 4.1**
Let $(\mathcal{S}, <)$ be a totally ordered set, then $(\mathfrak{M}(\mathcal{S}), \ll_{\mathfrak{R}})$ and $(\mathfrak{M}(\mathcal{S}), \ll_{\mathcal{S}})$ are totally ordered.

Proof:
Let $(\mathcal{S}, \prec)$ be partially ordered. Then by (v) and (vi), we have $\prec\prec_{DM} \subseteq \prec\prec_{\mathfrak{R}}$ and $\prec\prec_{DM} \subseteq \prec\prec_{\mathcal{S}}$ . Consequently given a totally ordered set $(\mathcal{S}, <)$, we have $\ll_{DM} \subseteq \ll_{\mathfrak{R}}$ and $\ll_{DM} \subseteq \ll_{\mathcal{S}}$. But $(\mathfrak{M}(\mathcal{S}), \ll_{DM})$ is total given a totally ordered set $(\mathcal{S}, <)$. Hence, $(\mathfrak{M}(\mathcal{S}), \ll_{\mathfrak{R}})$ and $(\mathfrak{M}(\mathcal{S}), \ll_{\mathcal{S}})$ must be totally ordered.

**Proposition 4.2**
The following hold:
(i) $\prec\prec^{\boldsymbol{n}} \subset \prec\prec^{\infty} \subset \prec\prec_{\boldsymbol{HO}}$ (ii) $\prec_{\boldsymbol{\varepsilon}} \subset \prec_{\boldsymbol{HO}}$ and (iii) $\prec^{\boldsymbol{n}} \subset \prec^{\infty}$

Proof:
Following the definition of $\prec\prec^{\boldsymbol{n}}$, $\prec\prec^{\infty}$, $\prec_{\varepsilon}$ and $\prec\prec_{\boldsymbol{HO}}$ it is easy to see that (i) and (ii) hold.

For (iii), Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \prec^n N$.

By definition, there is a finite $\prec\prec^n$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_m$ such that $M = M_0, N = M_m$ and $M_{i-1} \prec\prec^n M_i$ , $i = 1,2, \ldots, m$.

It follows that $M_{i-1} \prec\prec^{\infty} M_i$ , $i = 1,2, \ldots, m$. Thus, there is a finite $\prec\prec^{\infty}$ chain from $M$ to $N$ i.e., a sequence $M_{0,\ldots,}M_m$ such that $M = M_0, N = M_m$ and $M_{i-1} \prec\prec^{\infty} M_i$ , $i = 1,2, \ldots, m$. Therefore, by definition, $M \prec^{\infty} N$ and $\prec^{\boldsymbol{n}} \subset \prec^{\infty}$ we have, $M\ {}_{Der}{\succcurlyeq\succcurlyeq}\ N \rightarrow M\ {}_{DM}{\succ\succ}\ N$ for any $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \neq N$ (viii)
(see Dershowitz (1987) for details).

However, in general,
$\prec\prec_{DM} \nsubseteq \preccurlyeq\preccurlyeq_{Der}$; since $[3,4,2,1,1] \prec\prec_{DM} [3,3,4,0]$ but
$[3,4,2,1,1] \#_{\preccurlyeq\preccurlyeq_{Der}} [3,3,4,0]$ .

We have $f_A(>) =_{HO}{>>}$ whenever $A = I_n$ where $I_n$ is an identity matrix of order $n$. (ix)
However, in general, $f_A(\succ) \neq_{HO} {\succ\succ}$, for example $[y, y] f_A(\succ) [x]$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{S} = \{x, y\}$ such that $x \succ y$; but $[y, y]\ _{HO}\succ\succ [x]$ is not true.

We have, $\prec\prec_{Zan} = \prec\prec_{DM}$ (see Zantema (1992)) (x)
We have, $\prec\prec_{BN} = \prec\prec_{DM}$ (see Kusakari (2000) and Ruiz-Reina et al. (2000 )) (xi)

**Proposition 4.3**
$\prec\prec_{HO} \subseteq \prec\prec_{Zan}$.
Proof:
For $M, N \in \mathfrak{M}(\mathcal{S})$, we have $M \prec\prec_{HO} N$ if and only if $M \neq N$ and
$[M(y) > N(y) \rightarrow (\exists x \in \mathcal{S}) y \prec x$ and $M(x) < N(x)]$.

Now, let $z \in \mathcal{S}$ such that $x \prec z\ \forall x \in \mathcal{S}$. We show that $M(z) < N(z)$.

Let $M(z) > N(z)$. Then, by definition, there must exist an element $p \in \mathcal{S}, z \prec p$ such that $M(p) < N(p)$ (a contradiction, since $\forall x \in \mathcal{S}, x \prec z$ ). Hence, $M(z) < N(z)$. In particular,
$M \prec\prec_{HO} N$ if and only if $M \neq N$ and $[M(y) > N(y) \rightarrow (\exists x \in \mathcal{S}) y \prec x$ and
$M(x) < N(x)$ and $M(z) < N(z)$ if $x \prec z\ \forall x \in \mathcal{S}]$
Thus, $M \prec\prec_{Zan} N$. In this case, we have $\prec\prec_{HO} \subseteq \prec\prec_{Zan}$.

**Corollary 4.4**
$\prec\prec_{HO} = \prec\prec_{Zan}$
Proof:
We have, from (iii) and (x), $\prec\prec_{Zan} = \prec\prec_{DM} = \prec\prec_{HO}$
Thus, $\prec\prec_{HO} = \prec\prec_{Zan}$.

**Proposition 4.5**
For all $M, N \in \mathfrak{M}(\mathcal{S})$
$M \neq N \wedge M \prec\prec_{Mar} N \rightarrow M \prec\prec_{HO} N$.

Proof:
Let $M, N \in \mathfrak{M}(\mathcal{S})$ such that $M \neq N$ and $M \prec\prec_{Mar} N$.
We show that $M \prec\prec_{HO} N$.
By definition, $M \prec\prec_{Mar} N$ if and only if $f_x(M) \leq f_x(N)$ for all $x \in \mathcal{S}$
where $f_x(M) = \sum_{y \succcurlyeq x} M(y)$ and $f_x(N) = \sum_{y \succcurlyeq x} N(y)$.
Let $M(z) > N(z)$.
By definition, $f_z(M) \leq f_z(M)$; i.e, $\sum_{y \succcurlyeq z} M(y) \leq \sum_{y \succcurlyeq z} N(y)$.
Thus, there exists $p \in \mathcal{S}$ such that $p \succ z$ and $M(p) < N(p)$.
Otherwise, $M(p) \geq N(p)$ for all $p \succcurlyeq z$ and $\sum_{p \succcurlyeq z} M(p) \geq \sum_{p \succcurlyeq z} N(p)$ (a contradiction). Thus, if $M(z) > N(z)$, we have $p \in \mathcal{S}$ such that $M(p) < N(p)$. Hence, $M \prec\prec_{HO} N$.
Note that $\preccurlyeq\preccurlyeq_{Weh}$ allows for weaker relation between objects of $X$ and $Y$, but $\prec\prec_{DM}$ needs to be strict. In particular, relaxing the weaker relation between elements of $X$ and $Y$, we have
$\prec\prec_{Weh} = \prec\prec_{DM}$ (xii)
(see Wehrman (2006) for details).
However, in general, $\prec\prec_{DM} \subset \preccurlyeq\preccurlyeq_{Weh}$ holds. (xiii)

In view of the aforesaid comprehensive study of mset orderings on $\mathfrak{M}(\mathcal{S})$ induced by a partial order on $\mathcal{S}$, and the results obtained in (i)-(xiii) and 4.2-4.4, a summary of the results pertaining to comparison of mset orders is as follows:

(i) $\prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$
(ii) $\prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan} \subseteq \prec\prec_{\mathcal{S}}$
(iii) $\prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan} \subseteq \prec\prec_{\mathcal{R}}$
(iv) $\prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan} \subset \prec\prec_{Weh}$
(v) ${}^{1}\prec\prec \subseteq \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$
(vi) $\prec\prec^{\boldsymbol{n}} \subset \prec\prec^{\infty} \subset \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$
(vii) $\prec_{\boldsymbol{\varepsilon}} \subseteq \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$

(viii) $f_A(\prec) = \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$ whenever $A = I_n$ where $I_n$ is an identity matrix of order $n$

However, relaxing the weaker relations on $\mathcal{S}$ and $\mathfrak{M}(\mathcal{S})$ , we have

(ix) $\prec\prec_{Mar} \subset \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$

(x) $\prec\prec_{Der} \subset \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$ and

(xi) $\prec\prec_{Weh} = \prec\prec_{BN} = \prec\prec_{DM} = {}^{*}\prec\prec = \prec\prec_{*} = \prec\prec_{HO} = \prec\prec_{Zan}$

## 5 IMPLEMENTATION AND EFFICIENCY

With the proposed efficient implementation of the ordering $\prec\prec_{DM}$ in Jouannaud and Lescanne (1982), we propose the same efficient implementation for the equivalent orderings $f_A(\prec)$, $\prec\prec_{BN}$, $\prec\prec_{DM}$, ${}^{*}\prec\prec$, $\prec\prec_{*}$, $\prec\prec_{HO}$, $\prec\prec_{Zan}$ whenever $A = I_n$ where $I_n$ is an identity matrix of order $n$

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