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VALUE OF THE INFINITE SERIES $\sum_{n=1}^{\infty}\frac{1}{n^3}$

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ABSTRACT: *The value of the infinite series of the sum of reciprocals of the cubes has remained unknown so far .It has not been expressed as a multiple of π^3. I have tried to find it by computation.*

***KEYWORD*:** Infinite series
NOTATIONS: Notations are very simple.

I. INTRODUCTION

A natural question is whether Zeta(3) is a rational multiple of π^3. Although the values of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}$ are known to us, the value of $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is unknown to us. I have tried to find out by computation. (π=22/7) has been taken into consideration.

II. METHOD OF ANALYSIS

Fourier sine series for x^2 gives a well known result,

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----+\infty=\frac{\pi^3}{32}$$

Now $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----+\infty$ can be written as

$$(1+\frac{1}{5^3}+\frac{1}{9^3}+----+\infty)-(\frac{1}{3^3}+\frac{1}{7^3}+\frac{1}{11^3}+----+\infty)=\frac{\pi^3}{32}$$

$\frac{\pi^3}{32}$ is the difference of two infinite series. Here the difference of two infinite series is a multiple of π^3.Hence the addition of these infinite series must be a multiple of π^3.If we are able to express $\frac{\pi^3}{32}$ as the difference of two numbers precisely then we can find the value of $1+\frac{1}{3^3}+\frac{1}{5^3}+----+\infty$.

It has been found that $\frac{\pi^3}{32}=\frac{2695\pi^3}{82756}-\frac{871\pi^3}{662048}$

Now $(1+\frac{1}{5^3}+\frac{1}{9^3}+----+\infty)-(\frac{1}{3^3}+\frac{1}{7^3}+\frac{1}{11^3}+----+\infty)=\frac{\pi^3}{32}=\frac{2695\pi^3}{82756}-\frac{871\pi^3}{662048}$

Equating equal parts, we get $1+\frac{1}{5^3}+\frac{1}{9^3}+----+\infty \;\frac{2695\pi^3}{82756}$ ---------(1)

and $\frac{1}{3^3}+\frac{1}{7^3}+\frac{1}{11^3}+----+\infty=\frac{871\pi^3}{662048}$ ---------- (2)

Adding (1) and (2),we get

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{9^3}+----+\infty \;\frac{2695\pi^3}{82756}+\frac{871\pi^3}{662048}$$

$$Or,\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}=\frac{(21560+871)\pi^3}{662048}$$

$$Or,\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}=\frac{22431\pi^3}{662048} \text{ -------------(3)}$$

$$Now,\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}+\sum_{n=1}^{\infty}\frac{1}{(2n)^3}$$

$$Or,\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}+\sum_{n=1}^{\infty}\frac{1}{8n^3}$$

$$Or,(1-\frac{1}{8})\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}$$

$$Or,\frac{7}{8}\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}$$

$$Or,\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{8}{7}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}$$

$$Or,\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{8}{7}(\frac{22431\pi^3}{662048}) \text{ -------------- from(3)}$$

$$Finally,\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{22431\pi^3}{579292}$$

III. CONCLUSION

From this method we find that $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{22431\pi^3}{579292}$

$$Also, \frac{22431\pi^3}{579292} = 1.20205.....$$

Here $\sum_{n=1}^{\infty} \frac{1}{n^3}$ becomes equal to Apery's constant.

REFERENCE

(1) George F. Simons, *Differential Equations With Applications And Historical Notes*, Tata McGraw-Hill Publishing company limited, New Delhi.