International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 3 Issue 7 || November. 2015 || PP-40-44

Probabilistic diameter and its properties.

Dr. Ayaz Ahmad
Reader,Deptt. of Mathematics Millat College, Darbhanga, BIHAR, PIN;846004

Abstract: *In this paper ,we discuss on probabilistic diameter and some of its basic properties.*
Key Words: *Probabilistic diameter, Probabilistic distance, Distribution function.*

I. Introduction

Probabilistic metric spaces were first introduced by K. Menger in 1942 and reconsidered by him in the early 1950's B. Schweizer and A. Sklar have been studying these spaces, and have developed their theory in depth. In probabilistic metric spaces the notion of distance between two points x and y is replaced by a distribution function F_{xy}. Thus, the distance between points as being probabilistic with $F_{xy}(t)$ representing the probability that the distance between x and y is less than t.

Definition: 1.1. Let (S, F,T) denote a Menger space with a continuous t-norm and A be a nonempty subset of S. The function D_A, defined by

$$D_A(x) = \sup_{t<x}\left[\inf_{p,q\in A} F_{pq}(t)\right]$$, is called the probabilistic diameter of A.

Properties of the probabilistic diameter.

Defenition.1.2.A nonempty subset A of S is bounded if $\mathrm{Sup}_x\, D_A(x) = 1$, semi-bounded if $0 < \mathrm{Sup}\, D_A(x) < 1$, and unbounded if $D_A = 0$.

P_1) The function D_A is a distribution function

P_2) If A is a nonempty subset of S, then $D_A = H$ if and only if A consists of a single point.

P_3) If A and B are nonempty subsets of S and $A \subseteq B$, then $D_A \geq D_B$.

Theorem.1.3. If A and B are two nonempty subsets of S such that $A \cap B = \phi$,

then $D_{A\cup B}(x + y) \geq T(D_A(x), D_B(y))$............ (1.1)

Proof. Let x and y be given. To establish (1.1) we first show that

$$\inf_{p,q\in A\cup B} F_{pq}(x+y) \geq T\left(\inf_{p,q\in A} F_{pq}(x), \inf_{p,q\in B} F_{pq}(y)\right) \qquad \text{............ (1.2)}$$

There are two distinct cases to consider:

Case (1).

$$\inf_{p,q\in A\cup B} F_{pq}(x+y) = \inf_{\substack{P\in A\\ q\in B}} F_{pq}(x+y) \qquad \text{.................(1.3)}$$

Now for any triple of points p,q and r in S, we have

$F_{pq}(x + y) \geq T(F_{pr}(x), F_{rq}(y))$.

Taking the Infimum of both sides of this inequality as p ranges over A, q ranges over B and r ranges over $A \cap B$

$$\inf_{p,q\in A\cup B} F_{pq}(x+y) \geq \inf_{\substack{p\in A\\ q\in B\\ r\in A\cap B}} T(F_{pr}(x), F_{rq}(y)).$$

However, since T is continuous and non decreasing ,we obtain,

$$\inf_{p,q\in A\cup B} F_{pq}(s+y) \geq T\left(\inf_{p,r\in A} F_{pr}(x), \inf_{r,q\in B} Frq(y)\right).$$

Case (2).

$$\inf_{p,q\in A\cup B} F_{pq}(x+y) < \inf_{\substack{p\in A\\ q\in B}} F_{pq}(x+y).$$

In this case of the equalities, $\inf_{p,q\in A\cup B} F_{pq}(x+y) = \inf_{p,q\in A} F_{pq}(x+y)$ Or

$$\underset{p,q \in A\cup B}{Inf} F_{pq}(x+y) = \underset{p,q\in B}{Inf} F_{pq}(x+y)$$

must hold. If the first equality holds, we have $\underset{p,q\in A\cup B}{Inf} F_{pq}(x+y) \geq T\left(\underset{p,q\in A}{Inf} F_{pq}(x), H(y)\right) \geq T\left(\underset{p,q\in A}{Inf} F_{pq}(x), \underset{p,q\in B}{Inf} F_{pq}(y)\right)$

The same argument works for the second equality.
This establishes (1.2.).
Finally using the fact that the rectangle

{(s, t): 0 $\leq \delta \leq x, 0 \leq t \leq y$ }

is contained in the triangle {(s, t): s, t$\geqq$ 0, s + t < x + y},
the inequality (1.2) and the continuity of T. we have

$$D_{A\cup B}(x+y) = \underset{S+t<x+y}{Sup}\left[\underset{p,q\in A\cup B}{Inf} F_{pq}(s+t)\right] \geq \underset{\substack{S<x\\ t<y}}{Sup}\left[\underset{p,q\in A\cup B}{Inf} F_{pq}(x+t)\right]$$

$$\geq T\left(\underset{S<X}{Sup}\left[\underset{p,q\in A}{Inf} F_{pq}(s)\right], \underset{t<v}{Sup}\left[\underset{p,q\in B}{Inf} F_{pq}(t)\right]\right)$$

$= T(D_A(x), D_B(y))$.

Theorem.1.4. If A is a nonempty subset of S, then $D_A = D_{\bar{A}}$, where $\bar{A}$ denotes the closure of A in the ε – λ topology on S.

Proof. Since A $\subseteqq \bar{A}$, if follows from property P_1 that DA $\geq D_{\bar{A}}$.

Let η > 0 be given. In view of the uniform continuity of $\mathfrak{I}$ with respect to the Levy metric L there exists an

ε > 0 and a λ > 0 such that for any four points p_1, p_2, p_3 and p_4 in S, $L(F_{p_1p_2}, F_{p_3p_4}) < \eta$
When ever $F_{p_1p_3}(\varepsilon) > 1 - \lambda$ and $F_{p_2p_4}(\varepsilon) > 1 - \lambda$.

Next, with each point $\bar{P}$ and $\bar{A}$ associate a point P($\bar{p}$) in $\bar{A}$ such that

$F_{p(\bar{p})\bar{p}}(\varepsilon) > 1 - \lambda$.

Then, in view of the above for any pair of points $\bar{p}$ and $\bar{q}$ A,

$L(F_{p(\bar{p})q(\bar{q})}, F_{\bar{p}\bar{p}}) < \eta$.

In particular, for all t we have, $F_{p(\bar{p})q(\bar{q})}(t-\eta) - \eta \leq F_{\bar{p}\bar{q}}(t)$.

Let A^{η} = {p($\bar{p}) : \bar{p} \in \bar{A})$. Then since $A^{\eta} \subseteqq$ A,

$$\underset{\bar{P},\bar{q}\in \bar{A}}{Inf} F_{\bar{P}\bar{q}}(t) \geq \underset{\bar{p},\bar{q}\in\bar{A}}{Inf} p(\bar{p})q(\bar{q})(t-\eta)-\eta$$

$$= \underset{p,q\in A\eta}{Inf} F_{pq}(t-\eta)-\eta \geq \underset{p,q\in A}{Inf} F_{pq}(t-\eta)-\eta.$$

Now, taking the suprimum for t < x of the above inequality yields

$$D_{\bar{A}}(x) = \underset{t<x}{Sup}\left[\underset{\bar{p},\bar{q}\in\bar{A}}{Inf} F_{\bar{p}\bar{q}}(t)\right] \geq \underset{t>x}{Sup}\left[\underset{p,q\in A}{Inf} F_{pq}(t-\eta)\right] - \eta$$

$$= \underset{t<x-\eta}{Sup}\left[\underset{p,q\in A}{Inf} F_{pq}(t)\right] - \eta = D_A(x - -\eta) - \eta$$

Since the above inequality is valid for all η and,
since D_A is left continuous.

It follows that $D_{\bar{A}}(x) \geq D_A(x)$.Hence $D_{\bar{A}}(x) = D_A(x)$, Hence, the proof is complete.

Definition.1.5. Let A and B be nonempty subsets for S. The probabilistic distance between A and B is the function F_{AB} defined by

$$F_{AB}(x) = \sup_{t<x} T\left(\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(t) \right], \inf_{q\in B}\left[\sup_{p\in B} F_{pq}(t) \right]\right) \quad \text{......... (1.4)}$$

The following are the properties of F_{AB}.

P_4. F_{AB} is a distribution function.

P_5. If A and B are nonempty subsets of S, then $F_{AB} = F_{BA}$.

P_6. If A is a nonempty subset of S, then $F_{AA} = H$.

Theorem.1.6. If A and B are nonempty subsets of S, then $F_{AB} = F_{\bar{A}\bar{B}}$.

Proof. It is sufficient to show that FAB = $F_{A\bar{B}}$ since this result together with property P_5 yields.

Hence, $F_{AB} = F_{A\bar{B}} = F_{\bar{B}A} = F_{\bar{B}\bar{A}} = F_{\bar{A}\bar{B}}$.

Now, we first show that $F_{A\bar{B}} \leqq F_{AB}$. Since $B \subseteqq \bar{B}$ for all t,

$$\inf_{q\in B}\left[\sup_{p\in A} F_{pq}(t) \right] \geqq \inf_{\bar{q}\in \bar{B}}\left[\sup_{p\in A} F_{p\bar{q}}(t) \right] \quad \text{............. (1.5)}$$

Let $\eta > 0$ be given. The argument given in the proof of Theorem(1.4) establishes that for each point $\bar{q} \in \bar{B}$, there exists a point q($\bar{q}$) in B such that for all t, $F_{p\bar{q}}(t-\eta) - \eta \leqq F_{pq(\bar{q})}(t)$.

Let B_η = {q($\bar{q}$) : $\bar{q} \in \bar{B}$ }. Since $B_\eta \subseteqq$ B we have,

$$\sup_{\bar{q}\in \bar{B}} F_{p\bar{q}}(t-\eta) - \eta \leqq \sup_{\bar{q}\in \bar{B}} (t) = \sup_{q\in B_\eta} F_{pq}(t) \leqq \sup_{q\in B} F_{pq}(t).$$

Consequently, $$\inf_{p\in A}\left[\sup_{\bar{q}\in \bar{B}} F_{p\bar{q}}(t-\eta) \right] - \eta \leq \inf_{p\in A}\left[\sup_{p\in B} F_{pq}(t) \right].$$

Now ,taking the supremum on t < x of the above inequality,yields for any η,

$$f(x) \overset{df}{=} \sup_{t<x}\left(\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(t) \right]\right) \geqq \sup_{x<x}\left(\inf_{p\in A}\left[\sup_{\bar{q}\in \bar{B}} F_{p\bar{q}}(t-\eta) \right]\right) - \eta$$

$$= \sup_{t>x-\eta}\left(\inf_{q\in A}\left[\sup_{\bar{q}\in \bar{B}} F_{p\bar{q}}(t) \right]\right) - \eta \underline{\underline{df}} \; g(x-\eta) - \eta.$$

since both f and g are left-continuous and η is arbitrary, it follows that

f(x) $\geqq$ g(x). This together with (1.2.5), and the continuity of T yields.

$$F_{AB}(x) = T\left\{ \sup_{t<x}\left(\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(t) \right]\right), \sup_{t<x}\left(\inf_{q\in B}\left[\sup_{p\in A} F_{pq}(y) \right]\right)\right\}$$

$$\geq T\left\{ \sup_{t<x}\left(\inf_{p\in A}\left[\sup_{\bar{q}\in \bar{B}} F_{p\bar{q}}(t) \right]\right), \sup_{t<x}\left(\inf_{\bar{q}\in \bar{B}}\left[\sup_{p\in A} F_{p\bar{q}}(t) \right]\right)\right\}$$

$$= \sup_{t<x} T\left(\inf_{p\in A}\left[\sup_{\bar{q}\in \bar{B}} F_{p\bar{q}}(t) \right], \inf_{\bar{q}\in \bar{B}}\left[\sup_{p\in A} F_{p\bar{q}}(t) \right]\right) = F_{A\bar{B}}(x).$$

A similar argument shows that $F_{A\bar{B}} \geqq F_{AB}$.

Theorem.1.7. If A and B are nonempty subsets of S, then $F_{AB} = H$, if and only if $\overline{A} = \overline{B}$.

Proof. Suppose $F_{AB} = H$ and let $\varepsilon > 0$ be given. Then

$$1 = F_{AB}(\varepsilon) = T\left\{ \sup_{t<\varepsilon}\left(\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(t)\right]\right), \sup_{t<\varepsilon}\left(\inf_{q\in B}\left[\sup_{p\in A} F_{pq}(t)\right]\right)\right\}$$

$$= \sup_{t<\varepsilon}\left(\inf_{q\in B}\left[\sup_{p\in A} F_{pq}(t)\right]\right) = \inf_{q\varepsilon B}\left[\sup_{p\varepsilon A} F_{pq}(\varepsilon)\right].$$

So that for any $q \in B$ and every $\lambda > 0$ there exists a point p in A for which $F_{pq}(\varepsilon) > 1 - \lambda$. Consequently, q is an accumulation point of A and we have $B \subseteq \overline{A}$. A similar argument shows that $A \subseteq \overline{B}$. Conversely, suppose $\overline{A} = \overline{B}$.

Then in view of P_6 and Theorem 1.7

We have, $F_{AB} = F_{\overline{A}\,\overline{B}} = F_{\overline{A}\,\overline{A}} = H$.

Theorem.1.8. If A, B and C are nonempty subsets of S, then for any x and y $F_{AB}(x - y) \geq T(F_{AC}(x), F_{BC}(y))$.

Proof. Let u and v be given. Then for any triple of points p, q and r in S we have $F_{pq}(u+v) \geq T(F_{pr}(u), F_{qr}(v))$. Making use of the continuity and monotonicity of T we have the following inequality :

$$\sup_{q\in B} F_{pq}(u+v) \geq T\left(\sup_{r\varepsilon C} F_{pr}(u), \inf_{r\in C}\left[\sup_{q\in B} F_{qr}(v)\right]\right).$$

Consequently,

$$\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(u+v)\right] \geq T\left(\inf_{p\in A}\left[\sup_{r\in C} F_{pr}(u)\right], \inf_{r\in C}\left[\sup_{q\in B} F_{rq}(v)\right]\right).$$

Similarly,

$$\inf_{p\in B}\left[\sup_{q\in A} F_{pq}(u+v)\right] \geq T\left(\inf_{p\in C}\left[\sup_{r\in A} F_{pr}(u)\right], \inf_{r\in B}\left[\sup_{q\in C} F_{rq}(v)\right]\right)$$

Therefore, since T is associative,

we have

$$T\left(\inf_{p\in A}\left[\sup_{r\in B} F_{pq}(u+v)\right], \inf_{r\in B}\left[\sup_{q\in A} F_{pq}(u \_ v)\right]\right)$$

$$\geq T\left\{ T\left(\inf_{p\in A}\left[\sup_{r\in C} F_{pr}(u)\right], \inf_{r\in C}\left[\sup_{p\in A} F_{pr}(u)\right]\right),\right.$$

$$\square\ T\left.\left(\inf_{q\in B}\left[\sup_{r\in C} F_{qr}(v)\right], \inf_{r\in C}\left[\sup_{q\in B} F_{qr}(v)\right]\right)\right\}.$$

So, we have $F_{AB}(x+ y) = \sup_{u+v<x+y} T\left(\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(u+v)\right], \inf_{q\in B}\left[\sup_{p\in A} F_{pq}(u+v)\right]\right)$

$$\geq \sup_{\substack{u<x\\ v<y}} T\left(\inf_{p\in A}\left[\sup_{q\in B} F_{pq}(u+v)\right], \inf_{q\in B}\left[\sup_{q\in B} F_{pq}(u+v)\right]\right)$$

$$= T\left\{ \sup_{u<x} T\left(\inf_{p\in A}\left[\sup_{r\in C} F_{pr}(u)\right], \inf_{r\in C}\left[\sup_{p\in A} F_{pr}(u)\right]\right),\right.$$

$$\geq \sup_{v<y} T\left(\inf_{p\in B}\left[\sup_{r\in C} F_{qr}(u)\right], \inf_{r\in C}\left[\sup_{p\in B} F_{qr}(u)\right]\right)\Bigg\} = T(F_{AC}(x), F_{BC}(y)).$$

References:

[1] R. Egbert, Cartesian products of statistical metric spaces, Amer. Math. Soc. Notices 10 (1963), 266-267.

[2] V. Istratescu and I. Vaduva, Products of statistical metric spaces (Romanian), Acad. R. P. Romine Stud. Cerc. Mat. 12 (1961), 567-574.

[3] K. Menger, Statistical metrics, Proc. Nat. Acad. Of Science. U.S.A. 28 (1942), 535-537.

[4] K.Menger, Probabilistic geometry, Pro. Nat. Acad. Of Science. U.S.A. 37 (1951), 226-229.

[5] K.Menger, Geometrie generale (Chap. VII), Memorial des Sciences Mathematiques, No. 124, Paris 1954.

[6] B. Schweizer, Equivalence relations in probabilistic metric spaces, Bulletin of the Polytechnic Inst. Of Jassy 10 (1964), 67-70.

****