International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 4 Issue 3 || March. 2016 || PP-13-26

Necessary and Sufficient Conditions for Oscillations of Neutral Delay Difference Equations with Several Coefficients

A. Murugesan[1] , P. Sowmiya [2]
[1,2] *Department of Mathematics, Government Arts College (Autonomous),*
Salem-636 007, Tamil Nadu, India.

ABSTRACT *: In this paper, we discuss the oscillatory behavior of all solutions of the first order neutral delay difference equations with several positive and negative coefficients*

$$\Delta\left[x(n)+\sum_I p_i x(n-\tau_i)-\sum_J r_j x(n-\rho_j)\right]+\sum_K q_k x(n-\sigma_k)=0, \quad n\ge n_o, \qquad (*)$$

where I, J and K are initial segments of natural numbers, p_i, r_j, q_k are positive numbers, τ_i, ρ_j are positive integers and σ_k is a nonnegative integer for $i\in I$, $j\in J$ and $k\in K$. We establish a necessary and sufficient conditions for the oscillation of all solutions of (*) is that its characteristic equation

$$(\lambda-1)\left(1+\sum_I p_i\lambda^{-\tau_i}-\sum_J r_j\lambda^{-\rho_j}\right)+\sum_K q_k\lambda^{-\sigma_k}=0$$

has no positive roots

.

AMS Subject Classifications : 39A10, 39A12.
KEYWORDS : *Neutral, delay difference equation, oscillation, positive and negative coefficients.*

I. INTRODUCTION

Neutral delay difference equations are difference equations in which the highest order difference of the unknown function appears with the argument n (present state) as well as one or more retarded arguments (past histories). Recently some investigations have appeared which are concerned with the oscillation as well as the nonoscillation behavior of neutral delay difference equations, see, for example [3,7-9] . For further oscillation results on the oscillatory behavior of solutions of neutral difference equation we refer the readers to the monographs [1, 2, 4-6].

In this paper, we consider the following first order neutral delay difference equation with several positive and negative coefficients

$$\Delta\left[x(n)+\sum_I p_i x(n-\tau_i)-\sum_J r_i x(n-\rho_j)\right]+\sum_K q_k x(n-\sigma_k)=0, \quad n\ge n_o, \qquad (1.1)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, I, J and K are initial segments of natural numbers, p_i, r_j, q_k are positive real numbers, τ_i, ρ_j are positive integers and σ_k is a nonnegative integer for $i\in I$, $j\in J$ and $k\in K$. In the case where I = ϕ or J = ϕ or both I = ϕ and J = ϕ, difference equations of the following special forms are included respectively:

$$\Delta\left[x(n)-\sum_J r_j x(n-\rho_j)\right]+\sum_K q_k x(n-\sigma_k)=0, \qquad (1.2)$$

$$\Delta\left[x(n)+\sum_I p_i x(n-\tau_i)\right]+\sum_K q_R x(n-\sigma_k)=0, \qquad (1.3)$$

and

$$\Delta\, x(n)+\sum_K q_k x(n-\sigma_k)=0\ . \qquad (1.4)$$

Also in the case where $K = \phi$, the equation (1.1) yields

$$\Delta\left[x(n)+\sum_I p_i x(n-\tau_i)-\sum_J r_i x(n-\rho_j)\right]=0$$

and there exist a (nonoscillatory) solution $\{x(n)\}$ where $x(n) = c$, c a constant. We therefore assume that $k \neq \phi$.

Our aim in this paper is to establish a necessary and sufficient condition under which all solutions of (1.1) (and therefore all solutions of (1.2), (1.3), and (1.4)) oscillate. Indeed, we prove that every solution of (1.1) oscillates if and only if its characteristic equation

$$F(\lambda)\equiv(\lambda-1)\left(1+\sum_I p_i\lambda^{-\tau_i}-\sum_J r_j\lambda^{-\rho_j}\right)+\sum_K q_k\lambda^{-\sigma_k}=0 \tag{1.5}$$

has no positive roots. That is, the oscillatory character of the solutions is determined by the roots of the characteristic equation.

The problem of asymptotic and oscillatory behavior of solutions of neutral delay difference equations is of both theoretical and practical interest. The equations of this type appear in networks containing lossless transmission lines. Such networks, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

Let us choose a positive integer $n^* > \max\{\tau_i, \rho_j, \sigma_k\}$. By a solution of (1.1) on $N(n_o)=\{n_o, n_o+1, \dots\}$, we mean a real sequence $\{x(n)\}$ which is defined on $n \geq n_o - n^*$ and which satisfies (1.1) for $n \in N(n_o)$. As is customary, a solution $\{x(n)\}$ of (1.1) on $N(n_o)$ is said to be oscillatory if for every positive integer $N_o > n_o$ there exists $n \geq N_o$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

In the sequel all functional inequalities that we write are assumed to hold eventually, that is for all sufficiently large n.

In the case where $I \neq \phi$, $J \neq \phi$ we can assume without loss of generality that

$$I = \{1, 2, \dots, \alpha\},\quad J = \{1, 2, \dots, \beta\},\quad K = 1, 2, \dots, \gamma\},$$

$$0 < \tau_1 < \tau_2 < \dots < \tau_\alpha,\quad 0 < \rho_1 < \rho_2 < \dots < \rho_\beta,$$

and $0 \leq \sigma_1 < \sigma_2 < \dots \sigma_\gamma$, and $\tau_i \neq \rho_j$ for $i \in I$ and $j \in J$

Since otherwise the terms in the brackets of (1.1) can be abbreviated. Also for our convenience we use the following notations

$$I_1=\{i\in I:\tau_i<\rho_1\},\qquad I_2=\{i\in I:\tau_i\geq\rho_1\},$$

$$J_1=\{j\in J:\rho_j<\tau_\alpha\},\qquad J_2=\{j\in J:\rho_j\geq\tau_\alpha\},$$

$$K_1=\{k\in K:\sigma_k<\tau_\alpha\},\qquad K_2=\{k\in K:\sigma_k\geq\tau_\alpha\},$$

$$K_3=\{k\in K:\sigma_k>\rho_1\},\qquad K_4=\{k\in K:\sigma_k\geq\rho_1\},$$

and

$$P=\sum_I p_i,\quad P_1=\sum_{I_1} p_i,\quad R=\sum_J r_j,\quad R_1=\sum_{J_1} r_j,$$

$$Q=\sum_K q_k,\quad Q_1=\sum_{K_1} q_k,\quad Q_3=\sum_{K_3} q_k.$$

From the above definition it is clear that

$$I = I_1 \cup I_2, \qquad J = J_1 \cup J_2, \qquad K_1 \cup K_2 = K = K_3 \cup K_4 .$$

II. SOME USEFUL LEMMAS

In this section we establish some useful lemmas which will be used in the proof of our main theorem.

Lemma. 1. Let $\{x(n)\}$ be a nonoscillatory solution of (1.1). Then it also has a nonoscillatory solution $\{\omega(n)\}$ such that either

$$\omega(n) > 0,\ \Delta\omega(n) < 0,\ \Delta^2\omega(n) > 0, \text{ and } \lim_{n\to\infty}\omega(n) = \lim_{n\to\infty}\Delta\omega(n) = 0 \text{ (Class - I)}$$

or

$$\omega(n) > 0,\ \Delta\omega(n) > 0,\ \Delta^2\omega(n) > 0, \text{ and } \lim_{n\to\infty}\omega(n) = \lim_{n\to\infty}\Delta\omega(n) = \infty \text{ (Class - II).}$$

Proof. Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive solution of (1.1). Set

$$z(n) = x(n) + \sum_I p_i x(n-\tau_i) - \sum_J r_i x(n-\rho_j)$$

and

$$\omega(n) = z(n) + \sum_I p_i z(n-\tau_i) - \sum_J r_j z(n-\rho_j) .$$

We can easily show that $\{z(n)\}$and $\{\omega(n)\}$ are the solutions of (1.1), and they are eventually strictly monotone sequences. We have

$$\Delta z(n) = -\sum_K q_k x(n-\sigma_k) < 0, \tag{2.1}$$

$$\Delta\omega(n) = -\sum_K q_k z(n-\sigma_k), \tag{2.2}$$

and

$$\Delta^2\omega(n) = -\sum_K q_k \Delta z(n-\sigma_k) > 0 . \tag{2.3}$$

From (2.1) it follows that $\{z(n)\}$ is strictly decreasing and so either

$$\lim_{n\to\infty} z(n) = -\infty \tag{2.4}$$

or

$$\lim_{n\to\infty} z(n) = L \in R . \tag{2.5}$$

Assume that (2.4) holds. Then (2.2) leads to

$$\lim_{n\to\infty}\Delta\omega(n) = \infty$$

which implies that $\omega(n) > 0$ and $\lim_{n\to\infty}\omega(n) = \infty$, that is ,$\{\omega(n)\}$ belongs to Class II.

Next, assume that (2.5) holds. Then

$$\lim_{n\to\infty}\left(\Delta\left[z(n) + \sum_I p_i z(n-\tau_i) - \sum_J r_j z(n-\rho_j)\right]\right)$$

$$=\lim_{n\to\infty}\left(-\sum_{K} q_k z(n-\sigma_k)\right)=-QL.$$

First we will prove that L = 0. For otherwise

$$\lim_{n\to\infty}\left[z(n)+\sum_{I} p_i z(n-\tau_i)-\sum_{J} r_i z(n-\rho_j)\right]=\begin{cases}-\infty, & \text{if } L>0\\ +\infty, & \text{if } L<0\end{cases}$$

which contradicts (2.5). Thus we have established that $\Delta z(n)<0$ and $\lim_{n\to\infty} z(n)=0$. This implies that $z(n)>0$ and from (2.2) we see that $\Delta\omega(n)<0$. By the definition of $\omega(n)$ it follows that $\lim_{n\to\infty}\Delta\omega(n)=0$ which together with $\Delta\omega(n)<0$ imply that $\omega(n)>0$. Clearly, $\lim_{n\to\infty}\Delta\omega(n)=0$, for otherwise $\lim_{n\to\infty}\Delta\omega(n)<0$ which contradicts the fact that $\omega(n)>0$. Therefore in this case $\{\omega(n)\}$ belongs to class I. This completes the proof.

Lemma 2. If the characteristic equation (1.5) has no positive roots, then

$$\tau_\alpha<\max\{\rho_\beta,\sigma_\gamma\}. \tag{2.6}$$

Proof. Since $F(1)=Q>0$, $F(+\infty)=+\infty$ and (1.5) has no positive roots, it follows that $\lim_{\lambda\to\infty} F(\lambda)$ must be positive or $+\infty$. But if $\rho_\beta<\tau_\alpha$ and $\sigma_\gamma\le\tau_\alpha$, we have $\lim_{\lambda\to 0} F(\lambda)=-\infty$. Moreover by our assumption $\tau_\alpha\ne\rho_\beta$ and therefore $\tau_\alpha\ge\max\{\rho_\beta,\sigma_\gamma\}$ implies that (1.5) has no positive root. This is impossible and the proof is complete.

Lemma 3. (a) Let $\{x(n)\}\in$ Class I. Then there exists a solution $\{\omega(n)\}$ of (1.1) which belongs to Class I, such that the set

$$\wedge^{+}(\omega)=\left\{\lambda>1:\Delta\omega(n)+\left(1-\frac{1}{\lambda}\right)\omega(n)\le 0\right\}\ne\phi.$$

(b) Let $\{x(n)\}\in$ Class II. Then there exists a solution $\{\omega(n)\}$ of (1.1) which belongs to Class II, such that

$$\wedge^{-}(\omega)=\left\{\lambda>1:-\Delta\omega(n)+(\lambda-1)\omega(n)\le 0\right\}\ne\phi.$$

Proof. (a) Let $\{x(n)\}\in$ Class I. Set

$$\omega(n)=x(n)+\sum_{I} p_i x(n-\tau_i)-\sum_{J} r_j x(n-\rho_j)+\sum_{K} q_k\sum_{n-N}^{n-\sigma_k-1} x(s), \tag{2.7}$$

Where $N=\max_{i,j,k}\{\tau_i,\rho_j,\sigma_k\}$. It is easy to see that $\{\omega(n)\}$is a solution of (1.1) which belongs to Class I and that $\Delta\omega(n)=-\sum_{K} q_k x(n-N)$ or $\Delta\omega(n)+Qx(n-N)=0$. Now, Since $\{x(n)\}$ is positive and decreasing, (2.7) yields, $\omega(n)<x(n-N)+Px(n-N)+\sum_{K} q_k(N-\sigma_k)(n-N)\le(1+P+QN)x(n-N)$.

Thus

$$0=\Delta\omega(n)+Qx(n-N)\ge\Delta\omega(n)+\frac{Q}{1+P+QN}\omega(n)$$

$$=\Delta\omega(n)+\left(1+\frac{Q}{1+P+QN}-1\right)\omega(n)$$

$$\geq \Delta\omega(n) + \left(1 - \frac{1}{1 + \frac{Q}{1+P+QN}}\right)\omega(n)$$

which implies that $1 + \frac{Q}{1+P+QN} \in \wedge^{+}(\omega) \neq \phi$.

(b) Let $\{x(n)\} \in$ Class II. Set

$$\omega(n) = -x(n) - \sum_{I} p_i x(n-\tau_i) + \sum_{J} r_j x(n-\rho_j) + \sum_{K} q_k \sum_{n-\sigma_k}^{n-\eta-1} x(s), \tag{2.8}$$

where $\eta = \min_{i,j,k}\{\tau_i, \rho_j, \sigma_k\}$. It is easy to prove that $\{\omega(n)\}$is a solution of (1.1) which belongs to Class II and that $\Delta\omega(n) = \sum_{K} q_k(x-\eta)$ or $-\Delta\omega(n) + Q(x-\eta) = 0$. Now, since $\{x(n)\}$is positive and increasing, (2.8) yields.

$$\omega(n) < Rx(n-\eta) + \sum_{K} q_k(\sigma_k - \eta)x(n-\eta)$$

$$\leq (R + \sum_{K} q_k\sigma_k)x(n-\eta)$$

$$< (R + Q\sigma_{\gamma})x(n-\eta).$$

Thus,

$$0 = -\Delta\omega(n) + Qx(n-\eta) \geq -\Delta\omega(n) + \frac{Q}{R+Q\sigma_{\gamma}}\omega(n),$$

or

$$-\Delta\omega(n) + \left(1 + \frac{Q}{R+Q\sigma_{\gamma}} - 1\right)\omega(n) \leq 0$$

which implies that $1 + \frac{Q}{R+Q\sigma_{\gamma}} \in \wedge^{-}(\omega)$, that is , $\wedge^{-}(\omega) \neq \phi$. The proof of the lemma is complete.

Lemma 4. (a) Let $\{x(n)\} \in$ Class I for which the set $\wedge^{+}(x) \neq \phi$. If for a given $\omega > 0$ there exists M > 0 such that $x(n) > Mx(n-\omega)$, then the number $\lambda_0 > 1$ which satisfies $\lambda_0^{-\omega} = M$ is an upper bound of $\wedge^{+}(x)$.

(b) Let $x(n) \in$ Class II for which the set $\wedge^{-}(x) \neq \phi$. If for given $\omega > 0$ there exits $M > 0$ such that $x(n) < Mx(n-\omega)$, then the number $\lambda_0 > 1$ which satisfies $\lambda_0^{\omega} = M$ is an upper bound of $\wedge^{-}(x)$.

Proof. (a) Otherwise $\lambda_o \in \wedge^{+}(x)$ which means that $\Delta x(n) + \left(1 - \frac{1}{\lambda}\right)x(n) \leq 0$; that is, $\Delta\left(\lambda_0^{n} x(n)\right) \leq 0$ and therefore the sequence $\{\lambda_0^{n} x(n)\}$ is decreasing. Thus

$$\lambda_0^{n} x(n) \leq \lambda_0^{(n-\omega)} x(n-\omega)$$

or

$$x(n) \geq \lambda_0^{-\omega} x(n-\omega) = Mx(n-\omega),$$

which is a contradiction and the proof in case (a) is complete.

(b) Otherwise $\lambda_o \in \wedge^-(x)$. which implies that $-\Delta x(n) + (\lambda_0 - 1)x(n) \le 0$; that is, $\Delta\left(\lambda_0^{-n}x(n)\right) \ge 0$ and therefore the sequence $\{\lambda_0^{-n}x(n)\}$ is increasing. Thus

$$\lambda_0^{-n}x(n) \ge \lambda_0^{-(n-\omega)}x(n-\omega)$$

or

$$x(n) \ge \lambda_0^{\omega}x(n-\omega) = Mx(n-\omega),$$

which is a contradiction and the proof in case (b) is complete.

Lemma 5. Assume that $\tau_\alpha < \max\{\rho_\beta, \sigma_\gamma\}$. Then we have the following:

(a) Let $\{x(n)\} \in$ Class I for which $\wedge^+(x) \ne \phi$. Then the set $\wedge^+(x)$ has an upper bound which is independent of *x*.

(b) Let $\{x(n)\} \in$ Class II for which $\wedge^-(x) \ne \phi$. Then the set $\wedge^-(x)$ has an upper bound which is independent of *x*.

Proof. (a) Let $\{x(n)\} \in$ Class I. Set

$$z(n) = x(n) + \sum_I p_i x(n-\tau_i) + \sum_J r_j x(n-\rho_j) \tag{2.9}$$

which also belongs to Class I. Because of the assumption, we consider the following two cases:

Case 1. $\rho_\beta > \tau_\alpha$. Then, taking into account that $\{x(n)\} \in$ Class I, we have

$$0 < x(n) + \sum_I p_i x(n-\tau_i) - \sum_J r_j x(n-\rho_j)$$

$$< x(n+\tau_\alpha) + \sum_I p_i x(n-\tau_\alpha) - r_\beta x(n-\rho_\beta)$$

$$< (1+P)x(n-\tau_\alpha) - r_\beta x(n-\rho_\beta).$$

Thus,

$$x(n-\tau_\alpha) > \frac{r_\beta x(n-\rho_\beta)}{1+p} \quad \text{or} \quad x(n) > \frac{r_\beta}{1+P}x\left[n-(\rho_\beta-\tau_\alpha)\right].$$

In this case, by Lemma 5(a), the positive number

$$\lambda_1 \equiv \exp\left\{\frac{1}{\rho_\beta-\tau_\alpha}\ln\left(\frac{1+P}{r_\beta}\right)\right\} \tag{2.10}$$

is an upper bound of $\wedge^+(x)$.

Case 2. $\sigma_\gamma > \tau_\alpha$. From (1.1) and (2.9), we have

$$\Delta z(n) + \sum_K q_k x(n-\sigma_k) = 0$$

or

$$\Delta z(n) + q_\gamma x(n-\sigma_\gamma) \le 0.$$

Summing the last inequality from $n-\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]$ to n–1 where [·] denotes the greatest integer function, we have,

$$z\left(n-\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]\right)>q_\gamma\left[\frac{\sigma_\gamma-z_\alpha}{2}\right]x(n-\sigma_\gamma),$$

or

$$z(n)>q_\gamma\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]x\left(n-\sigma_\gamma+\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]\right).$$

But also from (2.9) we have

$$x(n-\tau_\alpha)>\frac{z(n)}{1+P}$$

and combining the last two inequalities, we get

$$x(n-\tau_\alpha)>\frac{q_\gamma}{1+P}\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]x\left(n-\sigma_\gamma+\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]\right),$$

or

$$x(n)>\frac{q_\gamma}{1+P}\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]x\left(n-\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]\right).$$

Put $M=\frac{q_\gamma}{1+P}\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]$ and $\omega=\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]$.

We have $x(n)>Mx(n-\omega)$. As before, the positive number λ_2 where $\lambda_0^{-\omega}=M$ is an upper bound for $\wedge^+(\omega)$.

That is ,

$$\lambda_2\equiv\exp\left\{\frac{1}{\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]}\ln\left(\frac{1+P}{q_\gamma\left[\frac{\sigma_\gamma-\tau_\alpha}{2}\right]}\right)\right\}\tag{2.11}$$

is an upper bound for $\wedge^+(x)$.

(b) Let $\{x(n)\}\in$ Class II. Set

$$z(n)=-x(n)-\sum_I p_i x(n-\tau_i)+\sum_J r_j x(n-\rho_j)\tag{2.12}$$

which also belongs to Class II. Since z(n) > 0, we have

$$x(n)+\sum_I p_i x(n-\tau_i)<\sum_J r_j x(n-\rho_j)$$

and, taking into account that $\{x(n)\}$ is positive and increasing, the last inequality yields

$$x(n) < \sum_J r_j x(n-\rho_1) = Rx(n-\rho_1)$$

Then, by Lemma 4(b), the positive number

$$\lambda_3 \equiv \exp\left(\frac{1}{\rho_1}\ln R\right)$$

is an upper bound for $\wedge^-(x)$.

Remark. In the cases of equation (1.2), (1.3) and (1.4) the above Lemmas and also the main Theorem are modified appropriately. For example, in the case of equations (1.3) and (1.4) Class II is empty and therefore case (b) does not appear in Lemmas 3, 4, 5 and in the proof the following main theorem. On the other hand the proofs remain the same by modifying appropriately (2.7), (2.8), (2.9) and (2.12), (3.1) and (3.12).

III. MAIN RESULT

Our main result is the following:

Theorem. The necessary and sufficient condition for the oscillation of all solutions of (1.1) is that its characteristic equation (1.5) has no positive roots.

Proof. The theorem will be proved in the contrapositive form: there is a nonoscillatory solution of (1.1) if and only if the characteristic equation (1.5) has a positive root. Assume first that (1.5) has a positive root λ. Then (1.1) has the nonoscillatory solution $\{x(n)\}$ where $x(n)=\lambda^n$.

Assume, conversely, that there is a nonoscillatory solution of (1.1) and, for the sake of contradiction, that the equation (1.5) has no positive root. Then by Lemma 1, it also has a nonoscillatory solution $\{x(n)\}$ which belongs to either to Class I or to Class II. We consider the following cases.

(i) *The case $\{x(n)\} \in$ Class I.* For this solution $\{x(n)\}$, by Lemma 3(a), we can assume without loss of generality that $\wedge^+(x) \neq \phi$. Let $\lambda_4 \in \wedge^+(x)$. Also, by Lemma 5(a), there is a positive number, $\lambda_o > 1$, such that $\wedge^+(\omega)$ is bounded above by λ_o.

Let $\lambda \in \wedge^+(x)$ and consider the sequence $\{u(n)\}$ where

$$u(n) \equiv Tx(n) = x(n) + \sum_I p_i x(n-\tau_i) - \sum_J r_j x(n-\rho_j)$$

$$+ \sum_{K_1} q_k \sum_{s=n-\tau_\alpha}^{\infty} x(s) + \left(1-\frac{1}{\lambda}\right)\sum_{J_1} r_j \sum_{s=n-\tau_\alpha}^{\infty} x(s). \tag{3.1}$$

We can easily show that $\{u(n)\}$ is a solution of (1.1) and it can be easily seen that $\{u(n)\} \in$ Class I.

Since (1.5) has no positive root and $F(0) = F(+\infty) = +\infty$ it follows that

$$m = \min_{\lambda \in (0,\infty)} F(\lambda) > 0.$$

That is, for all $\lambda \in (0, \infty)$,

$$(\lambda - 1)\left(1 + \sum_I p_i \lambda^{-\tau_i} - \sum_J r_j \lambda^{-\rho_j}\right) + \sum_K q_k \lambda^{-\sigma_k} \geq m \tag{3.2}$$

and replacing λ by $\frac{1}{\lambda}$, we get

$$\left(\frac{1}{\lambda}-1\right)\left(1+\sum_{I} p_i\lambda^{\tau_i}-\sum_{J} r_j\lambda^{\rho_j}\right)+\sum_{K} q_k\lambda^{\sigma_k} \ge m$$

or

$$\left(1-\frac{1}{\lambda}\right)\left(1+\sum_{I} p_i\lambda^{\tau_i}-\sum_{J} r_j\lambda^{\rho_j}\right)-\sum_{K} q_k\lambda^{\sigma_k} \le -m. \tag{3.3}$$

We will show that $\lambda(1+\mu)\in\wedge^{+}(u)$ where

$$\mu=\frac{m}{1+\lambda_0^{\tau_\alpha}\left(P+R_1+\dfrac{Q_1}{1-\dfrac{1}{\lambda_4}}\right)}.$$

To this end it sufficies to show that

$$\Delta u(n)+\left(1-\frac{1}{\lambda(1+\mu)}\right)u(n)\le 0.$$

Define $\phi(n)=\lambda^{n}x(n)$. Then

$$\Delta\phi(n)=\lambda^{n+1}\left[\Delta x(n)+\left(1-\frac{1}{\lambda}\right)x(n)\right]\le 0$$

and therefore $\{\phi(n)\}$ is decreasing sequence. From (1.1) and (3.1), we have

$$\Delta u(n)=-\sum_{K} q_k x(n-\sigma_k)-\sum_{K_1} q_k x(n-\tau_\alpha)-\left(1-\frac{1}{\lambda}\right)\sum_{J_1} r_j x(n-\tau_\alpha)$$

$$=-\sum_{K_1} q_k x(n-\sigma_k)-\sum_{K_2} q_k x(n-\sigma_k)-\sum_{K_1} Q_1 x(n-\tau_\alpha)-\left(1-\frac{1}{\lambda}\right)R_1 x(n-\tau_\alpha). \tag{3.4}$$

Also from (3.1), we have

$$u(n)=x(n)+\sum_{I} p_i x(n-\tau_i)-\sum_{J_1} r_j x(n-\rho_j)$$

$$-\sum_{J_2} r_j x(n-\rho_j)-\sum_{K_1} q_k\sum_{s=n-\tau_\alpha}^{\infty} x(s)+\left(1-\frac{1}{\lambda}\right)\sum_{J_1} r_j\sum_{s=n-\tau_\alpha}^{\infty} x(s). \tag{3.5}$$

Now for $\lambda\in\wedge^{+}(x)$ with $\lambda\ge\lambda_4$ and taking into account (3.4), (3.5) and the fact that $x(n)=\lambda^{-n}\phi(n)$, we have,

$$\Delta u(n)+\left(1-\frac{1}{\lambda(1+\mu)}\right)u(n)$$

$$\le \Delta u(n)+(1+\mu)\left(1-\frac{1}{\lambda(1+\mu)}\right)u(n)$$

$$=\Delta u(n)+\left(1-\frac{1}{\lambda}+\mu\right)u(n)$$

$$= \lambda^{-n}\left\{\left[-\sum_{K_1} q_k\lambda^{\sigma_k}\phi(n-\sigma_k)-\sum_{K_2} q_k\lambda^{\sigma_k}\phi(n-\sigma_k)\ -Q_1\lambda^{\tau_\alpha}\phi(n-\tau_\alpha)-\left(1-\frac{1}{\lambda}\right)R_1\lambda^{\tau_\alpha}\phi(n-\tau_\alpha)\right]\right.$$

$$\left.+\left(1-\frac{1}{\lambda}+\mu\right)\left[\phi(n)+\sum_{I} p_i\lambda^{\tau_i}\phi(n-\tau_i)\ -\sum_{J_1} r_j\lambda^{\rho_j}\phi(n-\rho_j)-\sum_{J_2} r_j\lambda^{\rho_j}\phi(n-\rho_j)\right]\right\}$$

$$+\left(1-\frac{1}{\lambda}+\mu\right)\left[\sum_{K_1} q_k\sum_{s=n-\tau_\alpha}^{\infty}\lambda^{-s}\phi(s)\ +\left(1-\frac{1}{\lambda}\right)\sum_{J_1} r_j\sum_{s=n-\tau_\alpha}^{\infty}\lambda^{-s}\phi(s)\right].$$

Using the fact that $\{Q(n)\}$ is decreasing and taking into account the definition of K_1, K_2, J_1, J_2, we obtain

$$\Delta u(n)+\left(1-\frac{1}{\lambda(1+\mu)}\right)u(n)$$

$$\le \lambda^{-n}\phi(n-\tau_\alpha)\left[-\sum_{K_2} q_k\lambda^{\sigma_k}-Q_1\lambda^{\tau_\alpha}-\left(1-\frac{1}{\lambda}\right)R_1\lambda^{\tau_\alpha}\right.$$

$$+\left(1-\frac{1}{\lambda}+\mu\right)+\left(1-\frac{1}{\lambda}+\mu\right)\sum_{I} p_i\lambda^{\tau_i}$$

$$\left.-\left(1-\frac{1}{\lambda}+\mu\right)\sum_{J_2} r_j\lambda^{\rho_j}\right]$$

$$+\left[-\lambda^{-n}\sum_{K_1} q_k\lambda^{\sigma_k}\phi(n-\sigma_k)+\left(1-\frac{1}{\lambda}\right)\sum_{K_1} q_k\sum_{s=n-\tau_\alpha}^{\infty}\lambda^{-s}\phi(s)\right] \tag{3.6}$$

$$+\left(1-\frac{1}{\lambda}+\mu\right)\left[-\lambda^{-n}\sum_{J_1} r_J\lambda^{\rho_j}\phi(n-\sigma_j)+\left(1-\frac{1}{\lambda}\right)\sum_{J_1} r_J\sum_{s=n-\tau_\alpha}^{\infty}\lambda^{-s}\phi(s)\right] \tag{3.7}$$

$$+\mu\sum_{K_1} q_k\sum_{s=n-\tau_\alpha}^{\infty}\lambda^{-s}\phi(s). \tag{3.8}$$

Summing the inequality $\Delta x(n)+\left(1-\frac{1}{\lambda}\right)x(n)\le 0$ from $n-\sigma_k$ to ∞ and taking into account that $\lim_{n\to\infty} x(n)=0$, we have

$$-x(n-\sigma_k)+\left(1-\frac{1}{\lambda}\right)\sum_{s=n-\sigma_k}^{\infty} x(s)\le 0.$$

Thus

$$-\sum_{K_1} q_k\lambda^{-n+\sigma_k}\phi(n-\sigma_k)+\left(1-\frac{1}{\lambda}\right)\sum_{K_1} q_k\sum_{s=n-\sigma_k}^{\infty}\lambda^{-s}\phi(s)\le 0$$

and therefore the quantity in (3.6), call it A_1, we obtain

$$A_1\le\left(1-\frac{1}{\lambda}\right)\sum_{K_1} q_k\sum_{s=n-\tau_\alpha}^{n-\sigma_{k-1}}\lambda^{-s}\phi(s)$$

$$\le \left(1-\frac{1}{\lambda}\right)\phi(n-\tau_\alpha)\sum_{K_1} q_k \sum_{s=n-\tau_\alpha}^{n-\sigma_{k-1}} \lambda^{-s}$$

$$= \lambda^{-n}\phi(n-\tau_\alpha)\left[Q_1\lambda^{\tau_\alpha} - \sum_{K_1} q_k\lambda^{\sigma_k}\right]. \tag{3.9}$$

Similarly, for the quantity inside the brackets in (3.7), call it A_2, we have

$$A_2 \le \left(1-\frac{1}{\lambda}\right)\sum_{J_1} r_j \sum_{s=n-\tau_\alpha}^{n-\rho_{j-1}} \lambda^{-s}\phi(s)$$

$$\le \lambda^{-n}\phi(n-\tau_\alpha)\sum_{J_1} r_j\left(\lambda^{\tau_\alpha}-\lambda^{\rho_j}\right)$$

$$= \lambda^{-n}\phi(n-\tau_\alpha)\left[R_1\lambda^{\tau_\alpha} - \sum_{J_1} r_j\lambda^{\rho_j}\right]. \tag{3.10}$$

Also in (3.8)

$$\sum_{s=n-\tau_\alpha}^{\infty} \lambda^{-s}\phi(s) \le \phi(n-\tau_\alpha)\frac{\lambda^{-n+\tau_\alpha}}{\left(1-\frac{1}{\lambda}\right)} = \lambda^{-n}\phi(n-\tau_\alpha)\frac{\lambda^{\tau_\alpha}}{\left(1-\frac{1}{\lambda}\right)}. \tag{3.11}$$

Using the above inequalities (3.9), (3.10), and (3.11), we have

$$\Delta u(n) + \left(1-\frac{1}{\lambda(1+\mu)}\right)u(n) \le \lambda^{-n}\phi(n-\tau_\alpha)\left\{\left(1-\frac{1}{\lambda}\right)\left[1+\sum_{I} p_i\lambda^{\tau_i} - \left(\sum_{J_1} r_j\lambda^{\rho_j} + \sum_{J_2} r_j\lambda^{\rho_j}\right)\right]\right.$$

$$\left. -\left(\sum_{K_1} q_k\lambda^{\sigma_k} + \sum_{K_2} q_k\lambda^{\sigma_k}\right) + \mu\left[1+\sum_{I} p_i\lambda^{\tau_i} - \left(\sum_{J_1} r_j\lambda^{\rho_j} + \sum_{J_2} r_j\lambda^{\rho_j}\right) + R_1\lambda^{\tau_\alpha} + Q_1\frac{\lambda^{\tau_\alpha}}{1-\frac{1}{\lambda}}\right]\right\}$$

$$\le \lambda^{-n}\phi(n-\tau_\alpha)\left\{-m + \mu\left(1+P\lambda^{\tau_\alpha} + R_1\lambda^{\tau_\alpha} + Q_1\frac{\lambda^{\tau_\alpha}}{1-\frac{1}{\lambda}}\right)\right\}$$

$$\le \lambda^{-n}\phi(n-\tau_\alpha)\left\{-m + \mu\left(1+\lambda_0^{\tau_\alpha}\left(P+R_1+\frac{Q_1}{1-\frac{1}{\lambda_4}}\right)\right)\right\}$$

$$= 0$$

which implies that $\lambda(1+\mu) \in \wedge^+(u)$. Now set $x_0 \equiv x, x_1 \equiv Tx_0 \equiv u, x_2 \equiv Tu_1$, and in general $x_m = Tx_{m-1},\ m = 1,2,\ldots$ and observe that for $\lambda \in \wedge^+(x) \equiv \wedge^+(x_o) \Rightarrow \lambda(1+\mu) \in \wedge^+(u) \equiv \wedge^+(x_1)$ and after m steps $\lambda(1+\mu)^m \in \wedge^+(x_m),\ m = 1,2,\ldots$, which is a contradiction since λ_o is common upper bound for all $\wedge^+(x_n)$. This completes the proof in case (i).

(ii) *The case {x(n)}∈ Class II.* By Lemma 3(b) we can assume that $\wedge^{-}(x) \neq \phi$. Let $\lambda_5 \in \wedge^{-}(x)$. Also by Lemma 5(b), there is a positive number λ_o>1. Such that $\wedge^{-}(x)$ is bounded above by λ_o. Let $\lambda \in \wedge^{-}(x)$ and consider the sequence {v(n)} defined by

$$v(n) = -x(n) - \sum_{I} p_i x(n-\tau_i) + \sum_{J} r_i x(n-\rho_j)$$

$$+\sum_{K_3} q_k \sum_{s=n_o}^{n-\rho_i-1} x(s) + (\lambda-1)\sum_{I_1} p_i \sum_{s=n_o}^{n-\rho_i-1} x(s) + C, \tag{3.12}$$

where $C = \left(\frac{1}{Q}\right)(Q_3 + (\lambda-1)P_1)\left[-x(n_o) - \sum_{I} p_i x(n_0-\tau_i) + \sum_{J} r_i x(n_o-\rho_j)\right].$

We can easily show that $\{v(n)\}$is a solution of (1.1) which belongs to Class II.

As in case (i), $m = \min_{\lambda \in (0,\infty)} F(\lambda) > 0.$ We will show that $\lambda + \mu \in \wedge^{-}(x)$ where $\mu = \dfrac{m}{2\left(R + P_1 + \dfrac{Q_3}{\lambda_s - 1}\right)}$.

It suffices to show that $-\Delta v(n) + (\lambda+\mu-1)v(n) \leq 0$. Define the sequence {φ(n)} by φ(n)=$\lambda^{-n}x(n)$. Then $\Delta\varphi(n) = \lambda^{-n-1}(\Delta x(n) - (\lambda-1)x(n)) \geq 0$ and therefore {φ(n)} is increasing sequence. From (1.1) and (3.12), we obtain

$$-\Delta v(n) = -\sum_{K} q_k x(n-\sigma_k) - \sum_{K_3} q_k x(n-\rho_1) - (\lambda-1)\sum_{I} p_i x(n-\rho_1)$$

$$= -\sum_{K_3} q_k x(n-\sigma_k) - \sum_{K_4} q_k x(n-\sigma_k) - Q_3 x(n-p_1) - (\lambda-1)P_1 x(n-\rho_1). \tag{3.13}$$

Also, from (3.12), we have

$$v(n) = -x(n) - \sum_{I_1} p_i x(n-\tau_i) - \sum_{I_2} p_i x(n-\tau_i) + \sum_{J} r_j x(n-\rho_j) + \sum_{K_3} q_k \sum_{s=n_o}^{n-\rho_1-1} x(s)$$

$$+(\lambda-1)\sum_{I_1} p_i \sum_{s-n_o}^{n-\rho_1-1} x(s) + C. \tag{3.l4}$$

Now for $\lambda \in \wedge^{-}(x)$ with $\lambda \geq \lambda_5$ and taking into account (3.13) and (3.14), and the fact that $x(n) = \lambda^n \varphi(n)$, we have

$$-\Delta v(n) + (\lambda+\mu-1)v(n)$$

$$= \lambda^n \left\{\left[-\sum_{K_3} q_k \lambda^{-\sigma_k}\varphi(n-\sigma_k) - \sum_{K_4} q_k \lambda^{-\sigma_k}\varphi(n-\sigma_k) - Q_3\lambda^{-\rho_1}\varphi(n-P_1) - (\lambda-1)P_1\lambda^{-\rho_1}\right]\right.$$

$$+(\lambda+\mu-1)\left[-\varphi(n) - \sum_{I_1} p_i\lambda^{-\tau_i}\varphi(n-\tau_i)\right.$$

$$\left. - \sum_{I_2} p_i\lambda^{-\tau_i}\varphi(n-\tau_i) + \sum_{J} r_j\lambda^{-\rho_j}\varphi(n-\rho_j)\right]$$

$$+(\lambda+\mu-1)\left[\sum_{K_3} q_k \sum_{s=n_o}^{n-\rho_1-1} \lambda^s \varphi(s) + (\lambda-1)\sum_{I_1} p_i \sum_{s=n_o}^{n-\rho_1-1} \lambda^s \varphi(s) + C\right]\Bigg\}.$$

Using the fact that {φ(n} is increasing and taking into account the definition of K_3, K_4, I_1, I_2, we obtain

$$-\Delta v(n) + (\lambda+\mu-1)v(n)$$

$$\leq \lambda^n \varphi(n-\rho_1) - \left[\sum_{K_4} q_k \lambda^{-\sigma_k} Q_3 \lambda^{-\rho_1} - (\lambda-1)P_1\lambda^{-\rho_1} + (\lambda+\mu-1)\left(-1-\sum_{I_2} p_i \lambda^{-\tau_i} + \sum_J r_j \lambda^{-\rho_j}\right)\right]$$

$$+\left[-\lambda^n \sum_{K_3} q_k \lambda^{-\sigma_k}\varphi(n-\sigma_k) + (\lambda-1)\sum_{K_3} q_k \sum_{s=n_o}^{n-\rho_1-1}\lambda^s\varphi(s)\right] \tag{3.15}$$

$$+(\lambda+\mu-1)\left[-\lambda^n \sum_{I_1} p_i \lambda^{-\tau_i}\varphi(n-\tau_i) + (\lambda-1)\sum_{I_1} p_i \sum_{s=n_o}^{n-\rho_1-1}\lambda^s\varphi(s)\right] \tag{3.16}$$

$$+\mu\sum_{K_3} q_k \sum_{s=n_o}^{n-\rho_1-1}\lambda^s\varphi(s) + (\lambda+\mu-1)C. \tag{3.17}$$

Summing the inequality $-\Delta x(n) + (\lambda-1)x(n) \leq 0$ from n_o to $n-\sigma_k-1$ and taking into account that $x(n) > 0$, we have

$$-x(n-\sigma_k) + (\lambda-1)\sum_{s=n_o}^{n-\sigma_k-1} x(s) \leq 0.$$

Thus

$$-\sum_{K_1} q_k \lambda^{n-\sigma_k}\varphi(n-\sigma_k) + (\lambda-1)\sum_{K_3} q_k \sum_{s=n_o}^{n-\sigma_k-1}\lambda^s\varphi(s) \leq 0$$

and therefore for the quantity in (3.15), call it B_1, we have

$$B_1 \leq (\lambda-1)\sum_{K_3} q_k \sum_{s=n-\sigma_k}^{n-\rho_1-1}\lambda^s\varphi(s)$$

$$\leq (\lambda-1)\varphi(n-\rho_1)\sum_{K_3} q_k \sum_{s=n-\sigma_k}^{n-\rho_1-1}\lambda^s$$

$$= \lambda^n\varphi(n-\rho_1)\left(Q_1\lambda^{-\rho_1} - \sum_{K_3} q_k\lambda^{-\sigma_k}\right). \tag{3.18}$$

Similarly, for the quantity inside the brackets in (3.16), call it B_2, we have,

$$B_2 \leq (\lambda-1)\sum_{I_1} p_i \sum_{s=n-\tau_i}^{n-\rho_1-1}\lambda^s\varphi(s)$$

$$\leq (\lambda-1)\varphi(n-\rho_1)\sum_{I_1} p_i \sum_{s=n-\tau_i}^{n-\rho_1-1}\lambda^s$$

$$= \lambda^n \varphi(n - p_1)\left(P_1\lambda^{-\rho_1} - \sum_{I_1} p_i\lambda^{-\tau_i} \right). \tag{3.19}$$

Also in (3.17)

$$\sum_{s=n_o}^{n-\rho_1-1} \lambda^s \varphi(s) \le \varphi(n-\rho_1) \sum_{s=n_o}^{n-\rho_1-1} \lambda^s = \lambda^s \varphi(n-\rho_1) \frac{\lambda^{-\rho_1}}{\lambda - 1}. \tag{3.20}$$

Using the above inequalities (3.18), (3.19) ,and (3.20), we have

$$-\Delta v(n) + (\lambda - 1)v(n)$$

$$= \lambda^n \varphi(n-\rho_1)\left\{ (\lambda - 1)\left[-1 - \left(\sum_{I_1} p_i^{-\tau_i}\lambda + \sum_{I_2} p_i\lambda^{-\tau_i} \right) + \sum_J r_j\lambda^{-\rho_j} \right] - \left(\sum_{K_3} q_k\lambda^{-\sigma_k} + \sum_{K_4} q_k\lambda^{-\sigma_k} \right) \right.$$

$$\left. + \mu\left[-1 - \left(\sum_{I_1} p_i\lambda^{-\tau_i} + \sum_{I_2} p_i\lambda^{-\tau_i} \right) + \sum_J r_j\lambda^{-\rho_j} + P_i\lambda^{-\rho_1} + \frac{Q_3\lambda^{-\rho_1}}{\lambda - 1} \right] \right\} + (\lambda + \mu - 1)C.$$

and, inview of (3.2)

$$-\Delta v(n) + (\lambda + \mu - 1)v(n)$$

$$\le \lambda^n \varphi(n-\rho_1)\left[-m + \mu\left(\sum_J r_j\lambda^{-\rho_j} + P_1\lambda^{-\rho_1} + \frac{Q_3\lambda^{-\rho_1}}{\lambda - 1} \right) \right] + (\lambda - \mu - 1)C$$

$$= \lambda^{\rho_1} x(n-\rho_1)\left[-m + \mu\left(R + P_1 + \frac{Q_3}{\lambda - 1} \right)\lambda^{-\rho_1} + (\lambda + \mu - 1)\frac{C\lambda^{-\rho_1}}{x(n-\rho_1)} \right].$$

As $\lim_{n\to\infty} x(n) = \infty$, it follows that for *n* sufficiently large

$$-m + (\lambda + \mu - 1)\lambda^{-\rho_1} \frac{C}{x(n-\rho_1)} \le \frac{-m}{2}.$$

Thus

$$-\Delta v(n) + (\lambda + \mu - 1)v(n) \le \lambda^{\rho_1} x(n-\rho_1)\left[\frac{-m}{2} + \mu(R + P_1 + \frac{Q_3}{\lambda_s^{-1}} \right] = 0$$

which implies that $\lambda + \mu \in \wedge^{-}(v)$ and, as in case (i), we are lead to a contradiction.

This proof of the theorem is complete.

REFERENCES

[1] R.P. Agarwal, Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 1999.

[2] R.P. Agarwal and P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer, Dodrecht, 1997.

[3] M.P. Chen, B.S. Lalli and J.S. Yu, Oscillation in neutral delay difference equations with variable coefficients, Comput. Math. Appl. 29(1995), 5-11.

[4] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[5] A. Murugesan and P. Sowmiya, Necessary and sufficient conditions for oscillations of first order neutral delay difference equations with constant coefficients, International Journal of Advanced Mathematical Sciences, 13(2015), 12-24.

[6] A. Murugesan and P. Sowmiya, Necessary and sufficient conditions for oscillations of first order neutral delay difference equations, Global Journal of Mathematical Analysis, 3(2015), 61-72.

[7] Ö. Öcalan and O. Duman, Oscillation analysis of neutral difference equations with delays, Chaos Solitons Fractals, 39(2009), 261-270.

[8] X.H. Tang and X. Lin, Necessary and sufficient conditions for oscillation of first-order nonlinear neutral difference equations, Comput. Math. Appl. 55 (2008), 1279-1292.

[9] E. Thandapani, R. Arul and P.S. Raja, Oscillation of first order neutral delay difference equations, Appl. Math. E-Notes, 3(2003), 88-94.