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# A Study of Some Systems of Linear and Nonlinear Partial Differential Equations (Pdes) Using Reduced Differential Transform Method

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***ABSTRACT:*** *In this paper, we introduce the solution of systems of linear and nonlinear partial differential equations subject to the initial conditions by using reduced differential transformation method. The proposed method was applied to three systems of linear and nonlinear partial differential equations, leading to series solutions with components easily computable. The results obtained are indicators of the simplicity and effectiveness of the method.*
***KEYWORDS:*** *Linear and Nonlinear PDEs, Reduced Differential Transform Method, System of Equations.*

## I. INTRODUCTION

Partial Differential Equations (PDEs) have numerous applications in various fields of science and engineering such as fluid mechanic, thermodynamic, heat transfer and physics [1].
Systems of PDEs, linear or nonlinear have attracted much concern in studying evolution equations that describe wave propagation, in investigating shallow water waves and in examining the chemical reaction-diffusion model of Brusselator. The general ideas and the essential features of these systems are of wide applicability.
The commonly used methods are the method of characteristics and the Riemann invariants among other methods. The existing techniques encountered some difficulties in terms of the size of computational work needed especially when the system involves several partial differential equations [2].
To avoid the difficulties that usually arise from traditional strategies, the reduced differential transform method [3] form a reasonable basis for studying systems of partial differential equations.
The method, as would be seen later, has a useful attraction in that solution is presented in a rapidly convergent power series with easily computable components.

## II. BASIC IDEAS OF REDUCED DIFFERENTIAL TRANSFORM METHOD

Suppose that $u(x,t)$ is a function of two variables which is analytic and $k-times$ continuously differentiable with respect to time $t$ and space $x$ in our domain of interest.
Assume we can represent this function as a product of two single variable functions $u(x,t)= f(x)g(t)$.
From the definitions of differential transform method, the function can be represented as

$$u(x,t)=\sum_{i=0}^{\infty}F(i)x^i.\sum_{j=0}^{\infty}G(j)t^j=\sum_{k=0}^{\infty}U_k(x)t^k \qquad (2.1)$$

Where $U_k(x)$ is the transformed function, which can be defined as

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \qquad (2.2)$$

Thus from equations $(2.1)$ and $(2.2)$, we can deduce

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k \qquad (2.3)$$

Considering equations $(2.1)-(2.3)$, it is clear that the concept of the RDTM is derived from the power series expansion.
The summary of the fundamental transformation properties of RDTM are shown in the table below:

**Table 1:** Basic transformations of RDTM

| *Functional form* | *Transformed form* |
|---|---|
| $u(x,t)$ | $U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$ |
| $w(x,t)=u(x,t)\pm v(x,t)$ | $W_k(x)=U_k(x)\pm V_k(x)$ |
| $w(x,t)=\lambda u(x,t)$ | $W_k(x)=\lambda U_k(x)$ |
| $w(x,t)=u(x,t)v(x,t)$ | $W_k(x)=\sum_{i=0}^{k}U_i(x)V_{k-i}(x)=\sum_{i=0}^{k}V_i(x)U_{k-i}(x)$ |
| $w(x,t)=p(x,t)q(x,t)r(x,t)$ | $W_k(x)=\sum_{i=0}^{k}\sum_{j=0}^{i}P_j(x)Q_{i-j}(x)R_{k-i}(x)$ |
| $w(x,t)=x^m t^n$ | $W_k(x)=x^m\delta(k-n), \delta(k-n)=\begin{cases}1, k=n\\0, k\neq n.\end{cases}$ |
| $w(x,t)=x^m t^n u(x,t)$ | $W_k(x)=x^m U_{k-n}(x)$ |
| $w(x,t)=\frac{\partial^n}{\partial t^n}u(x,t)$ | $W_k(x)=\frac{(k+n)!}{k!}U_{k+n}(x)$ |
| $w(x,t)=\frac{\partial^n}{\partial x^n}u(x,t)$ | $W_k(x)=\frac{\partial^n}{\partial x^n}U_k(x)$ |
| $w(x,t)=\frac{\partial^{n+m}}{\partial x^n \partial t^m}u(x,t)$ | $W_k(x)=\frac{\partial^n}{\partial x^n}\frac{(k+m)!}{k!}U_{k+m}(x)$ |

## III. APPLICATIONS

In this section, we apply the RDTM to three numerical examples of system of linear and nonlinear partial differential equations to show the efficiency of the method.

**Example 3.1**: Consider the non-homogenous linear system of partial differential equations

$$\begin{aligned}u_t - v_x - u + v &= -2\\ v_t + u_x - u + v &= -2\end{aligned} \tag{3.1}$$

Subject to the initial conditions

$$\begin{aligned}u(x,0) &= 1+e^x\\ v(x,0) &= -1+e^x\end{aligned} \tag{3.2}$$

Where the exact solutions are [4]

$$\begin{aligned}u(x,t) &= 1+e^{x+t}\\ v(x,t) &= -1+e^{x-t}\end{aligned} \tag{3.3}$$

Applying the basic properties of the RDTM to (3.1) and (3.2) we obtain the following recursive relations:

$$\begin{aligned}U_{k+1}(x) &= \frac{1}{k+1}\left(\frac{\partial}{\partial x}V_k(x)+N_k(x)\right)\\ V_{k+1}(x) &= \frac{1}{k+1}\left(-\frac{\partial}{\partial x}U_k(x)+N_k(x)\right)\end{aligned} \tag{3.4}$$

Where $N_k(x)$ is the transformed form of $u-v-2$ . The first few nonlinear terms are:

$$N_0 = u_0 - v_0 - 2$$
$$N_1 = u_1 - v_1$$
$$N_2 = u_2 - v_2, \ldots$$

And

$$u_0 = 1 + e^x \qquad v_0 = -1 + e^x \tag{3.5}$$

Where the $U_k(x)$, $V_k(x)$ are the transform function of the t-dimensional spectrum.
Now, substitute equation (3.5) into equation (3.4) to obtain the following:

$$U_1(x) = e^x \qquad V_1(x) = -e^x$$
$$U_2(x) = \frac{1}{2}e^x \qquad V_2(x) = \frac{1}{2}e^x$$
$$U_3(x) = \frac{1}{6}e^x \qquad \text{And} \qquad V_3(x) = -\frac{1}{6}e^x$$
$$U_4(x) = \frac{1}{24}e^x \qquad V_4(x) = \frac{1}{24}e^x$$
$$\ldots \qquad \ldots$$
$$U_n(x) = \frac{1}{n!}e^x \qquad V_n(x) = \frac{(-1)^n}{n!}e^x$$

Finally, the differential inverse transform of $u_k(x)$ , $v_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \ldots$$
$$= 1 + e^x + te^x + \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^x + \ldots$$
$$= 1 + \left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \ldots + \frac{1}{n!}t^n + \ldots\right)e^x = 1 + e^{x+t}$$

Similarly

$$v(x,t) = \sum_{k=0}^{\infty} V_k(x)t^k = -1 + e^{x-t}$$

Which is the exact solution.

**Example 3.2**: Consider the nonlinear Boussinesq equation:

$$\begin{aligned} u_t + v_x + uu_x &= 0 \\ v_t + (vu)_x + u_{xxx} &= 0 \end{aligned} \tag{3.6}$$

Subject to the initial conditions

$$\begin{aligned} u(x,0) &= 2x \\ v(x,0) &= x^2 \end{aligned} \tag{3.7}$$

Where the exact solutions are [5]

$$u(x,t)=\frac{2x}{1+3t}$$
$$v(x,t)=\frac{x^2}{(1+3t)^2} \qquad (3.8)$$

Applying the basic properties of the RDTM to equations (3.6) and (3.7), we obtain the recursive relations:

$$(k+1)U_{k+1}(x)=-\frac{\partial}{\partial x}V_k(x)-\sum_{i=0}^{k}U_i(x)\frac{\partial}{\partial x}U_{k-i}(x)$$
$$(k+1)V_{k+1}(x)=-\frac{\partial}{\partial x}\left(\sum_{i=0}^{k}U_i(x)V_{k-i}(x)\right)-\frac{\partial^3}{\partial x^3}U_k(x)$$

i.e.

$$U_{k+1}(x)=\left(\frac{-1}{k+1}\right)\left[\ \frac{\partial}{\partial x}V_k(x)+\sum_{i=0}^{k}U_i(x)\frac{\partial}{\partial x}U_{k-i}(x)\ \right]$$
$$V_{k+1}(x)=\left(\frac{-1}{k+1}\right)\left[\ \frac{\partial}{\partial x}\left(\sum_{i=0}^{k}U_i(x)V_{k-i}(x)\right)+\frac{\partial^3}{\partial x^3}U_k(x)\ \right] \qquad (3.9)$$

And $\quad U_0(x)=2x,\ V_0(x)=x^2. \qquad (3.10)$

Now, substituting equation $(3.10)$ into equation $(3.9)$, we obtain the following values successively:

$$\begin{aligned}&U_1(x)=-6x && V_1(x)=-6x^2\\ &U_2(x)=18x && V_2(x)=27x^2\\ &U_3(x)=-54x &\text{And}\quad& V_3(x)=-108x^2\\ &U_4(x)=162x && V_4(x)=405x^2\\ &\dots && \dots\\ &U_n(x)=(-1)^n3^n2x && V_n(x)=(-1)^n3^n(n+1)x^2\end{aligned}$$

Finally, the differential inverse transform of $U_k(x),\ V_k(x)$ gives

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=U_0(x)+U_1(x)t+U_2(x)t^2+U_3(x)t^3+\dots$$
$$=2x-6xt+18xt^2-54xt^3+162xt^4+\dots+(-1)^n3^n2xt^n+\dots$$
$$=\sum_{k=0}^{\infty}(-1)^k3^k2xt^k=\frac{2x}{1+3t}$$

And

$$v(x,t)=\sum_{k=0}^{\infty}V_k(x)t^k=V_0(x)+V_1(x)t+V_2(x)t^2+V_3(x)t^3+\dots$$
$$=x^2-6x^2t+27x^2t^2-108x^2t^3+405x^2t^4+\dots+(-1)^n3^n(n+1)x^2t^n+\dots$$
$$=\sum_{k=0}^{\infty}(-1)^k3^k(k+1)x^2t^k=\frac{x^2}{(1+3t)^2}$$

Which is the exact solution.

**Example 3.3**: Consider the coupled Burger's equation

$$\begin{aligned} u_t - u_{xx} - 2uu_x + (uv)_x &= 0 \\ v_t - v_{xx} - 2vv_x + (uv)_x &= 0 \end{aligned} \tag{3.11}$$

Subject to the initial conditions

$$\begin{aligned} u(x,0) &= \sin x \\ v(x,0) &= \sin x \end{aligned} \tag{3.12}$$

Where the exact solutions are [6]

$$u(x,t) = v(x,t) = \sin x.e^{-t} \tag{3.13}$$

Applying the RDTM to (3.11) and (3.12) we obtain the following recursive relations:

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + 2\sum_{i=0}^{k} U_i(x)\frac{\partial}{\partial x}U_{k-i}(x) - \frac{\partial}{\partial x}\left(\sum_{i=0}^{k} U_i(x)V_{k-i}(x)\right)$$

$$(k+1)V_{k+1}(x) = \frac{\partial^2}{\partial x^2}V_k(x) + 2\sum_{i=0}^{k} V_i(x)\frac{\partial}{\partial x}V_{k-i}(x) - \frac{\partial}{\partial x}\left(\sum_{i=0}^{k} U_i(x)V_{k-i}(x)\right)$$

i.e.

$$\begin{aligned} U_{k+1}(x) &= \left(\frac{1}{k+1}\right)\left[\frac{\partial^2}{\partial x^2}U_k(x) + 2\sum_{i=0}^{k} U_i(x)\frac{\partial}{\partial x}U_{k-i}(x) - \frac{\partial}{\partial x}\left(\sum_{i=0}^{k} U_i(x)V_{k-i}(x)\right)\right] \\ V_{k+1}(x) &= \left(\frac{1}{k+1}\right)\left[\frac{\partial^2}{\partial x^2}V_k(x) + 2\sum_{i=0}^{k} V_i(x)\frac{\partial}{\partial x}V_{k-i}(x) - \frac{\partial}{\partial x}\left(\sum_{i=0}^{k} U_i(x)V_{k-i}(x)\right)\right] \end{aligned} \tag{3.14}$$

And $\quad U_0(x) = \sin x,\; V_0(x) = \sin x. \qquad (3.15)$

Now, substituting equation $(3.15)$ into equation $(3.14)$, we obtain the following values successively:

$$U_1(x) = -\sin x \qquad V_1(x) = -\sin x$$

$$U_2(x) = \frac{1}{2}\sin x \qquad V_2(x) = \frac{1}{2}\sin x$$

$$U_3(x) = -\frac{1}{6}\sin x \qquad \text{And} \qquad V_3(x) = -\frac{1}{6}\sin x$$

$$U_4(x) = \frac{1}{24}\sin x \qquad V_4(x) = \frac{1}{24}\sin x$$

$$\ldots \qquad \ldots$$

$$U_n(x) = \frac{(-1)^n}{n!}\sin x \qquad V_n(x) = \frac{(-1)^n}{n!}\sin x$$

Finally, the differential inverse transform of $U_k(x),\; V_k(x)$ gives:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=U_0(x)+U_1(x)t+U_2(x)t^2+U_3(x)t^3+\ldots$$

$$=\sin x-\sin xt+\frac{1}{2}\sin xt^2-\frac{1}{6}\sin xt^3+\ldots+\frac{(-1)^n}{n!}\sin xt^n+\ldots$$

$$=\sin x\left(1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\ldots+\frac{(-1)^n}{n!}t^n+\ldots\right)=\sin x.e^{-t}.$$

And

$$v(x,t)=\sum_{k=0}^{\infty}V_k(x)t^k=V_0(x)+V_1(x)t+V_2(x)t^2+V_3(x)t^3+\ldots$$

$$=\sin x-\sin xt+\frac{1}{2}\sin xt^2-\frac{1}{6}\sin xt^3+\ldots+\frac{(-1)^n}{n!}\sin xt^n+\ldots$$

$$=\sin x.e^{-t}.$$

Which is the exact solution.

## IV. CONCLUSION

In this paper, the RDTM was implemented for solving the non-homogenous linear system, the nonlinear 1+1 dimensional Boussinesq equation and the coupled Burger's equation.

The exact solutions of the three systems of linear and non-linear partial differential equations were obtained by the application of RDTM, which constitute a significant improvement over existing techniques. This powerful method can be utilised to tackle complex situations arising in the real world.

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