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On Triplet of Positive Integers Such That the Sum of Any Two of Them is a Perfect Square

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ABSTRACT : *In this article we discussed determination of distinct positive integers a, b, c such that a + b, a + c, b + c are perfect squares. We can determine infinitely many such triplets. There are such four tuples and from them eliminating any one number we obtain triplets with the specific property. We can also obtain infinitely many such triplets from a single triplet.*

KEYWORDS - *Perfect squares, Pythagorean triples, Triplet of positive integers.*

I. Introduction

Number theory has been a subject of study by mathematicians from the most ancient of times (3000 B. C.). The Greeks had a deep interest in Number Theory. Euclid's great text, The Elements, contains a fair amount of Number Theory, which includes the infinitude of the primes, determination of all primitive Pythagorean triples, irrationality of $\sqrt{2}$ etc.

A remarkable aspect of Number Theory is that there is something in it for everyone from Puzzles as entertainment for leymen to many open problems for scholars and mathematicians. For details one may refer [1] or any standard book on Number Theory.

Perfect square numbers are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$… and for any $n \in \mathbb{N} = \{1, 2, 3, 4…\}$, any positive integer k, with $n^2 < k < (n + 1)^2$, is not a perfect square.

k-tuple of positive integers ($k \geq 3$), for any $n \in \mathbb{N}$, $(2n^2, 2n^2,…, 2n^2)$ is such that sum of any two coordinates is $4n^2$, a perfect square. Such k- tuples are infinitely many.

For any $m, n \in \mathbb{N}$ with $m^2 > 2n^2$, $m \neq 2n$, $k \geq 3$, the k- tuple $(m^2 - 2n^2, 2n^2, 2n^2…, 2n^2)$ is such that sum of its any two coordinates is a perfect square. Such k -tuples are infinitely many and first coordinate is different from other coordinates. Above are trivial examples of k-tuples in which sum of any two coordinates is a perfect square.

II. Determination of a Triplet (a, b, c) of Distinct Positive Integers Such That |a – b|, |a – c|, |b – c| are Perfect Squares

We know all primitive Pythagorean triples $(2st, s^2 - t^2, s^2 + t^2)$ where s > t are any positive integers relatively prime with different parity. For details please refer [2], [3] or any standard book on Number Theory.

Consider any $k \in \mathbb{N}$. Then triples (infinitely many) $(k, (2st)^2 + k, (s^2 + t^2)^2 + k)$, $(k, (s^2 - t^2)^2 + k), (s^2 + t^2)^2 + k)$ are such that, in any such triples, difference of any two out of three is a perfect square. For example

(1, 10, 26), (1, 17, 26), (2, 11, 27), (2, 18, 27)…

(1, 26, 170), (1, 145, 170), (2, 27, 171), (2, 146, 171)…

For any integer $k > \max \{(2st)^2, (s^2 - t^2)^2\}$, the triples (infinitely many)

$(k, k - (2st)^2, k + (s^2 - t^2)^2)$, $(k, k + (2st)^2, k - (s^2 - t^2)^2)$ are such that, in any such triples, difference of any two out of three is a perfect square. For example

(17, 1, 26), (17, 33, 8), (18, 2, 27), (18, 34, 9)…

(145, 289, 120), (145, 1, 170), (146, 290, 121), (146, 2, 171)…

We consider $k > (2st)^2$ or $k > (s^2 - t^2)$ to obtain such triples. For example (10, 26, 1), (26, 1, 170).

If $(a, b, c) \in \mathbb{N}^3$ is such that $|a – b|, |a – c|, |b – c|$ are perfect squares, then so is $(a + k, b + k, c + k)$ for all $k \in \mathbb{N}$.

III. Determination of a Triplet (a, b, c) of Distinct Positive Integers Such That a + b, a + c, b + c are Perfect Squares

We consider $a < b < c$ i.e. $a + b < a + c < b + c$ and particular cases:

Case1. Let $a + b = p^2$, $a + c = (p + 1)^2$ and $b + c = (p + 2)^2$, $p \in \mathbb{N}$.

Then $b - a = 2p + 3$, $c - a = 4p + 4$

$\Rightarrow b = a + 2p + 3$, $c = a + 4p + 4$ and $b + c = 2a + 6p + 7 = (p + 2)^2$

$\Rightarrow$ $2a = p^2 - 2p - 3$ i.e. $a = \frac{p^2-2p-3}{2} = \frac{(p-a)^2}{2} - 2$

For p = 5, $a = \frac{25-10-3}{2} = 6 \in \mathbb{N}$ and then b= 6 + 10 + 3 = 19, c = 6 + 20 + 4 = 30

$\Rightarrow$ (6, 19, 30) is a triplet for which sum of any two different coordinates is a perfect square.

For p= 7, $a = \frac{19-14-3}{2} = 16 \in \mathbb{N}$ and then b= 16 + 14 + 3 = 33, c = 16 + 28 + 4 = 48

$\Rightarrow$ (16, 33, 48) is a triplet for which sum of any two different coordinates is a perfect square.

Taking p = 9 we get a = 30, b = 51, c = 70.

Taking p = 11 we get a = 30, b = 73, c = 96.

In general for any $n \in \mathbb{N}$, $n \geq 3$;,$p = 2n - 1$ gives

$a = 2(n^2 - 2n)$, $b = 2(n^2 - 1) + 3$, $c = 2(n^2 + 2n) \in \mathbb{N}$ such that

$a + b = (2n - 1)^2$, $a + c = (2n)^2$, $b + c = (2n + 1)^2$ are perfect squares. Such triplets (a, b, c) are infinitely many and gcd (a, b, c) = 1 (since gcd(a + b, a + c, b + c) = 1).

Case2. Let $a + b = p^2$, $a + c = (p + 1)^2$ and $b + c = (p + 3)^2$, $p \in \mathbb{N}$

Then $b - a = 4p + 8$, $c - a = 6p + 9$, i.e. $b = a + 4p + 8$, $c = a + 6p + 9$.

$\Rightarrow$ $b + c = 2a + 10p + 17 = (p + 3)^2$ i.e. $a = \frac{p^2-4p-8}{2}$.

For p = 6, a = 2 $\in \mathbb{N}$ and b = 2 + 24 + 8 = 34, c = 2 + 36 + 9 = 47

For p = 8, a = 12, b = 52, c = 69

For p = 10, a = 26, b = 74, c = 95

For p = 12, a = 44, b = 100, c = 125 etc.

Thus we have triplets (2, 34, 47), (12, 52, 69), (26, 74, 95), (44, 100, 125) etc such that sum of any two different coordinates is a perfect square.

In general for any $n \in \mathbb{N}$; $n \geq 3$, $p = 2n$ gives

$a = 2(n^2 - 2n - 2)$, $b = 2n^2 + 4n + 4$ and $c = 2n^2 + 8n + 5$ and

$a + b = (2n)^2$, $a + c = (2n+1)^2$, $b + c = (2n+3)^2$ are perfect squares and here gcd(a, b, c) = 1 since gcd (a + b, a + c, b + c) = 1 etc.

Case3. Let $a + b = p^2$, $a + c = (p + 2)^2$, $b + c = (p + 3)^2$, $p \in \mathbb{N}$

Then $b - a = 2p + 5$, $c - a = 6p + 9$.

$\Rightarrow$ $b = a + 2p + 5$; $c = a + 6p + 9$ and hence $2a + 8p + 14 = b + c = (p + 3)^2$

$\Rightarrow$ $a = \frac{p^2-2p-5}{2} = \frac{(p-1)^2}{2} - 3 \in \mathbb{N}$ for $p = 2n - 1$ where $n \in \mathbb{N}$ and $n \geq 3$.

Then $a = 2n^2 - 4n - 1$, $b = 2n^2 + 2$, $c = 2n^2 + 8n + 2$ and $a + b = (2n - 1)^2$,

$a + c = (2n + 1)^2$, $b + c = (2n + 2)^2$ are perfect squares for all integers $n \geq 3$ and

gcd (a, b, c) = 1. Thus (5, 20, 44), (15, 34, 66), (29, 52, 92) etc. are triplets of positive integers in each of which sum of any two is a perfect square.

Case4. Let $a + b = p^2$, $a + c = (p + 1)^2$, $b + c = (p + 4)^2$, $p \in \mathbb{N}$

Then $b - a = 6p + 15$, $c - a = 8p + 16$

$\Rightarrow$ $b = a + 6p + 15$, $c = a + 8p + 16$.

$\therefore$ $2a + 14p + 31 = b + c = (p + 4)^2$

Then $a = \frac{p^2-6p-15}{2} = \frac{(p-3)^2}{2} - 12 \in \mathbb{N}$ for $p = 2n - 1$ where $n \in \mathbb{N}$ and $n \geq 5$ any.

In this case $a = 2n^2 - 8n - 4$, $b = 2n^2 + 4n + 5$, $c = 2n^2 + 8n + 4$ are positive integers with

$a + b = (2n - 1)^2$, $a + c = (2n)^2$, $b + c = (2n + 3)^2$ and gcd(a, b, c) = 1 for all $n \in \mathbb{N}$ $n \geq 5$.

For n = 5, (a, b, c) = (6, 75, 94) is a triplet of positive integers such that sum of any two coordinates is a perfect square etc.

Case5. Let $a + b = p^2$, $a + c = (p + 3)^2$, $b + c = (p + 4)^2$, $p \in \mathbb{N}$
Then $b - a = 2p + 7$, $c - a = 8p + 16$
$\Rightarrow b = a + 2p + 7$, $c = a + 8p + 16$
$\therefore 2a + 10p + 23 = b + c = (p + 4)^2$
$\Rightarrow a = \frac{p^2-2p-7}{2} = \frac{(p-1)^2}{2} - 4 \in \mathbb{N}$ for $p = 2n - 1$, $n \in \mathbb{N}$ and $n \geq 3$
Then $a = 2n^2 - 4n - 2$, $b = 2n^2 + 3$, $c = 2n^2 + 12n + 6$ and $a + b = (2n - 1)^2$,
$a + c = (2n + 2)^2$, $b + c = (2n + 3)^2$ are perfect squares for all integers $n \geq 3$ and
$\gcd(a, b, c) = 1$.
For $n = 3$, $(a, b, c) = (4, 21, 60)$,
For $n = 4$, $(a, b, c) = (14, 35, 86)$ etc.
Case6. Let $a + b = p^2$, $a + c = (p + 2)^2$, $b + c = (p + 4)^2$, $p \in \mathbb{N}$
Then $b - a = 4p + 12$, $c - a = 8p + 16$
$\Rightarrow b = a + 4p + 12$, $c = a + 8p + 16$
$2a + 12p + 28 = b + c = (p + 4)^2$ and hence $a = \frac{p^2-4p-12}{2} = \frac{(p-2)^2}{2} - 8 \in \mathbb{N}$ for all $p = 2n$, $n \in \mathbb{N}$ and $n \geq 4$.
In this case $a = 2n^2 - 4n - 6$, $b = 2n^2 + 4n + 6$, $c = 2n^2 + 12n + 10$ and $a + b = (2n)^2$,
$a + c = (2n + 2)^2$, $b + c = (2n + 4)^2$ are perfect squares and $\gcd(a, b, c) = 2$.
For $n = 4$, $(a, b, c) = (10, 54, 90)$.
For $n = 5$, $(a, b, c) = (24, 76, 120)$.

In this manner taking $a + b = p^2$, $a + c = q^2$, $b + c = r^2$ where p, q, r are positive integers with
$p < q < r$ and taking q, r with $q - p$, $r - p$ as some positive integers we obtain various triples of positive integers
(a, b, c) where sum of any two coordinates is a perfect square.

Note: There are infinitely many triplets of distinct positive integers (a, b, c) such that in each of them, sum of any two coordinates is a perfect square and they are not obtained by above six cases.
For example, (i) $(a, b, c) = (18, 882, 2482)$, where $a + b = 30^2$, $a + c = 50^2$, $b + c = 58^2$.
(ii) $(a, b, c) = (130, 270, 1026)$, where $a + b = 20^2$, $a + c = 34^2$, $b + c = 36^2$ and so on.

IV. Other Triplets by Using a Triplet

If (a, b, c) is a triplet of distinct positive integers such that sum of any two numbers is a perfect squares then so is true for (n^2a, n^2b, n^2c) for each $n \in \mathbb{N}$.
For example (6, 19, 30) is a triplet of positive integers such that sum of any two coordinates is a perfect square. Then $(6 \times 4, 19 \times 4, 30 \times 4)$, $(6 \times 9, 19 \times 9, 30 \times 9)$, $(6 \times 16, 19 \times 16, 30 \times 16)$ etc. are such triplets.
We consider now knowing a specific triplet (a, b, c) and referring process discussed in section 3, we obtain infinitely many triplets with the specific property.
Example1. (18, 882, 2482) is a triplet where $18 + 882 = 30^2$,
$18 + 2482 = 50^2 = (30 + 20)^2$, $882 + 2482 = 58^2 = (30 + 28)^2$.
Let $a < b < c$ in $\mathbb{N}$ and $a + b = p^2$, $a + c = (p + 20)^2$, $b + c = (p + 28)^2$, $p \in \mathbb{N}$
Then $b - a = 16p + 384$, $c - a = 56p + 784$.
$\Rightarrow b = a + 16p + 384$, $c = a + 56p + 784$
$\therefore 2a + 72p + 1168 = b + c = (p + 28)^2$
$\Rightarrow a = \frac{(p-8)^2}{2} - 224 \in \mathbb{N}$ for $p = 2n \in \mathbb{N}$ and $n \geq 15$.
Then $a = 2n^2 - 16n - 192$, $b = 2n^2 + 16n + 192$, $c = 2n^2 + 96n + 592$
where $a + b = (2n)^2$, $a + c = (2n + 20)^2$, $b + c = (2n + 28)^2$ are perfect squares for all $n \in \mathbb{N}$, $(n \geq 15)$.
For $n = 15$, $(a, b, c) = (18, 882, 2482)$
For $n = 16$, $(a, b, c) = (64, 960, 2640)$
For n = 17, 18, 19, we obtain triplets of positive integers, in each sum of any two coordinates is a perfect square.

Example2. (130, 270, 1026) is a triplet where $130 + 270 = 20^2$,
$130 + 1026 = 34^2 = (20 + 14)^2$, $270 + 1026 = 36^2 = (20 + 16)^2$.
Let $a < b < c$ in N and $a + b = p^2$, $a + c = (p + 14)^2$, $b + c = (p + 16)^2$, $p \in \mathbb{N}$. Then
$b - a = 4p + 60$, $c - a = 32p + 256$
$\Rightarrow$ $b = a + 4p + 60$, $c = a + 32p + 256$. $2a + 36p+ 316 = b + c = (p + 16)^2$
$\therefore\ a = \frac{p^2-4p-60}{2} = \frac{(p-2)^2}{2} - 32 \in \mathbb{N}$ for $p = 2n$, $n \in \mathbb{N}$ and $n \geq 6$.
Then $a = 2n^2 - 4n - 30$, $b = 2n^2 + 4n + 30$, $c = 2n^2 + 60n + 226$ and $a + b = (2n)^2$, $a + c = (2n + 14)^2$,
$b + c = (2n + 16)^2$ are perfect squares for $n \in \mathbb{N}$ and $n \geq 6$.
For n = 6; (a, b, c) = (18, 126, 658)
For n = 7; (a, b, c) = (40, 156, 744) etc.

V. Determination of Triplets from Four Tuples

If (a, b, c, d) is a four tuple of distinct positive integers such that a + b, a + c, a + d, b + c, b + d, c + d, are perfect squares, then (a, b, c), (a, b, d), (a, c, d), (b, c, d) are triplets of distinct positive integers such that sum of any two of integers from the triplets is a perfect square.
(18, 882, 2482, 4742), (4190, 10210, 39074, 83426), (7070, 29794, 71330, 172706),
(55967, 78722, 27554, 10082), (15710, 86690, 157346, 27554) are four tuples of distinct positive integers such that sum of any two of them is a perfect square. For this one may refer [1].
Then (4190, 10210, 39074), (4190, 10210, 83426), (4190, 39074, 83426), (10210, 39074, 83426) etc. are triplets of distinct positive integers such that sum of any two of them is a perfect square. Using following identity [4]
$(m^2_1 + n^2_1)(m^2_2 + n^2_2) = (m_1m_2 + n_1n_2)^2 + (m_1n_2 - m_2n_1)^2 \ldots \quad (*)$
One can obtain four tuple of distinct rational numbers such that sum of any two of them is a perfect square. If in such a four tuple at least three are positive, then we obtain a triplet of distinct positive integers such that sum of any two is a perfect square.
In this method, we have to obtain distinct positive integers $p_1, p_2, p_3, p_4, p_5, p_6$ such that
$p_1^2+p_2^2 = p_3^2+p_4^2 = p_5^2+p_6^2$ and obtain distinct numbers a, b, c, d such that
$\{a + b, a + c, a + d, b + c, b + d, c + d\} = \{ p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2 \}$
Let a, b, c, d be rational numbers with $a < b < c < d$ with sum of any two of them is a square of integers. Here we have
$a + b < a + c < a + d < b + d < c + d$,
$a + b < a + c < b + c < b + d < c + d$.
Let us consider $b + c < a + d$. Then we have
$a + b < a + c < b + c < a + d < b + d < c + d \quad \ldots (**)$

Example3. Now $5 = 1^2 + 2^2$, $10 = 1^2 + 3^2$, $13 = 2^2 + 3^2$.
By (*), $5 \times 13 = (2 + 6)^2 + (3 - 1)^2 = (3 + 4)^2 + (6 - 2)^2$
$\Rightarrow$ $65 = 8^2 + 1^2 = 7^2 + 4^2$
Now $10 \times 65 = (1^2 + 3^2)(8^2 + 1^2) = (1^2 + 3^2)(7^2 + 4^2)$
By (*), $650 = (8 + 3)^2 + (24 - 1)^2 = (1 + 24)^2 + (8 - 3)^2$
$= (7 + 12)^2 + (21 - 4)^2 = (21 + 4)^2 + (12 - 7)^2$
$\Rightarrow$ $650 = 25^2 + 5^2 = 19^2 + 17^2 = 11^2 + 23^2$.
Now Consider $a + b = 5^2$, $a + c = 11^2$, $b + c = 17^2$, $a + d = 19^2$, $b + d = 23^2$, $c + d = 25^2$ and here $a + b + c + d = 650$.
We have $c + b = 289$; $c - b = 96$ and hence
$2b = 193$, $2c = 385$, $d = 23^2 - b = 432.5$, $a = 5^2 - b = -71.5$
Here we have (a, b, c, d) = (- 71.5, 96.5, 192.5, 432.5) with sum of any two of them is a perfect square
($a < b < c < d$), then so is for $(n^2a, n^2b, n^2c, n^2d) \in \mathbb{N}$ for any n.
Taking n = 2, we get (-286, 386, 770, 1730) is a four-tuple of integers such that sum of any two of them is a perfect square.
Thus we have a triplet (386, 770, 1730) such that sum of any two of them is a perfect square.

Example 4. We have, $65 = 8^2 + 1^2 = 7^2 + 4^2$ and $5^2 = 2^2 + 1^2$
$\Rightarrow$ $65 \times 5 = (8^2 + 1^2)(2^2 + 1^2) = (7^2 + 4^2)(2^2 + 1^2)$ i.e. $325 = 10^2 + 15^2 = 6^2 + 17^2 = 1^2 + 18^2$ by (*).
Let a, b, c, d be rational numbers with $a < b < c < d$ and
$a + b < a + c < b + c < a + d < b + d < c + d$ with $a + b = 1^2$, $a + c = 6^2$, $b + c = 10^2$, $a + d = 15^2$, $b + d = 17^2$, $c + d = 18^2$.
Then $b - a = 64$, and as $b + a = 1$, so $a = -31.5$, $b = 32.5$ and then $c = 67.5$, $d = 256.5$
(4a, 4b, 4c, 4d) = (-126, 130, 270, 1026) is a four tuple of integers such that sum of any two of them is a perfect square.
Thus we have a triplet (130, 270, 1026) of distinct positive integers such that sum of any two of them is a perfect square.

Example 5[4]. $8125 = 30^2 + 85^2 = 50^2 + 75^2 = 58^2 + 69^2$.
Let a, b, c, d be numbers such that $a < b < c < d$ with
$a + b < a + c < b + c < a + d < b + d < c + d$ and $a + b = 30^2$, $a + c = 50^2$, $b + c = 58^2$, $a + d = 69^2$, $b + d = 75^2$, $c + d = 85^2$. Solving above for positive values of a, b, c, d we get, a = 18, b = 882, c = 2482, d = 4743.
Then (a, b, c, d) = (18, 882, 2482, 4743) is a four tuple of distinct positive integers such that the sum of any two of them is a perfect square.
Hence following are triplets of distinct positive integers in which sum of any two coordinates is a perfect square.
(18, 882, 2482), (18, 882, 4743), (18, 2482, 4743), (882, 2482, 4743).

VI. Conclusion

We conclude that, by taking $a + b = p^2$, $a + c = q^2$, $b + c = r^2$ where p, q, r are positive integers with $p < q < r$ and taking q, r with $q – p$, $r - p$ as some positive integers we obtain various triples of positive integers (a, b, c) where sum of any two coordinates is a perfect square. The set of triplets

$$S = \{(a, b, c) \in \mathbb{N}^3 \mid a + b, a + c, b + c \text{ are perfect squares}\}$$

is an infinite set. Since $\mathbb{N}^3$ is a countable set, so such triplets are countably infinite (denumerable). The set $\{(a, b, c) \in S \mid \gcd(a, b, c) = 1\}$ is also countably infinite.

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