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Matrix Product (Modulo-2) Of Cycle Graphs

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ABSTRACT: *Let G be simple graph of order n. $A(G)$ is the adjacency matrix of G of order $n \times n$. The matrix $A\ (G)$ is said to graphical if all its diagonal entries should be zero. The graph Γ is said to be the matrix product (mod-2) of G and $\bar{G}$ if $A(G)$ and $A(\bar{G})$(mod-2) is graphical and Γ is the realization of $A(G)\ A(\bar{G})$(mod-2). In this paper, we are going to study the realization of the Cycle graph G and any $k-$ regular subgraph of $\bar{G}$. Also some interesting characterizations and properties of the graphs for each the product of adjacency matrix under (mod-2) is graphical.*
Keywords: *Adjacency matrix, Matrix product, Graphical matrix, Graphical realization, Cycle.*

I. INTRODUCTION

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$ the open neighborhood of v is the set $N(v) = \{u \in V/uv \in E\}$ and the closed Neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = U_{v \in s}\ N(v)$ and the closed Neighborhood of $S\ is\ N[S] = N(S) \cup S$

A set $S \subseteq V$ is a dominating set if $N(S) = V - S\ or$ equivalently, every vertex $in\ V/S$ is adjacent to at least one vertex in S.

In this paper we considered the graph as connected simple and undirected. Let G be any graph its vertices are denoted by $\{v_1, v_2, \ldots, v_n\}$ two vertices $v_i\ and\ v_j, i \neq j$ are said to be adjacent to each other if there is an edge between them. An adjacency between the vertices v_i and v_j is denoted by $v_i \sim_G v_j\ and\ v_i \nsim_G v_j$ denotes that v_i is not adjacent with v_j in the graph G. The adjacency matrix of G is a Matrix $A(G) = (a_{ij}) \in M_n\ (R)\ in\ which\ a_{ij} = 1\ if\ v_i\ and\ v_j$ are adjacent, and $a_{ij} = 0$ otherwise, given two graphs G and H have the same set of vertices $\{v_1, v_2, \ldots, v_n\}, G \cup H$ represents the union of graphs G and H having the same vertex set and two vertics are adjacent in $G \cup H$ if they are adjacent in at least one of G and H. Graphs G and H having the same set of vertices are said to be edge disjoint, if $u \sim_G v$ implies that $u \nsim_H v$ equivalently, H is a subgraph of G and G is a sub graph of H.

II. MATRIX PRODUCT (MODULO-2) OF CYCLE GRAPHS

Definition : 2.1

A walk of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i. A walk in which all the Vertices are distinct is called a path. A path containing n vertices is denoted by P_n. A closed path is called a cycle. Generally C_n denoted a cycle with n vertices.

Definition : 2.2

Let G be a graph with n vertices, m edges, the incidence matrix A of G is an $n \times m$ matrix A=(a_{ij}), where n represents the number of rows correspond to the vertices and m represents the columns correspond to edges such that

$$(a_{ij}) = \begin{cases} 1 & if\ v_i\ and\ \ v_j\ are\ adjacent\ in\ G \\ 0 & otherwise \end{cases}$$

It is also called vertex-edge incidence matrix and is denoted by $\wedge(G)$.

Definition : 2.3

A symmetric $(0,1) -$ Matrix is said to be graphical if all its diagonal entries $a_{ij} = 0$ for an $i = j$.

If B is a graphical matrix such that B=A(G) for some graph G, Then we often say that G is the realization of graphical matrix B.

Definition : 2.4

Let us Consider any two graphs G and H having same set of vertices. A graph Γ is said to be the matrix product of G and H. If $A(G)\ A(H)$ is graphical and Γ is the realization of $A(G)\ A(H)$. We shall extend the above definition of matrix product of graphs when the matrix multiplications is considered over the integers modulo-2.

Definition : 2.5

The graph Γ is said to be a matrix product (mod-2) of graphs G and $\bar{G}$ if $A(G)\ A(\bar{G})$(mod-2) is graphical and Γ is the realization of $A(G)\ A(\bar{G})$ (mod-2).

Definition : 2.6

Given graphs G and H on the same set of vertices $\{v_1, v_2, ... v_n\}$, two vertices $v_i\ and\ v_j\ (i \neq j)$ are said to have a GH path if there exists a vertex v_k, different from $v_i\ and\ v_j$ such that $v_i \sim_G v_k\ and\ v_k \sim_H v_j$.

Definition : 2.7

A graph is a parity graph if for any two induced paths joining the same pair of vertices the path lengths have the same parity (odd or even).

$\bar{G}$, **Lemma: 2.8**

If G is a cycle graph with length 3, then $\bar{G}$ is a null graph.

Lemma: 2.9

Let G be a cycle graph with length 4 and $\bar{G}$ is a complement of G. Then $A(G)A(\bar{G}) = A(G)$.

Lemma: 2.10

If G is a cycle graph with length 4 then $\bar{G}$ is a disconnected graph.

Lemma: 2.11

Let G be a cycle graph with length 5 and $\bar{G}$ is a complement of G. The realization of $A(G)A(\bar{G}) = G \cup \bar{G}$.

Lemma: 2.12

Let G be a cycle graph and $\bar{G}$ is a complement of G.Then $A(G)A(\bar{G})$ is graphical.

Proof:

Let $C_n = \{v_1, v_2, ... v_n\}$ v_i is adjacent with v_{i-1} and v_{i+1} such that, $v_n = v_0$.

Let (a_{ij}) is the adjacent matrix of G, and (b_{ij}) is the adjacent matrix of

Then, each $(a_{ij}) = \begin{cases} 1 & if\ \ j = i+1\ \ and\ j = i-1 \\ 0 & otherwise \end{cases}$ and

each $(b_{ij}) = \begin{cases} 1 & if\ \ j = i+1\ \ and\ \ j = i+1 \\ 0 & otherwise \end{cases}$

Therefore, $A(G)A(\bar{G}) = \{(C_{ij}) = 0\ \ if\ i = j; \quad i = 1,2, \dots .n\ \}$

Hence all diagonal values are zero, so cycle graph is graphical.

Hence the proof.

Lemma 2.13

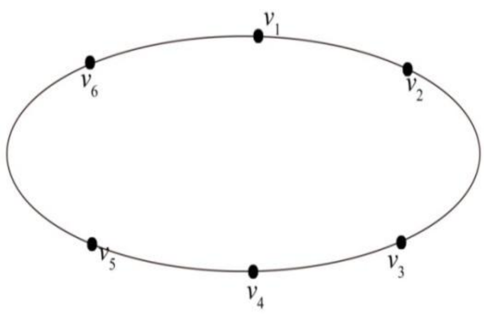
The $(i,j)^{th}$ $(i \neq j)$ entry of the matrix product $A(G)\ A(H)$ is either o or 1 depending on whether the number of GH paths from v_i to v_j is even or odd, respectively.

Example: 2.14

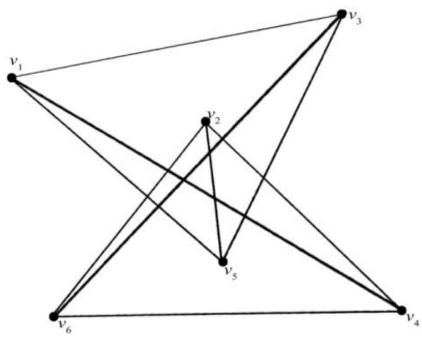
Consider a cycle graph G with length 6 and its complement $\bar{G}$ and Γ as shown in figure 1 Note that

$$A(G)A(\bar{G}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

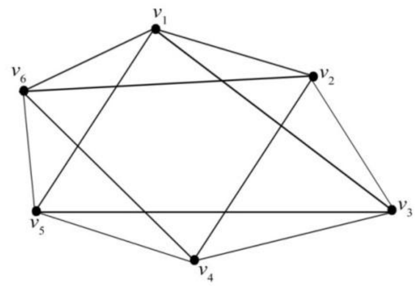
Figure 3 is the graph realizing $A(G)\ A(\bar{G})$ is graphical



G



$\bar{G}$



Γ

Figure: 1

A cycle graph G with length 6 for which Γ is the graph realizing $A(G)A(\bar{G})$.

Theorem: 2.15

For any cycle graph G and its complement $\bar{G}$ on the set of vertices are equivalent:

(i) The matrix product $A(G)A(\bar{G})$ is graphical.

(ii) For every i and j, $1 \leq i, j \leq n$, $deg_G v_j - deg_G v_j \equiv 0\ (mod\text{-}2)$.

(iii) The graph G is parity regular.

Proof:

Note that $(ii) \Leftrightarrow (iii)$ follows from the definition of parity regular graphs. Now, we shall prove $(i) \Leftrightarrow (ii)$.

Let $(A(G))_{ij} = (a_{ij})$.

From the definitions of the complement of a graph and GH path, $H = \bar{G}$ implies that

$deg_G v_i =$ Number of walks of length 2 from v_i in G +Number of $G\bar{G}$ paths from v_i to $v_j + a_{ij}$ ……(1)

Similarly,

$deg_G v_j =$Number of walks of length 2 from v_j to v_i in $G +$ Number of $G\bar{G}$ paths from v_j to $v_i + a_{ij}$ ……(2)

for every distinct pair of verticesv_i and v_j.

Since a $\bar{G}G$ path from v_j to v_i is a $\bar{G}G$ path from v_i to v_j, and comparing the right hand sides of (1) and (2), we get that $A(G)A(\bar{G})$ is graphical iff $deg_G v_i \equiv deg_G v_j\,(mod\text{-}2)$.

Hence the proof.

Remark 2.16

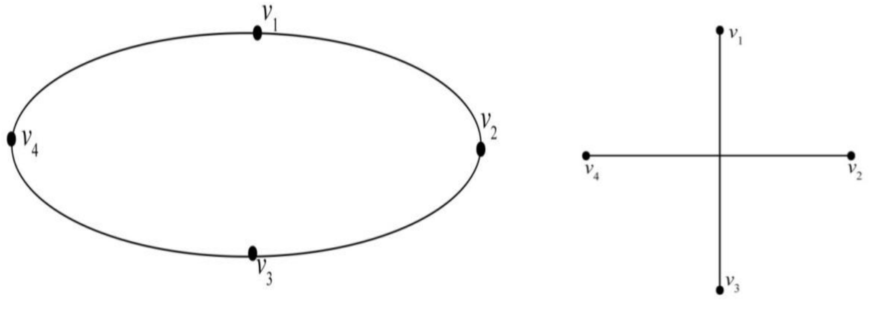
It is also possible for one to prove $(1) \Leftrightarrow (2)$, by taking $A(\bar{G}) = J - A(G) - I$ in the matrix products $A(G)A(\bar{G})$ and $A(\bar{G})A(G)$, where J is the $n \times n$ matrix with all 1's and I is the $n \times n$ identity matrix.

Remark: 2.17

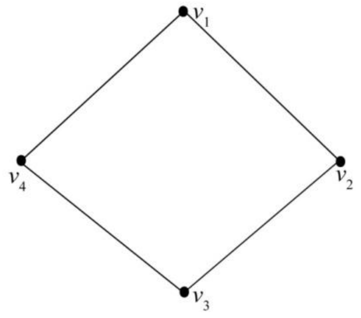
For any two graphs G and H such that $A(G)A(H)$ is graphical under ordinary matrix multiplication, it has been noted in equation (2) in Theorem 2.15 that any of G and H is connected implies that the other is regular.

When we consider matrix multiplication under modulo-2, we observe that the connected graphs G and $\bar{G}$.

Shown in the figure 2 deviate from said property, where both the graphs are just parity regular or neither of them are regular. Note that



G $\bar{G}$



Γ

Figure: 2

Γ is the graph realizing $A(G)A(\bar{G})$, where either G or $\bar{G}$ is regular.

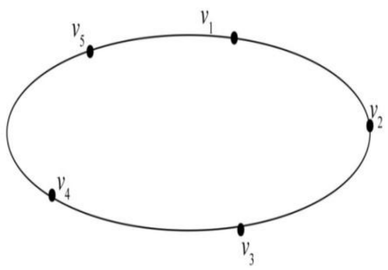
$$A(G)A(\bar{G}) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Realizing the graph Γ as shown in figure 6.

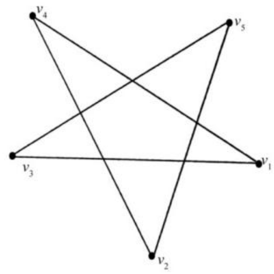
The graph G for which $A(G)A(\bar{G}) = A(G)$.When we consider the matrix multiplication with reference to modulo - 2.

Example: 2.18

Let G be a cycle graph with length 5 and $\bar{G}$ is a complement of G. Then realization of $A(G)$ and $A(\bar{G})$ is equal to $G \cup \bar{G}$.



G



$\bar{G}$

$\overline{G} =$

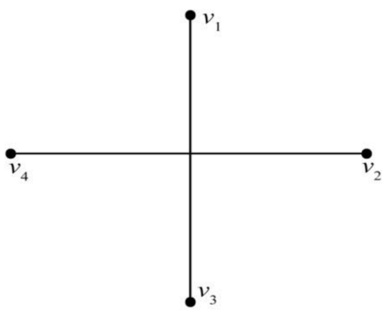


Figure: 3

$$A(G)A(\bar{G}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Therefore, the graph realizing $A(G)A(\bar{G}) = G \cup \bar{G}$.

Theorem : 2.19

Let G be a graph and its complement $\bar{G}$ defined on the set of vertices $\{v_1, v_2, \ldots\ v_n\}$. Then $A(G)A(\bar{G}) = A(G)$ iff $[A(G)]^2$ is either a null matrix or the matrix J with all entries equal to 1.

Proof :
Let, $A(G) = (a_{ij})$,
In theorem 2.15, taking $H = \bar{G}$,we get that, $A(G)A(\bar{G}) = A(G)$
The number of path in $G\bar{G}$ from v_i to v_j is (a_{ij}).
we have,
$deg_G v_i \equiv$ number of walks of length 2 from v_i to v_j in G(mod-2) for $i \neq j$ ……(B)
By theorem 2.15, G is a parity regular and therefore, $deg_G v_i - deg_G v_j \equiv 0\ (mod\text{-}2)$
we get that $(A(G))^2$ is either 0 $or\ J$. [∵ by (B)]
Conversely, suppose that $(A(G))^2$ is either $O\ or\ J$. If $(A(G))^2 = 0$ we get that the degree of all the vertices in G are even and $(A(G))^2 = J$ would mean that degree of all the vertices are odd. By taking $A(\bar{G}) = J - A(G) - I$
$= J + A(G) + I$ [since we know that the minus $(-)$ is the same as the plus (+) under modulo-2)]
Therefore, we get $A(G)A(\bar{G}) = A(G)(J + A(G) + I)$
$= A(G)J + (A(G))^2 + A(G)$
In each case, $(A(G))^2$ is $O\ or\ J$, we get that the right hand side of the above reduces to $A(G)$. Which also characterizes the graphs G with property $A(G)A(\bar{G}) = A(G)$ in terms of characteristics of $\bar{G}$.

Theorem:2.20
The product $\boldsymbol{A(G)\ A(H)}$ is graphical if and only if the following statements are true.
(i) For every $(\mathbf{1 \leq i \leq n})$, there are even number of vertices $\boldsymbol{v_k}$ such that $\boldsymbol{v_i \sim_G v_k}$ and $\boldsymbol{v_k \sim_H v_i}$
(ii) For each pair of vertices v_i and v_j $(i \neq j)$ the number of GH paths and HG paths from v_i to v_j have same parity.

Example: 2.21
Consider a cycle graph G with length 7 and its complement is shown in figure 4.

$$A(G)A(\bar{G}) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and the cocktail party graph shown in figure 4 is the graph realizing $A(G)\ A(\bar{G})$ is graphical.

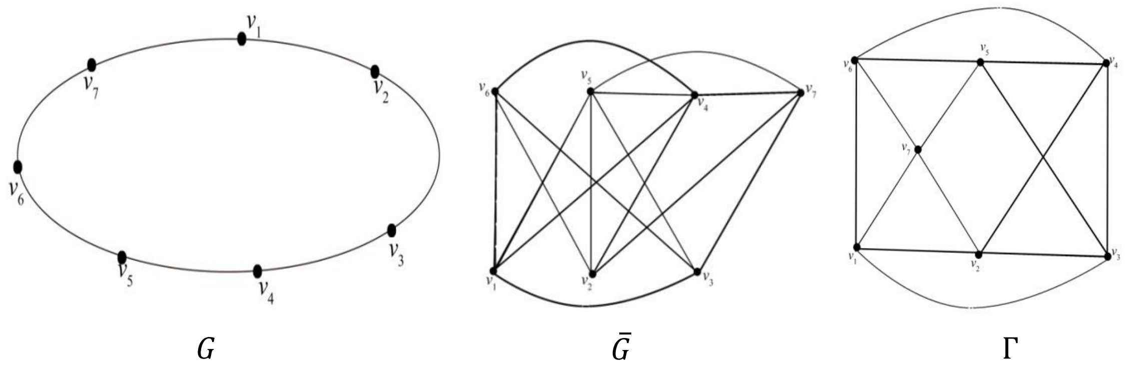


Figure : 4

Γ is the graph realizing $A(G)\ A(\bar{G})$ where either G or $\bar{G}$ is regular.

Remark: 2.22
For any two graphs G and H such that $A(G)\ A(H)\ is$ graphical under ordinary matrix multiplication, it has been noted that equation (2) of theorem 2.15 any subgraph H of G is connected implies that the other is regular
When we consider matrix multiplication under modulo-2, we observe that the connected graphs G and $\bar{G}$.
Shown in the figure 5 deviate from said property where both the graphs are just parity regular or neither of them are regular

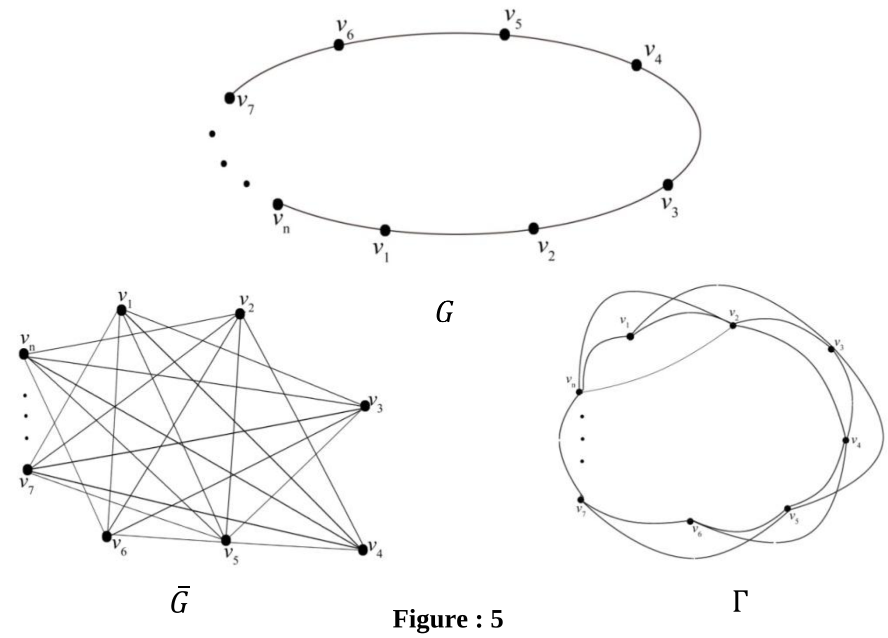


Figure : 5

$$A(G)A(\bar{G}) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \end{pmatrix}$$

Realization of the graph Γ is shown in figure 5.

Corollary: 2.23

Consider a graph such that $A(G) = A$ and $A(\bar{G}) = B$. Then the following statements are equivalent.

(i)$AB = A$

(ii)$B^2 = I$ or$J - I$

(iii) $\bar{G}$ is a graph with one of the following properties.

(a) Degree of each vertex is odd and the number of paths of length 2 between every pair of vertices is even.

(b) Degree of each vertex is even and the number of paths of length 2 between every pair of vertices is odd .

Proof:

$(i) \Rightarrow (ii)$ from the orem 2.19 we get $AB = A$ implies A^2 is either a null matrix or J

By taking $A = J - B - I$,we get that $A^2 = J^2 + B^2 + I^2$ further note that J^2 is either a null matrix or J itself depending on the number of vertices on which the graph defined is even or odd.

In both case, $A^2 = 0$ or J implies $B^2 = I$ or $J - I$.

$(ii) \Rightarrow (i)$ nothing that $J^2 = 0$ or J. itself by proper substitution for B^2 weget, $A^2 = J^2 + B^2 + I$

Is either a null matrix or J. So we obtain (1) from The orem 2.19.

As (iii) is graph theoretical interpretation of (ii),$(ii) \Leftrightarrow (iii)$ is trivial.

Hence the Proof.

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