International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 1 || January. 2017 || PP-43-55

A New Approach on the Log - Convex Orderings and Integral inequalities of the Log - Convex Ordering of the Triangular Fuzzy Random Variables

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ABSTRACT: *In this paper, we introduce a new approach on the convex orderings and integral inequalities of the convex orderings of the triangular fuzzy random variables. Based on these orderings, some theorems and integral inequalities are established.*
KEYWORDS: *Triangular fuzzy numbers, triangular fuzzy random variables, convex function, concave function, Hermite – Hadamard inequalities.*
AMS Classification: 60A86.

I. INTRODUCTION

The convex orders of random variables have found a wide field of applications in finance, economics, reliability, decision theory, comparison of experiments, epidemics, queueing, scheduling and other fields. The ordering is defined through the comparison of the expected values of convex functions. In this paper, we discuss about the convex orderings and the integral inequalities of the convex orderings of the triangular fuzzy random variables based on Kwakernaak's [1] fuzzy random variables. The association of the paper is as follows. In section 2, we dealt the basic notations and limits of the triangular fuzzy random variables with parameters mean μ and standard deviationσrespectively. We define the new definitions about the distribution function and survival functions are derived. In section 3, we prove some theorems of convex orderings and the integral inequalities of the convex orderings of the triangular fuzzy random variables are derived.

II. PRELIMINARIES

This section deals with some well – known definitions and notions which will be used in the next section.

Definition: 2.1 A fuzzy set A on the universal set X is defined as the set ordered pair
$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}$, where $\mu_A(x)$ is its membership function.

Definition: 2.2 The support of fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$.
That is, Support $(A) = \{x \in X / \mu_A(x) > 0\}$.

Definition: 2.3 The α-cut A_α of a fuzzy set A is the set consisting of those elements of the universe X whose membership values exceed the threshold level $\alpha \in [0, 1]$.
That is, $A_\alpha = \{x \in X / \mu_A(x) \geq \alpha\}$

Definition: 2.4 A fuzzy set A on the set R of real numbers said to be a fuzzy number, if:

i. A is a normal fuzzy set, i.e. there exists $x \in X$ such that $\mu_A(x) = 1$.
ii. A_α is a closed interval for every $\alpha \in [0,1]$, i.e. A_α is a convex subset of R.
iii. The membership function $y= \mu_A(x)$ is a piecewise continuous function.

Among the various shapes of fuzzy numbers, the triangular fuzzy number (TFN) is the most popular one. A TFN is defined as follows:

Definition: 2.5 The triangular fuzzy number is a fuzzy number represented with 3-tuples as follows: $A = (a_1, a_2, a_3)$. This representation is interpreted as membership function and holds the following conditions.

(i) a_1 to a_2 is increasing function
(ii) a_2 to a_3 is decreasing function
(iii) $a_1 \leq a_2 \leq a_3$ is a fuzzy number with membership function

$$\mu_A(x) = \begin{cases} 0 & for\ x < a_1 \\ \frac{x - a_1}{a_2 - a_1} & for\ a_1 \le x < a_2 \\ \frac{a_3 - x}{a_3 - a_2} & for\ a_2 \le x < a_3 \\ 0 & for\ x \ge a_3 \end{cases}$$

It is easy to check that the α-cut of a TFN A = (a_1, a_2, a_3) is of the form

$A_\alpha = [a_1^\alpha, a_3^\alpha]$ with $a_1^\alpha = (a_2 - a_1)\alpha + a_1$, $a_3^\alpha = -(a_3 - a_2)\alpha + a_3$

Definition: 2.6

In normal distribution, if the value of the standard deviation (σ) is very minimum, then the shape of the normal curve becomes very sharp peak as well as symmetric with respect to the mean (μ), so that the normal distribution curve is even.

Now, the ordered triple $(a_1=\mu - n\sigma,\ a_2=\mu,\ a_3 = \mu + n\sigma)$, n = 1, 2, 3 of the normal curve, it covered 99.73%. The remaining area is 0.27%. It spreads over outside of the curve $|X - \mu| \ge 3\sigma$ on both sides. It is nearly equal to zero and that the X – axis is asymptote to the curve. But, In triangular fuzzy number, every α-cut≥∈, ∈> 0 so that the normal curve can be treated as a triangular fuzzy number with base $(a_1=\mu-\sigma,\ a_2=\mu,\ a_3 = \mu+\sigma)$, Therefore, the triangular fuzzy number is also an even.

Definition: 2.7

If R be a triangular fuzzy random variable with mean μ_1, and standard deviation σ_1 and a function f: $[\mu - \sigma, \mu + \sigma] \rightarrow [0, 1]$ is said to be log – concave or multiplicatively concave function of the triangular fuzzy number, if log(f) is concave, or equivalently, then

$$P\{bR_\alpha^L + (1-b)R_\alpha^U\} \ge (P\{R_\alpha^L\})^{\mathbf{b}}(P\{R_\alpha^U\})^{\mathbf{1-b}}$$

$$P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \ge 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}$$
$$\ge (P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\})^{\mathbf{b}}(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \le 0)\})^{\mathbf{1-b}}, \qquad 0 \le b \le 1.$$

Similarly, $P\{bR_\alpha^L + (1-b)R_\alpha^U\} \le (P\{R_\alpha^L\})^{\mathbf{b}}(P\{R_\alpha^U\})^{\mathbf{1-b}}$

$$P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \ge 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}$$
$$\le (P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\})^{\mathbf{b}}(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \le 0)\})^{\mathbf{1-b}}, \qquad 0 \le b \le 1.$$

is called log – convex or multiplicatively convex function of the triangular fuzzy number. If for all R_α^L, $R_\alpha^U \in [\mu - \sigma, \mu + \sigma]$ and $0 \le b \le 1$.Moreover, since f = exp(log f), it follows that a log –convex function is convex, but the converse is not true. Apparently, it would seem that log – concave (log – convex) functions would be unremarkable because they are simply related to concave (convex) functions.

But they have some surprising properties. It is well known that the product of log – concave (log – convex) functions is also log – concave (log – convex). Moreover, the sum of log – convex functions is also log – convex, and a convergent sequence of log – convex (log – concave) functions has a log – convex (log – concave) limit function provided that the limit is positive. However, the sum of log – concave functions is not necessarily log – concave.

Definition: 2.8

If R be a triangular fuzzy random variable with mean μ_1, and standard deviation σ_1 and a function f: $[\mu_1 - \sigma_1, \mu_1 + \sigma_1] \rightarrow [0, 1]$ is said to be log - convex (or) multiplicatively convex function of the triangular fuzzy number (LCONF):

If $\log\left(P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \ge 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right)$ is convex (or equivalently) function of the triangular fuzzy number. Then:

$P\{bR_\alpha^L + (1-b)T_\alpha^U\} \le (P\{R_\alpha^L\})^b(P\{R_\alpha^U\})^{1-b}$

$P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \ge 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}$

$\le (P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\})^{\mathbf{b}}(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \le 0)\})^{\mathbf{1-b}},\ 0 \le b \le 1.$

Definition: 2.9

If R be a triangular fuzzy random variable with mean μ_1, and standard deviation σ_1 anda function f: $[\mu_1 - \sigma_1, \mu_1 + \sigma_1] \rightarrow [0, 1]$ is said to be geometric – arithmetically convex (or) GA - convex function of the triangular fuzzy number (GA - CONF)

If $\log(P\{(bR_\alpha^L - (\alpha-1)b\sigma 1 - b\mu 1) \ge 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\})$ is convex (or equivalently) function of the triangular fuzzy number

$$P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\}^{\mathbf{b}} P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \le 0\}^{\mathbf{1-b}}$$

$\leq$ bP $\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\}$+(1-b) P$\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$, $0 \leq b \leq 1$.
(i.e.,)P $\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\}^{\mathbf{b}}P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}^{\mathbf{1-b}}$

$\leq$ P $\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$

WhereP $\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\}^{\mathbf{b}}P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}^{\mathbf{1-b}}$and P $\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\}$+(1-b) P$\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$are respectively called the weighted geometric mean of P $\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\}$)and (P$\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$ and the weighted mean of P $\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\}$and P$\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$.

Definition: 2.10

The probability distribution function is given by $F(x) = \begin{bmatrix} 0 & x \leq a \\ \frac{(x-a)}{(b-a)} & a < x < b \\ 1 & x \geq b \end{bmatrix}$

Where $f(x) = \frac{1}{(b-a)}$ is the probability density function in $a \leq x \leq b$.
Now, we construct the distribution function of the triangular fuzzy number
(a, b, c) = $(\mu_1-\sigma_1, \mu_1, \mu_1+\sigma_1)$ is$\frac{(x-a)}{(b-a)} \geq \alpha$

$$\frac{X_\alpha^L-(\mu_1-\sigma_1)}{\sigma_1} \geq \alpha$$

$$X_\alpha^L-(\mu_1-\sigma_1) \geq \alpha\sigma_1$$

Therefore, P $\{(X_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$ which is the required distribution of the triangular fuzzy number and it is also called the membership function of the increasing function of the triangular fuzzy number.

Definition: 2.11
The complementary probability distribution or survival function is given by

$G(x) = \begin{bmatrix} 1 & x \leq b \\ 1-\frac{(x-b)}{(c-b)} & b < x < c \\ 0 & x \geq c \end{bmatrix}$

Where $g(x) = \frac{1}{(c-b)}$ is the probability density function in $b \leq x \leq c$ and F(x) = 1 - G(x).
Now, we construct the survival functions of the triangular fuzzy numbers
(a, b, c) = $(\mu_1-\sigma_1, \mu_1, \mu_1+\sigma_1)$ is as follows:

$$1-\frac{(x-b)}{(c-b)} = 1-\frac{(x-\mu_1)}{(\mu_1+\sigma_1)-\mu_1} \geq \alpha$$

$$\frac{\sigma_1-X_\alpha^L+\mu_1}{\sigma_1} \geq \alpha$$

$$\sigma_1-X_\alpha^L+\mu_1 \geq \alpha\sigma_1$$

$$-X_\alpha^L \geq \alpha\sigma_1 - \sigma_1 - \mu_1$$

$$X_\alpha^U \leq -(\alpha-1)\sigma_1+\mu_1$$

$$P\{(X_\alpha^U+(\alpha-1)\sigma_1-\mu_1) \leq 0\}$$

Which is the required survival function of the triangular fuzzy number and it is also called the membership function of the decreasing function of the triangular fuzzy number.
Obviously, Let f: $[\mu_1-\sigma_1, \mu_1+\sigma_1] \to [0, 1]$ is a convex function of the triangular fuzzy number. We say that f is an even function with respect to the point $\frac{[(\mu_1-\sigma_1)+(\mu_1+\sigma_1)]}{2} = \mu_1$
if $f((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-(r,\alpha)) = f(r,\alpha)$,for $(r,\alpha) \in [\mu_1-\sigma_1, \mu_1+\sigma_1]$.

III. PROPERTIES OF CONVEX ORDERINGS OF TRIANGULAR FUZZY NUMBER

In this section, we prove the following theorems by using the Hermite – Hadamard inequalities:
If f: $[a, b] \to \mathbb{R}$ is a convex function, then: $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$ (3.1)

Theorem: 3.1
If R and S be triangular fuzzy random variables with mean μ_1 and standard deviations σ_1 and $\frac{\sigma_1}{2}$also a function f: $[\mu_1-\sigma_1, \mu_1+\sigma_1] \to [0, 1]$ is a convex function of the triangular fuzzy number. Moreover g: $\left[\mu_1-\frac{\sigma_1}{2}, \mu_1+\frac{\sigma_1}{2}\right] \to [0, 1]$ is integrable and symmetric about $\frac{\left[\left(\mu_1-\frac{\sigma_1}{2}\right)+\left(\mu_1+\frac{\sigma_1}{2}\right)\right]}{2} = \mu_1$ (Say), i.e. $g\left(\left(\mu_1-\frac{\sigma_1}{2}\right)+\left(\mu_1+\frac{\sigma_1}{2}\right)-(r,\alpha)\right) = g(r,\alpha)$, $\forall(r,\alpha) \in \left(\left(\mu_1-\frac{\sigma_1}{2}\right), \left(\mu_1+\frac{\sigma_1}{2}\right)\right)$with

$$\int_{\left(\mu_1-\frac{\sigma_1}{2}\right)}^{\left(\mu_1+\frac{\sigma_1}{2}\right)} P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}\, d(S_\alpha^L,S_\alpha^U)>0$$

Then $P\{(bR_\alpha^L-b\mu 1)\geq 0 \vee ((1-b)R_\alpha^U-(1-b)\mu 1)\leq 0\}\leq$

$$\frac{\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1}\begin{pmatrix} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\} \\ P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}\end{pmatrix} d(R_\alpha^L,R_\alpha^U)}{\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}\, d(S_\alpha^L,S_\alpha^U)}$$

$$\leq \frac{P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}+P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}}{2} \quad (3.2)$$

Remark that $P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}=1$.
$\forall (S_\alpha^L, S_\alpha^U)\in (\mu_1-\frac{\sigma_1}{2}, \mu_1+\frac{\sigma_1}{2})$ and α = (0, 1). Hence, we obtain the Hermite – Hadamard inequality.

Theorem: 3.2
If R and S be triangular fuzzy random variables with mean μ_1 and standard deviations σ_1 and $\frac{\sigma_1}{2}$ also a function f: $[\mu_1-\sigma_1, \mu_1+\sigma_1]\to[0,1]$ is a convex function of the triangular fuzzy number. Moreover g: $\left[\mu_1-\frac{\sigma_1}{2}, \mu_1+\right.$ σ1*2*→ [0, 1] is integrable and symmetric aboutμ*1*− σ*12*+μ*1*+ σ*12*2=μ1 suchthat

$$\int_{\mu_1-x}^{\mu_1+x} P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}\, d(S_\alpha^L,S_\alpha^U)>0,$$
$$\forall\, 0\leq(x,\alpha)\leq\left[\mu_1-\frac{\sigma_1}{2}, \mu_1+\frac{\sigma_1}{2}\right].$$

Then $P\{(bS_\alpha^L-b\mu_1)\geq 0 \vee ((1-b)S_\alpha^U-(1-b)\mu_1)\leq 0\}$

$$\leq \frac{\int_{\mu_1-x}^{\mu_1+x}\begin{pmatrix} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\} \\ P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}\end{pmatrix} d(R_\alpha^L,R_\alpha^U)}{\int_{\mu_1-x}^{\mu_1+x} P\{(bS_\alpha^L-(\alpha-1)b\frac{\sigma}{2}-b\mu)\geq 0 \vee ((1-b)S_\alpha^U+(\alpha-1)(1-b)\frac{\sigma}{2}-(1-b)\mu)\leq 0\}\, d(S_\alpha^L,S_\alpha^U)}$$

$$\leq \frac{1}{2}P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0\}+\frac{1}{2}P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\} \quad (3.3)$$

Definition: 3.3
Let R and T be two triangular fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively, then R is smaller than T in convex order (denoted by $R\leq_{cx}T$) if

$$E\left[P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right]$$
$$\leq E\left[P\{(bT_\alpha^L-(\alpha-1)b\sigma 2-b\mu 2)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_2-(1-b)\mu_2)\leq 0\}\right]$$

For all real convex functions R.

Definition: 3.4
Let R and T be two triangular fuzzy random variables with means μ_1, μ_2 and standard deviation σ_1, σ_2 respectively. Then R is smaller than T in increasing convex order (denoted by $R\leq_{icx}T$), if

$$E\left[P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right]$$
$$\leq E\left[P\{(bT_\alpha^L-(\alpha-1)b\sigma 2-b\mu 2)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_2-(1-b)\mu_2)\leq 0\}\right]$$

For all increasing real convex functions R.

Theorem: 3.5
Let R and T two triangular fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. (i.e.)$[(\mu_1-\sigma_1)\leq R\leq(\mu_1+\sigma_1)$and $(\mu_2-\sigma_2)\leq T\leq(\mu_2+\sigma_2)]$, the co-ordinate points of the dependent triangular fuzzy numbers are arranged as follows $[s\leq t\leq u\leq v]$ with the convex survival functions $\overline{F}$ and $\overline{G}$ respectively. (i.e.)$\overline{F}(r,\alpha)=P\{R\geq\mu_1\}$ and $\overline{G}(t,\alpha)=P\{T\geq\mu_2\}$. If whenever s ≤ t and u ≤ v.
Here, $s=\mu_1-\sigma_1$, $t=\mu_2-\sigma_2$, $u=\mu_1+\sigma_1$ and $v=\mu_2+\sigma_2$. If E{R} = E{T} ($\mu_1=\mu_2$ and $\sigma_1=\sigma_2$) then $R\leq_{cx}T$

$$\Leftrightarrow \int_{\mu_1}^{\mu_1+\sigma_1} P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\}\, d(r,\alpha)$$
$$\leq \int_{\mu_2}^{\mu_2+\sigma_2} P\{((1-b)T_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\}\, d(t,\alpha) \Leftrightarrow R\leq_{icx}T.$$

For a triangular fuzzy random variable R, with values in $\{\mu_1-\sigma_1, \mu_1+\sigma_1\}$.
let us denote $F(r,\alpha)=P\{R\leq\mu_1\}=P\{(R_\alpha^L-(\alpha-1)\sigma-\mu)\leq 0\}$- the distribution function of R;
$\overline{F}(r,\alpha)=P\{R\geq\mu_1\}=P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\}$ - the convex function of the survival function of R;
$G(t,\alpha)=P\{T\leq\mu_2\}=P\{(T_\alpha^L-(\alpha-1)\sigma-\mu)\leq 0\}$ - the distribution function of T;

$\overline{G}(t,\alpha)$ = P {T≥ μ_2 } =P $\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}$- the convex function of the survival function of T;

E {f(R)} = $\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} f(R)d\{F(r,\alpha)\}$,(r,α)∈ {$\mu_1$- σ_1, μ_1+σ_1} – the mean (or the expectation) of the triangular fuzzy random variable f(R). The inequality (3.1) is written as f(E[R]) ≤E {f(R)] ≤E {f (R*)} (3.3)

Where R is a triangular fuzzy random variable on [μ_1- σ_1 ≤ R ≤ μ_1+ σ_1], R* a triangular fuzzy random variable on {μ_1- σ_1 ≤ R* ≤ μ_1+σ_1} and f :{ μ_1- σ_1, μ_1+σ_1} → (0, 1) is a convex function. In fact, the right inequality of (3.3) say R $\le_{cx}$R*.

Lemma: 3.6

Let R be a triangular fuzzy random variable with mean $\mu_1$and standard deviationσ_1 (i.e.) R defined on the interval [μ_1- σ_1, μ_1+σ_1]. If $\overline{F}$ is the convex survival function of R.

Then $E[R] = (\mu_1 - \sigma_1) + \int_{\mu_1-\sigma_1}^{\mu_1} \bar{F}(r,\alpha)\, d(r,\alpha) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha) d(r,\alpha)$

Proof:

Let F (r, α) = 1 - $\overline{F}$(r, α),

Here, F (r, α) = P{R ≤ μ_1}= $P\{(R_\alpha^L - (\alpha-1)\sigma - \mu) \le 0\}$ be the distribution function of R and

$\overline{F}$(r, α) = P{R ≥ μ_1} = P $\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}$be the convex survival function of R.

Then E[R] = $\int_{-\infty}^{\infty}(r,\alpha)d\{F(r,\alpha)\}$

$$E[R] = \int_{-\infty}^{\mu_1-\sigma_1}(r,\alpha)d\{F(r,\alpha)\} + \int_{\mu_1-\sigma_1}^{\mu_1}(r,\alpha)d\{F(r,\alpha)\}$$
$$+ \int_{\mu_1}^{\mu_1+\sigma_1}(r,\alpha)d\{\bar{F}(r,\alpha)\} + \int_{\mu_1+\sigma_1}^{\infty}(r,\alpha)d\{\bar{F}(r,\alpha)\}$$
$$= (\mu_1 - \sigma_1)\, F(\mu_1 - \sigma_1) + [(r,\alpha)F(r,\alpha)]_{\mu_1-\sigma_1}^{\mu_1} - \int_{\mu_1-\sigma_1}^{\mu_1} F(r,\alpha)\, d(r,\alpha)$$
$$+[(r,\alpha)\bar{F}(r,\alpha)]_{\mu_1}^{\mu_1+\sigma_1} - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$= (\mu_1 - \sigma_1)\, F(\mu_1 - \sigma_1) + \mu_1\, F(\mu_1) - (\mu_1 - \sigma_1)\, F(\mu_1 - \sigma_1) - \int_{\mu_1-\sigma_1}^{\mu_1} F(r,\alpha)\, d(r,\alpha)$$
$$+ (\mu_1 + \sigma_1)\, \bar{F}(\mu_1 + \sigma_1) - \mu_1\bar{F}(\mu_1) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$= \mu_1\, F(\mu_1) - \int_{\mu_1-\sigma_1}^{\mu_1} F(r,\alpha)\, d(r,\alpha) - \mu_1 + \mu_1\, F(\mu_1) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$E[R] = \mu_1 - \int_{\mu_1-\sigma_1}^{\mu_1} F(r,\alpha)\, d(r,\alpha) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$= \mu_1 - \int_{\mu_1-\sigma_1}^{\mu_1} (1-\bar{F}(r,\alpha))\, d(r,\alpha) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$= \mu_1 - \int_{\mu_1-\sigma_1}^{\mu_1} d(r,\alpha) + \int_{\mu_1-\sigma_1}^{\mu_1} \bar{F}(r,\alpha)\, d(r,\alpha) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$= \mu_1 - (\mu_1 - (\mu_1 - \sigma_1)) + \int_{\mu_1-\sigma_1}^{\mu_1} \bar{F}(r,\alpha)\, d(r,\alpha) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$
$$= (\mu_1 - \sigma_1) + \int_{\mu_1-\sigma_1}^{\mu_1} \bar{F}(r,\alpha)\, d(r,\alpha) - \int_{\mu_1}^{\mu_1+\sigma_1} \bar{F}(r,\alpha)d(r,\alpha)$$

Hence the proof.

Lemma: 3.7

Let R and T two triangular fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. (i.e.)[$(\mu_1 - \sigma_1) \le R \le (\mu_1 + \sigma_1)$and $(\mu_2 - \sigma_2) \le T \le (\mu_2 + \sigma_2)$], the co-ordinate points of the dependent triangular fuzzy numbers are arranged as follows [s ≤ t ≤ u ≤ v] with the convex survival functions $\overline{F}$ and$\overline{G}$ respectively. If whenever s ≤ t andu ≤ v.Here,s = μ_1–σ_1,t = μ_2–σ_2, u = μ_1 + σ_1 andv = μ_2 + σ_2. Assume ($\mu_1 = \mu_2$ and σ_1= σ_2),

$$\int_{\mu_1}^{\mu_1+\sigma_1} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(r,\alpha)$$

$=\int_{\mu_2}^{\mu_2+\sigma_2} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(t,\alpha)$. ThenR $\le_{cx}$ T

if and only if

$$\int_{(r,\alpha)}^{\mu_1+\sigma_1} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(r,\alpha)$$
$$\le \int_{(t,\alpha)}^{\mu_2+\sigma_2} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(t,\alpha),$$

for all (r,α)∈ $[(\mu_1 - \sigma_1), (\mu_1 + \sigma_1)]$.

Proof:

By lemma: 3.6, we have

$$\int_{(r,\alpha)}^{\infty} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(r,\alpha)$$
$$= (\mu_1 - \sigma_1) - (r,\alpha) + \int_{\mu_1-\sigma_1}^{\mu_1} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(r,\alpha)$$
$$- \int_{\mu_1}^{\mu_1+\sigma_1} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \le 0\}d(r,\alpha)$$

$= (\mu_2 - \sigma_2) - (r,\alpha) + \int_{\mu_2-\sigma_2}^{\mu_2} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(t,\alpha)$

$- \int_{\mu_2}^{\mu_2+\sigma_2} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(t,\alpha),$

for $(r,\alpha) \le (\mu_1 - \sigma_1), (t,\alpha) \le (\mu_2 - \sigma_2)$

$= \int_{(r,\alpha)}^{\infty} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(r,\alpha)$

$\int_{(r,\alpha)}^{\infty} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(r,\alpha)$

$= \int_{(r,\alpha)}^{\mu_1+\sigma_1} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(r,\alpha)$

$\le \int_{(t,\alpha)}^{\mu_2+\sigma_2} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(t,\alpha)$

$= \int_{(t,\alpha)}^{\infty} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(t,\alpha),$

for $(\mu_1 - \sigma_1) \le (r,\alpha) \le (\mu_1 + \sigma_1), (\mu_2 - \sigma_2) \le (t,\alpha) \le (\mu_2 + \sigma_2)$

$\int_{(r,\alpha)}^{\infty} P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(r,\alpha) = 0$

$= \int_{(t,\alpha)}^{\infty} P\{((1-b)T_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(t,\alpha),$

for $(r,\alpha) \ge (\mu_1 + \sigma_1), (t,\alpha) \ge (\mu_2 + \sigma_2)$

Then, applying Theorem 3.5, we obtain the conclusion.

IV. INTEGRAL INEQUALITIES OF CONVEX ORDERINGS OF TRIANGULAR FUZZY NUMBER

In this Section, we will use the following notations: For R be a triangular fuzzy random variable with mean μ and standard deviation σ (i.e.,) it lies in $[\mu_1 - \sigma_1, \mu_1 + \sigma_1]$.

$A\left(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right)$

$= \frac{P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\} + P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}}{2}$

$G\left(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right)$

$= \sqrt{P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}\, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}}$

$H\left(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right)$

$= \frac{2P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}\, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}}{P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\} + P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}}$

$L\left(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right)$

$= \frac{P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\} - P\{(bR_\alpha^L - (\alpha-1)b\sigma 1 - b\mu 1) \ge 0\}}{\ln P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\} - \ln P\{(bR_\alpha^L - (\alpha-1)b\sigma 1 - b\mu 1) \ge 0\}}$

$P\left(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right)$

$$= \frac{1}{3}\left\{\begin{matrix} 2G\left((P\{(bR_\alpha^L - (\alpha-1)b\sigma 1 - b\mu 1) \ge 0), P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right) \\ + \\ A\left(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\right) \end{matrix}\right\}$$

Where A,G,H and L refer to the arithmetic, geometric, harmonic and logarithmic mean respectively

Note that

$\text{Min}\{P\{(bR_\alpha^U + (\alpha-1)b\sigma 1 - b\mu 1) \le 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\} \le$

Harmonic ≤ Geometric ≤ Logarithmic ≤ Classical Polya's ≤ Arithmetic

$\le \text{Max}\{P\{(bR_\alpha^U + (\alpha-1)b\sigma 1 - b\mu 1) \le 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}\}$ (4.1)

Theorem: 4.1

If R be a triangular fuzzy random variable with parameters mean μ and standard deviation σ and a function f: $[\mu_1 - \sigma_1, \mu_1 + \sigma_1] \to [0, 1]$ is a log - convex function of the triangular fuzzy number R. Then: $P\{(bR_\alpha^L - b\mu 1) \ge 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \le 0\}$

$\le \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L - (\alpha-1)b\sigma - b\mu) \ge 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\} d(r,\alpha)$

$\le \frac{P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \ge 0\} + P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \le 0\}}{2}$

Proof:

By the Hermite – Hadamard inequalities is $f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$

This type of inequalities is also hold when f is log – convex function of the triangular fuzzy number. Here, $f(r,\alpha) = P\{(bR_\alpha^L - (\alpha-1)b\sigma - b\mu) \ge 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \le 0\}$. Then: $P\{(bR_\alpha^L - b\mu 1) \ge 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \le 0\}$

$$\leq e^{\left[\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} \ln[P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\}]d(r,\alpha)\right]}$$

$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} G(f(r,\alpha), f((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-(r,\alpha))d(r,\alpha)$$

$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} G\big((f(r,\alpha), f(r,\alpha)\big)d(r,\alpha) \qquad \text{(Since R is even)}$$

$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} f(r,\alpha)d(r,\alpha)$$

$$= \frac{1}{2\sigma_1}[F(r,\alpha)]_{\mu_1-\sigma_1}^{\mu_1+\sigma_1}$$

$$= \frac{1}{2\sigma_1}[F(\mu_1+\sigma_1)-F(\mu_1-\sigma_1)]$$

$$= \frac{1}{2\sigma_1}\left[P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\} - P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}\right]$$

$$= L\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$$

$$\leq \frac{P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\} + P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}}{2}$$

$$= A\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$$

Where $G(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$

$$= \sqrt{P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0\}P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}}$$

is the geometric mean of the positive real values $(\mu_1-\sigma_1), (\mu_1+\sigma_1)$ of R and

$$L(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$$

$$= \frac{P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\} - P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0\}}{\ln P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\} - \ln P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0\}}$$

is the logarithmic mean of the positive real values $(\mu_1-\sigma_1), (\mu_1+\sigma_1)$ of the triangular fuzzy number R. $\Big(\text{for } (\mu_1-\sigma_1)=(\mu_1+\sigma_1), \text{we put } L\big((\mu_1-\sigma_1),(\mu_1-\sigma_1)\big) = (\mu_1-\sigma_1)\Big)$

Lemma: 4.2

If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ and f: $[\mu_1-\sigma_1, \mu_1+\sigma_1] \to [0, 1]$ is a log - convex function of the triangular fuzzy number R. Then the following inequality holds:

1. $$\sqrt{P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}\, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}}$$
$$\geq \begin{cases} (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\})^{\mathbf{1-b}}\big(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\big)^{\mathbf{b}}, & b\in[0,\frac{1}{2}] \\ (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\})^{\mathbf{b}}\big(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\big)^{\mathbf{1-b}}, & b\in(\frac{1}{2},1] \end{cases}$$

Here $0\leq b\leq 1$.the following inequality holds:

2. $$\sqrt{(P\{(bR_\alpha^L-(\alpha-1)b\sigma 1-b\mu 1)\geq 0)(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})}$$
$$\leq (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\})^{\mathbf{1-b}}\big(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\big)^{\mathbf{b}}$$
$$+(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\})^{\mathbf{b}}\big(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\big)^{\mathbf{1-b}}$$
$$\leq P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\} + P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$

Lemma: 4.3

If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ and f: $[\mu_1-\sigma_1, \mu_1+\sigma_1] \to [0, 1]$ is a log - convex function of the triangular fuzzy number R. Then the following inequality holds:

1. $G\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$
$$= \sqrt{P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}\, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}}$$
$$\leq \int_0^1 (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\})^{\boldsymbol{b}}\big(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\big)^{\boldsymbol{1-b}}\, d(b,\alpha)$$
$$= \int_0^1 (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\})^{\mathbf{1-b}}\big(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\big)^{\mathbf{b}}\, d(b,\alpha)$$
$$\leq \frac{P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\} + P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}}{2}$$

$= A (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

2. Classical Polya inequality:

$$\int_0^1 (P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\})^{\mathbf{b}} \left(P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}\right)^{\mathbf{1-b}} d(b,\alpha)$$

$$= L(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$$

$\leq P(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

Theorem: 4.4

If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ.Let f: $[\mu_1- \sigma_1, \mu_1+\sigma_1] \rightarrow [0, 1]$ be a log - convex function of the triangular fuzzy number R. Then the following inequality holds:

$\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma- b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \leq 0\} d(r,\alpha)$

$\leq P(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

$\leq A(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

Proof:

Let f: $[\mu_1- \sigma_1, \mu_1+\sigma_1] \rightarrow [0, 1]$ be a log - convex function of the triangular fuzzy number R.

we have $P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}$

$\leq (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{1-b}},$

$P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \geq 0 \vee (bR_\alpha^U + (\alpha-1)b\sigma_1-b\mu_1) \leq 0\}$

$\leq (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{1-b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{b}}, \quad 0 \leq b \leq 1.$

From the above two inequalities, we have

$$P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}$$
$$+ P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \geq 0 \vee (bR_\alpha^U + (\alpha-1)b\sigma_1-b\mu_1) \leq 0\}$$
$$\leq (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{1-b}}$$
$$+(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{1-b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{b}}$$

Integrating both sides of the above inequality over [0, 1] and by using lemma 1, we obtain

$\frac{2}{\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma- b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \leq 0\} d(r,\alpha)$

$=\int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}$

$+P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \geq 0 \vee (bR_\alpha^U + (\alpha-1)b\sigma_1-b\mu_1) \leq 0\}\} d(b,\alpha)$

$$\leq \int_0^1 \{(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{1-b}}$$

$+(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{1-b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{b}}\} d(b,\alpha)$

$= P(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

Which implies that

$\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma- b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \leq 0\} d(r,\alpha)$

$\leq L(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

$$\leq \frac{P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\} + P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\}}{2}$$

$= A(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

Where we have used the fact that

1. $\int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\} d(b,\alpha)$

$= \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma- b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \leq 0\} d(r,\alpha)$

2. $\int_0^1 P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \geq 0 \vee (bR_\alpha^U + (\alpha-1)b\sigma_1-b\mu_1) \leq 0\} d(b,\alpha)$

$= \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma- b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma-(1-b)\mu) \leq 0\} d(r,\alpha)$

3. $\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{1-b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{b}} d(b,\alpha)$

$= \int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1) \geq 0\})^{\mathbf{b}} \left(P\{(R_\alpha^U + (\alpha-1)\sigma_1-\mu_1) \leq 0\}\right)^{\mathbf{1-b}} d(b,\alpha)$

$\leq P(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1-(1-b)\mu_1) \leq 0\})$

Theorem: 4.5

If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ. Let f: $[\mu_1 - \sigma_1, \mu_1 + \sigma_1] \to [0, 1]$ be a log - convex function of the triangular fuzzy number R. Then the following inequality holds:

$$\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1}\left\{\begin{matrix} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\} \\ f(((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-x),\alpha)\end{matrix}\right\} d(r,\alpha)$$

$$\leq G^2\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$

Proof:

Let f: $[\mu_1 - \sigma_1, \mu_1 + \sigma_1] \to [0, 1]$ be a log - convex function of the triangular fuzzy number R.

we have $P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$

$$\leq (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}},$$

$$P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\geq 0 \vee (bR_\alpha^U+(\alpha-1)b\sigma_1-b\mu_1)\leq 0\}$$

$$\leq (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{1-b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{b}}, \quad 0\leq b\leq 1.$$

Using the above two inequalities, we have

1. $$\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1}\left\{\begin{matrix} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\} \\ f(((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-x),\alpha)\end{matrix}\right\} d(r,\alpha)$$

$$\leq \int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$

$$P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\geq 0 \vee (bR_\alpha^U+(\alpha-1)b\sigma_1-b\mu_1)\leq 0\}\}\ d(b,\alpha)$$

$$\leq \int_0^1 \{(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}$$

$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{1-b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{b}}\}\ d(b,\alpha)$$

$$= \int_0^1 P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\ d(b,\alpha)$$

$$= P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$

$$= G^2\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}) \qquad (4.2)$$

2. $$\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1}\left\{\begin{matrix} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\} \\ f(((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-r),\alpha)\end{matrix}\right\} d(r,\alpha)$$

$$\leq \int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$

$$P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\geq 0 \vee (bR_\alpha^U+(\alpha-1)b\sigma_1-b\mu_1)\leq 0\}\}\ d(b,\alpha)$$

$$\leq \frac{1}{2}\int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2$$

$$+P\{((1-b)R_\alpha^L-(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\geq 0 \vee (bR_\alpha^U+(\alpha-1)b\sigma_1-b\mu_1)\leq 0\}^2\}\ d(b,\alpha)$$

$$\leq \frac{1}{2}\int_0^1 \left\{\begin{matrix}\left((P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\right)^2 \\ +\left((P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\})^{\mathbf{1-b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{b}}\right)^2\end{matrix}\right\} d(b,\alpha)$$

$$\leq \frac{1}{2}\int_0^1 \left\{\left((P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\right)^2\right\} d(b,\alpha)$$

$$+\frac{1}{2}\int_0^1 \left\{\left((P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\})^{\mathbf{1-b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{b}}\right)^2\right\} d(b,\alpha)$$

$$= L\,((P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\})^{\mathbf{2}}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^{\mathbf{2}}) \qquad (4.3)$$

From the inequalities (4.2) and (4.3), we get

$$G^2\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$$

$$\leq L\,((P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\})^{\mathbf{2}}, \left(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right)^{\mathbf{2}})$$

$$= \frac{\left(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right)^{\mathbf{2}}-\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\leq 0)\}\right)^{\mathbf{2}}}{2(\text{In}P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}-\text{In}P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}} \qquad (4.4)$$

From (4.2), (4.3) and (4.4), we get the required result.

Theorem: 4.6
If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ.
Letf: $[\mu_1 - \sigma_1, \mu_1+\sigma_1] \to [0, 1]$ be a log - convex function of the triangular fuzzy number R. Then the following inequality holds:

$$P\{(bR_\alpha^L - b\mu 1) \geq 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \leq 0\}$$
$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L - (\alpha-1)b\sigma - b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \leq 0\}d(x,\alpha)$$
$$\leq P\,(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\})$$

Proof:
Let f: $[\mu_1 - \sigma_1, \mu_1+\sigma_1] \to [0, 1]$ be a log - convex function of the triangular fuzzy number R. we have

$$P\{(bR_\alpha^L - b\mu 1) \geq 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \leq 0\} = \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} f\left(\frac{(((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-r+r),\alpha)}{2}\right)d(r,\alpha)$$
$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} f^{\frac{1}{2}}(r,\alpha) f\left(\frac{(((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-r),\alpha)}{2}\right) f^{\frac{1}{2}}(r,\alpha)\,d(r,\alpha)$$
$$\leq \frac{1}{2\sigma_1}\left\{\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} f\,(((\mu_1-\sigma_1)+(\mu_1+\sigma_1)-r),\alpha))\; d(r,\alpha)\right\}^{\frac{1}{2}}\left\{\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} f(r,\alpha)\,d(r,\alpha)\right\}^{\frac{1}{2}}$$
$$= \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L - (\alpha-1)b\sigma - b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \leq 0\}\,d(r,\alpha)$$
$$= \frac{1}{2}\int_0^1 \{P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\}\,d(b,\alpha)$$
$$+\ \frac{1}{2}\int_0^1 P\{((1-b)R_\alpha^L - (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \geq 0 \vee (bR_\alpha^U + (\alpha-1)b\sigma_1 - b\mu_1) \leq 0\}\}\,d(b,\alpha)$$
$$\leq \frac{1}{2}\int_0^1 (P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\right)^{\mathbf{1-b}} d(b,\alpha)$$
$$+\ \frac{1}{2}\int_0^1 (P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0\})^{\mathbf{1-b}}\left(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\right)^{\mathbf{b}} d(b,\alpha)$$

By the Polya inequality and the above inequality, we get

$$P\{(bR_\alpha^L - b\mu 1) \geq 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \leq 0\}$$
$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L - (\alpha-1)b\sigma - b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \leq 0\}d(x,\alpha)$$
$$= \frac{1}{2}\begin{bmatrix} \int_0^1 \{P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\}\,d(b,\alpha) \\ + \int_0^1 P\{((1-b)R_\alpha^L - (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \geq 0 \vee (bR_\alpha^U + (\alpha-1)b\sigma_1 - b\mu_1) \leq 0\}\}\,d(b,\alpha) \end{bmatrix}$$
$$\leq P\,(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\})$$
$$\leq A\,(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\})$$

Therefore, we get

$$P\{(bR_\alpha^L - b\mu 1) \geq 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \leq 0\}$$
$$\leq \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L - (\alpha-1)b\sigma - b\mu) \geq 0 \vee P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma - (1-b)\mu) \leq 0\}d(r,\alpha)$$
$$\leq (P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\})$$

Hence, the inequalities proved.

Theorem: 4.7
If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ. Let f: $[\mu_1 - \sigma_1, \mu_1+\sigma_1] \to [0, 1]$ be an increasing log - convex function of the triangular fuzzy number R. Then the following inequality holds: $P\{(bR_\alpha^L - b\mu 1) \geq 0 \vee ((1-b)R_\alpha^U - (1-b)\mu 1) \leq 0\}$

$$L\,(P\{bR_\alpha^L - (\alpha-1)b\sigma_1 - \mu_1 b \geq 0\}, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\})$$
$$\leq \int_0^1 \{P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\}$$
$$(P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$
$$\leq \frac{1}{2}\begin{bmatrix} A\left(P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0\}^2, P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\}^2\right) \\ +L\left(P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0\}^2,\ P\{((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\}^2\right) \end{bmatrix}$$

Proof:
Let f: $[\mu_1 - \sigma_1, \mu_1+\sigma_1] \to [0, 1]$ be an increasing log - convex function of the triangular fuzzy number R. we have

$$P\{(bR_\alpha^L - (\alpha-1)b\sigma_1 - b\mu_1) \geq 0 \vee ((1-b)R_\alpha^U + (\alpha-1)(1-b)\sigma_1 - (1-b)\mu_1) \leq 0\}$$
$$\leq (P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\right)^{\mathbf{1-b}},\ 0 \leq b \leq 1.$$

Using the inequality 2xy $\leq x^2 + y^2$ (x, y) $\in$ R, we have

$$2P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$
$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}$$
$$\leq P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2$$
$$+(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{2b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{2(1-b)}}, \quad (4.5)$$

Integrating on both sides of the inequality over (0, 1), we obtain

$$1. \int_0^1\{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$
$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$
$$\leq \frac{1}{2}\left[\int_0^1\{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2 d(b,\alpha)\right]$$
$$+\frac{1}{2}\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{2b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{2(1-b)}} d(b,\alpha)$$
$$=\frac{1}{2}\left[\begin{array}{l}\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\}^2 d(r,\alpha)\\ +L\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\}^2, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2\right)\end{array}\right]$$
$$\leq \frac{1}{2}\left[\begin{array}{l}A\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\}^2, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2\right)\\ +L\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\}^2, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2\right)\end{array}\right] \quad (4.6)$$

$$2. \int_0^1\{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$
$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$
$$\geq \left[\int_0^1 P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2 d(b,\alpha)\right]$$
$$\left[\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}} d(b,\alpha)\right]$$
$$=L\left(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right)$$
$$\left[\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\} d(r,\alpha)\right]$$
$$\geq P\{(bR_\alpha^L-b\mu 1)\geq 0 \vee ((1-b)R_\alpha^U-(1-b)\mu 1)\leq 0\}$$
$$L\left(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right) \quad (4.7)$$

From (4.6) and (4.7), we get

$$P\{(bR_\alpha^L-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U-(1-b)\mu_1)\leq 0\}$$
$$L\left(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\right)$$
$$\leq \int_0^1\{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^2 d(b,\alpha)$$
$$\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}} d(b,\alpha)$$
$$\leq \int_0^1\{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$
$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$
$$\leq \frac{1}{2}\left[\int_0^1\{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\}^2 d(b,\alpha)\right]$$

$$+\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{2b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{2(1-b)}} d(b,\alpha)$$

$$\le \frac{1}{2}\begin{bmatrix} A\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0\}^2, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}^2\right) \\ +L\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0\}^2, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}^2\right)\end{bmatrix}$$

Hence, the proof.

Theorem: 4.8

If R be the triangular fuzzy random variable with parameters mean μ and standard deviation σ.

Let f: $[\mu_1\text{-}\ \sigma_1,\ \mu_1+\sigma_1] \to [0, 1]$ be an increasing log - convex function of the triangular fuzzy number R. Then the following inequality holds:$P\{(bR_\alpha^L-b\mu 1)\ge 0 \vee ((1-b)R_\alpha^U-(1-b)\mu 1)\le 0\}$

$L(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\ge 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\})$

$$\le \int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}$$

$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{b}}\left(P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$

$$\le \frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\ge 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\le 0\}^4\, d(x,\alpha)$$

$$+L\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0\}^4, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}^4\right)+\frac{1}{8}$$

Proof:

Let f: $[\mu_1\text{-}\ \sigma_1,\ \mu_1+\sigma_1] \to [0, 1]$ be an increasing log - convex function of the triangular fuzzy number R. we have

$P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}$

$$\le (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{1-b}}, \qquad 0\le b\le 1.$$

Using the inequality $xy \le x^4+y^4+\frac{1}{8}$,(x, y) ∈ R, we have

$$P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}$$
$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{1-b}}$$
$$\le P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}^4$$
$$+(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{4b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{4(1-b)}}+\mathbf{\frac{1}{8}}$$

Integrating on both sides of the above inequality over [0, 1], we obtain

1. $\int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}$

$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$

$$\le \left[\int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}^4 d(b,\alpha)\right]$$

$$+\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{4b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{4(1-b)}} d(b,\alpha)+\frac{1}{8}$$

$$=\frac{1}{2\sigma_1}\int_0^1 P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\ge 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\le 0\}^4\, d(b,\alpha)$$
$$+A\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0\}^2,\ P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}^2\right)$$
$$A(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\ge 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\})$$
$$L(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\ge 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\})+\frac{1}{8} \qquad (4.8)$$

And $\int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\}$

$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$

$$\ge \left[\int_0^1 P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\ge 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\} d(b,\alpha)\right]$$

$$\left[\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\le 0\}\right)^{\mathbf{1-b}} d(b,\alpha)\right]$$

$$=L(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\ge 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\le 0\})$$

$$\left[\frac{1}{2\sigma_1}\int_{\mu_1-\sigma_1}^{\mu_1+\sigma_1} P\{(bR_\alpha^L-(\alpha-1)b\sigma-b\mu)\geq 0 \vee P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma-(1-b)\mu)\leq 0\}\, d(x,\alpha)\right]$$

$$\geq P\{(bR_\alpha^L-b\mu 1)\geq 0 \vee ((1-b)R_\alpha^U-(1-b)\mu 1)\leq 0\}$$

$$L\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}) \quad (4.9)$$

From (4.8) and (4.9), we get

$$P\{(bR_\alpha^L-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U-(1-b)\mu_1)\leq 0\}$$

$$L\,(P\{bR_\alpha^L-(\alpha-1)b\sigma_1-\mu_1 b\geq 0\}, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\})$$

$$\leq \int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}\, d(b,\alpha)$$

$$\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\, d(b,\alpha)$$

$$\leq \int_0^1 \{P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee ((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}$$

$$(P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{1-b}}\}d(b,\alpha)$$

$$\leq \frac{1}{2}\left[\int_0^1 \begin{matrix}\left\{P\left\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0 \vee \left((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1\right)\leq 0\right\}\right\}^4 d(b,\alpha)\\ +\int_0^1 (P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0\})^{\mathbf{4b}}\left(P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\right)^{\mathbf{4(1-b)}} d(b,\alpha)+\frac{1}{8}\end{matrix}\right]$$

$$\leq \frac{1}{2}\left[\begin{matrix}A\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\}^4, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^4\right)\\ +L\left(P\{(bR_\alpha^L-(\alpha-1)b\sigma_1-b\mu_1)\geq 0\}^4, P\{((1-b)R_\alpha^U+(\alpha-1)(1-b)\sigma_1-(1-b)\mu_1)\leq 0\}^4\right)+\frac{1}{8}\end{matrix}\right]$$

Hence, the proof.

REFERENCES

[1]. H. Kwakernaak, Fuzzy random variables – I, Definitions and theorems, Information Science, Vol.15, 1978, 1 – 29.

[2]. Aurelia Florea, Eugen Paltanea and Dumitru Bala. "Convex ordering properties and applications", Journal of Mathematical Inequalities, Volume 9(4), 2015, 1245 – 1257.

[3]. Jaekeun Park, "Some Hermite – Hadamard – like Type Inequalities for Logarithmically Convex Functions", Int. journal of Math..Analysis, Volume 7(45), 2013, 2217 – 2233.

[4]. Mevlut Tunc, "Some integral inequalities for logarithmically convex functions", Journal of the Egyptian Mathematical Society. Volume 22, 2014, 177 – 181.

[5]. A. Nagoor Gani, and S.N. Mohamed Assarudeen "A New operation on triangular fuzzy number for solving fuzzy linear programming problem", Applied Mathematical Sciences, Vol.6(11), 2012, 525-532.

[6]. Stochastic ordering-from wikipedia, the free encyclopedia.