International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 2 || February. 2017 || PP-53-57

New Oscillation Criteria for Second Order Neutral Difference Equations

A. Murugesan[1] and K. Ammamuthu[2]

[1] *Department of Mathematics,Government Arts College (Autonomous), Salem – 636 007, Tamil Nadu, India.*
[2] *Department of Mathematics, Arignar Anna Government Arts College, Attur– 636 121, Tamil Nadu, India.*

ABSTRACT: *In this paper, we discuss the oscillatory properties of a class of second order neutral difference equation relating oscillation of these equation to existence of positive solutions to associated first order neutral difference inequalities. Our assumptions allow application to difference equations with delayed and advanced arguments, and not only. Examples are given to illustrate our results.*

MATHEMATIES SUBJECT CLASSIFICATION (2010): *39A10, 39A12*
KEYWORDS AND PHRASES: *Oscillation, nonoscillation, positive solutions, second order, delay, advanced, neutral difference equation.*

I. INTRODUCTION

In this paper, we discuss the oscillatory properties of a class of second order neutral difference equation of the form

$$\Delta\left(r(n)\Delta\big(x(n)+p(n)x(n+\tau)\big)\right)+q(n)x(n+\sigma)=0, \qquad n\geq n_0 \qquad (1.1)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$.

Throughout the paper the following conditions are assumed to be hold:

(i) $\{p(n)\}$ is a sequence of nonnegative real numbers and there exist a constant p such that $0\leq p(n)\leq p<\infty$;

(ii) $\{q(n)\}$ is a sequence of nonnegative real numbers and is not identically zero for sufficiently large values of n;

(iii) $\{r(n)\}$ is a sequence of postive real numbers;

(iv) τ and σ are integers.

(v) $\lim_{n\to\infty} R(n)<\infty$, where

$$R(n)=\sum_{s=n_0}^{n-1}\frac{1}{r(s)};$$

If $\{x(n)\}$ is a solution of (1.1),then its associated sequence $\{z(n)\}$ is defined by

$$z(n)=x(n)+p(n)x(n+\tau). \qquad (1.2)$$

By a solution of (1.1),we mean a real sequence $\{x(n)\}$ which is defined for $n^*\geq min\ \{n_0, n_0+\tau, n_0+\sigma\}$ and satisfies (1.1) for $n\geq n^*$. We consider only such solution which are nontrivial for all large n. A solution $\{x(n)\}$of (1.1) is said to be nonoscillatory if the terms $x(n)$ of the sequence are eventually positive or eventually negative. Otherwise it is called Oscillatory.

Recently,there has been an increasing interest in studying the oscillatory and asymptotic behavior of second order difference equation; see, for example [3,5-10] and the references cited therein. For the general background of difference equations one can refer to [1,2].

In [4], we established sufficient conditions under which every solution of the equation

$$\Delta\left(r(n)\Delta\big(x(n)-p(n)x(n-\tau)\big)\right)+q(n)f(x(n-\sigma))=0 \qquad (1.3)$$

is either oscillatory or tends to zero,

Our aim of this paper is to discuss the oscillatory behavior of solutions to equation (1.1).Our established results are discrete analogues of some well-known results due to [4].

In the sequel, for our convenience, when we write a fractional inequality without mentioning its domain of validity we assume that it holds for all sufficiently large values of n.

II. MAIN RESULTS

In this sequel, all inequalities are assumed to be hold eventually, that is, for all sufficiently large n. We also use the following notations.

$y(n) = -v(n) = r(n)\Delta z(n),$

$Q(n) = min\{q(n), q(n+\tau)\},$

and

$$\delta(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)}.$$

Theorem 2.1. *Let n_1 be large enough. Suppose that there exist two integers α and β such that $\alpha \le \sigma \le \beta$. If the first order neutral difference inequalities*

$$\Delta\big(y(n) + py(n+\tau)\big) + Q(n)\big(R(n+\alpha) - R(n+1)\big)y(n+\alpha) \le 0 \qquad (2.1)$$

and

$$\Delta\big(w(n) + pw(n+\tau)\big) - Q(n)\delta(n+\beta)w(n+\beta) \ge 0 \qquad (2.2)$$

has no positive solutions, then every solution of (1.1) is oscillatory.

Proof: Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). Then from (1.1) and (1.2), we have

$$\Delta\big(r(n)\Delta z(n)\big) + p\Delta\big(r(n+\tau)\Delta z(n+\tau)\big) + qx(n+\sigma) + pq(n+\tau)x(n+\tau+\sigma) = 0$$

or

$$\Delta\big(r(n)\Delta z(n)\big) + p\Delta\big(r(n+\tau)\Delta z(n+\tau)\big) + Q(n)z(n+\sigma) \le 0. \qquad (2.3)$$

Equation (1.1) yields that, for some n_1 large enough and for all $n \ge n_1$ either,

$$\Delta z(n) > 0, \qquad \Delta\big(r(n)\Delta z(n)\big) < 0, \qquad (2.4)$$

or

$$\Delta z(n) < 0; \qquad \Delta\big(r(n)\Delta z(n)\big) < 0. \qquad (2.5)$$

Assume that (2.4) holds. Inequality (2.3) and the fact that $\alpha \le \sigma$ yields

$$\Delta\big(r(n)\Delta z(n)\big) + p\Delta\big(r(n+\tau)\Delta z(n+\tau)\big) + Q(n)z(n+\alpha) \le 0. \qquad (2.6)$$

It follows from (2.4) that

$$z(n) \ge \sum_{s=n_1}^{n-1} \frac{r(s)\Delta z(s)}{r(s)} \ge r(n)\Delta z(n) \sum_{s=n_1}^{n-1} \frac{1}{r(s)} = y(n)\big(R(n) - R(n_1)\big). \; (2.7)$$

Using (2.7) in (2.6), we see that $\{y(n)\}$ is an eventually positive solution of the inequalities (2.1), which contradicts our assumption that (2.1) has no positive solution.

Consider now the second case. If follows from (2.5) that

$$\Delta z(s) \le \frac{r(n)\Delta z(n)}{r(s)} \text{ for all } s \ge n. \qquad (2.8)$$

Summing from n to $l-1$, we have

$$z(l) \le z(n) + r(n)\Delta z(n) \sum_{s=n}^{l-1} \frac{1}{r(s)}.$$

Taking limit $l \to \infty$, we get

$$z(n) + r(n)\Delta z(n)\delta(n) \ge 0$$

or

$$z(n) \ge -r(n)\Delta z(n)\delta(n). \qquad (2.9)$$

Using (2.9) in (2.3) and the fact that $\sigma \le \beta$, we have

$$\Delta\big(r(n)\Delta z(n)\big) + p\Delta\big(r(n+\tau)\Delta z(n+\tau)\big) + Q(n)z(n+\beta) \le 0. \qquad (2.10)$$

Then, $y(n) < 0$and by virtue of (2.9) and (2.10), we have

$$\Delta\big(y(n) + py(n+\tau)\big) - Q(n)\delta(n+\beta)y(n+\beta) \le 0,$$

or

$$\Delta\big(u(n) + pu(n+\tau)\big) - Q(n)\delta(n+\beta)u(n+\beta) \ge 0, \qquad (2.11)$$

which shows that $\{u(n)\}$ is a positive solution the inequality (2.2), which according to our assumption, has no positive solutions. This is a contradiction and the proof is completed.

Theorem 2.2. *Assume that $\tau \ge 0$ and n_1 be largeenough. Suppose that there exist two integers α and β such that $\alpha \le \sigma \le \beta$. If the first order difference inequalities*

$$\Delta g(n) + \frac{1}{1+p}Q(n)\big(R(n+\alpha) - R(n_1)\big)g(n+\alpha) \le 0 \qquad (2.12)$$

and

$$\Delta\Box(n) - \frac{1}{1+p}Q(n)\delta(n+\beta)\Box(n+\beta-\tau) \ge 0 \qquad (2.13)$$

have no positive solutions, thenevery solution of (2.13) is oscillatory.

Proof: Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). As in the proof of Theorem 2.1, one arrive at the inequality (2.3);
Equation (1.1) yields that, for some n_1 sufficiently large enough and for all $n \ge n_1$, either

$$\Delta z(n) > 0, \qquad \Delta\big(r(n)\Delta z(n)\big) < 0, \qquad (2.14)$$

or

$$\Delta z(n) < 0, \qquad \Delta\big(r(n)\Delta z(n)\big) < 0. \qquad (2.15)$$

Assume first that (2.14) holds. By repeating the procedure as we followed in Theorem 2.1 we arrive at the inequality (2.1). Set

$$v(n) = y(n) + p\,y(n+\tau). \qquad (2.16)$$

Using the fact that $\tau \ge 0$ and the decreasing nature of $\{y(n)\}$, we have

$$v(n) \le y(n)(1+p). \qquad (2.17)$$

Substituting (2.17) in (2.1), we see that $\{v(n)\}$ is a positive solution of (2.12), which contradicts our assumption.

Consider the second case. If follows from (2.15) as we have shown in Theorem 2.1, $\{u(n)\}$ is positive, increasing and satisfies (2.2). That is,

$$\Delta\big(u(n) + pu(n+\tau)\big) - Q(n)\big(\delta(n+\beta)u(n+\beta)\big) \ge 0. \qquad (2.18)$$

Set

$$w(n) = u(n) + p\,u(n+\tau). \qquad (2.19)$$

Using the fact that $\tau \ge 0$, we obtain

$$w(n) \le u(n+\tau)(1+p). \qquad (2.20)$$

Substituting (2.20) in (2.18), we see that $\{w(n)\}$ is a positive solution of (2.13), which leads to a contradiction. This completes the proof.

Combining Theorem 2.2 with the oscillation results presented in Gyori et al. [2], we obtain the following result.

Corollary 2.3. *Assume that $\tau \ge 0$. Suppose that there exist two integers α and β such that $\alpha \le 0$, $\beta > \tau + 1$ and $\alpha \le \sigma \le \beta$. If for all sufficiently large $n_1 \ge n_0$,*

$$\liminf_{n\to\infty} \sum_{s=n+\alpha}^{n-1} Q(s)\left(R(n+\alpha)-R(n_1)\right) > (1+p)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \qquad (2.21)$$

and

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\beta-\tau-1} Q(s)\delta(s+\beta) > (1+p)\left(\frac{\beta-\tau-1}{\beta-\tau}\right)^{\beta-\tau}, \qquad (2.22)$$

then every solution of (1.1) is oscillatory.

Proof: By [2, Theorem 7.5.2], assumption (2.21) ensures that the delay difference inequality (2.12) has no positive solution. On the other hand, by [2, Theorem 7.5.2] condition (2.22) guarantees that the advanced difference inequality (2.13) has no positive solution. Application of Theorem 2.2 fields the result.

Theorem2.4. *Assume that $\tau \le 0$ and let n_1 be an integer large enough. Suppose that there exist two integers α and β such that $\alpha \le \sigma \le \beta$. If the first order difference inequalities*

$$\Delta g(n) + \frac{1}{1+p} Q(n)\left(R(n+\alpha)-R(n_1)\right)g(n+\alpha-\tau) \le 0 \qquad (2.23)$$

and

$$\Delta \square(n) - \frac{1}{1+p} Q(n)\delta(n+\beta)\square(n+\beta) \ge 0 \qquad (2.24)$$

have no positive solutions, then every solution of (1.1) oscillatory.

Proof: Assume the contrary.Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). As in the proof of Theorem 2.1, one arrive at the inequality (2.3)
Equation (1.1) yields that, for some n_1 sufficiently large enough and for all $n \ge n_1$ either

$$\Delta z(n) > 0, \qquad \Delta\left(r(n)\Delta z(n)\right) < 0, \qquad (2.25)$$

or

$$\Delta z(n) < 0, \qquad \Delta\left(r(n)\Delta z(n)\right) < 0. \qquad (2.26)$$

Assume that (2.25) holds. As we proved in the Theorem 2.1, we deduce the inequality (2.1). Set
$v(n) = y(n) + py(n+\tau)$.

Then

$$v(n) \le (1+p)y(n+\tau). \qquad (2.27)$$

Using (2.27) in (2.1), we see that the inequality (2.23) has a positive solution $\{v(n)\}$, which contradicts our assumption.
Consider now the second case. If follows from (2.26) as we have shown in Theorem 2.1, $\{u(n)\}$ is positive increasing and satisfies (2.2). That is

$$\Delta\left(u(n) + pu(n+\tau)\right) - Q(n)\delta(n+\beta)u(n+\beta) \ge 0 \qquad (2.28)$$

Set

$$w(n) = u(n) + pu(n+\tau). \qquad (2.29)$$

By virtue of $\tau \le 0$, we have, from (2.29)

$$w(n) \le u(n)(1+p). \qquad (2.30)$$

Using (2.30) in (2.28), we obtain

$$\Delta w(n) - \frac{1}{1+p} Q(n)\delta(n+\beta)w(n+\beta) \ge 0.$$

This shows that the equation (2.24) has a positive solution $\{w(n)\}$ which contradicts our assumption. The proof is complete.
Combining Theorem 2.4 with results in Gyori et al. [2], we obtain the following oscillation results.

Corollary 2.5. *Assume that* $\tau \leq 0$, *and there exist two integers* α *and* β *such that* $\alpha \leq \tau, \beta > 1$ *and* $\alpha \leq \sigma \leq \beta$. *If, for all sufficiently large* $n_1 \geq n_0$

$$\liminf_{n\to\infty} \sum_{s=n-\tau+\alpha}^{n-1} Q(s)\left(R(s+\alpha) - R(n_1)\right) > (1+p)\left(\frac{\tau-\alpha}{\tau-\alpha+1}\right)^{\tau-\alpha+1}, \qquad (2.31)$$

and

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\beta-1} Q(s)\delta(s+\beta) > (1+p)\left(\frac{\beta-1}{\beta}\right)^{\beta}, \qquad (2.32)$$

then every solution of (1.1) is oscillatory.

Proof: By [2, Theorem 7.5.2], condition (2.31) ensures that the difference inequality (2.23) has no positive solution. On the other hand, it follows from [2, Theorem 7.5.2] that condition (2.32) guarantees that difference inequality (2.24) has no positive solutions. Application of Theorem 2.4 completes the proof.

Remark 2.6. *By using the following inequalities*
$y_1^\alpha + y_2^\alpha \geq (y_1 + y_2)^\alpha;\ for\ 0 < \alpha \leq 1$ *and* all $y_1, y_2 \epsilon [0,\infty)$
and
$y_1^\alpha + y_2^\alpha \geq \frac{1}{2^{\alpha-1}}(y_1 + y_2)^\alpha;\ for\ \alpha \geq 1$ *and* all $y_1, y_2 \epsilon [0,\infty)$,

results proved in this paper can be extended to a second order half-linear neutral difference equation
$\Delta(r(n)(\Delta[x(n) + p(n)x(n+\tau)])^\alpha) + q(n)x^\alpha(n+\sigma) = 0;$

where $\alpha > 0$ *is a ratio of odd positive integers.*

III. EXAMPLES

The following examples illustrate applications of our results in the previous section.
Example 3.1. *consider the following second order neutral difference equation*

$$\Delta\left[2^n \Delta\left(x(n) + \frac{1}{n+1}x(n+1)\right)\right] + 2^{n+2}x(n+2) = 0; \quad n = 0,1,2,\ldots \qquad (3.1)$$

where $r(n) = 2^n,\ p(n) = \frac{1}{n+1},\ q(n) = 2^{n+2}, \tau = 1$ *and* $\sigma = 2$. *We can choose* $\alpha = -1$ *and* $\beta = 3$. *Also we see that p=1 and* $Q(n) = 2^{n+2}$ *and* $\delta(n) = \frac{1}{2^{n-1}}$.

Also $R(n) = 2 - \frac{1}{2^{n-1}}$. *We can easily verify that*

$\lim_{n\to\infty} R(n) < \infty$,

$$\liminf_{n\to\infty} \sum_{s=n+\alpha}^{n-1} Q(s)\left(R(n+\alpha) - R(n_1)\right) = \infty$$

and

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\beta-\tau-1} Q(s)\delta(s+\beta) = 1 > (1+p)\left(\frac{\beta-\tau-1}{\beta-\tau}\right)^{\beta-\tau} = \frac{1}{2}.$$

Hence all the conditions of the Corollary 2.3 are satisfied. Here by Corollary 2.3 every solution of (3.1) is oscillatory.
Example 3.2. *consider the following second order neutral difference equation*

$$\Delta\left[e^n \Delta\left(x(n) + \frac{1}{n+2}x(n-1)\right)\right] + e^{n+2}x(n+1) = 0; \quad n = 0,1,2,\ldots. \qquad (3.2)$$

Here, we have $r(n) = e^n, p(n) = \frac{1}{n+2}, q(n) = e^{n+2},\ \tau = -1,$ *and* $\sigma = 1$. *Clearly we see that* $p = \frac{1}{2}$, $Q(n) = e^{n+1}$ *and* $\delta(n) = \frac{1}{e^{n-1}(e-1)}$, $\alpha = -2$ *and* $\beta = 2$. *We can easily show that*

$$R(n) = \frac{e^n - 1}{e^{n-1}(e-1)}$$

and

$$\lim_{n\to\infty} R(n) < \infty.$$

Also, one can verify that

$$\lim_{n\to\infty} inf \sum_{s=n-\tau+\alpha}^{n-1} Q(s)\,[R(s+\alpha) - R(n_1)] = \infty$$

and

$$\lim_{n\to\infty} inf \sum_{s=n+1}^{n+\beta-1} Q(s)\,\delta(s+\beta) = \frac{1}{e-1} > (1+p)\left(\frac{\beta-1}{\beta}\right)^{\beta} = 3/8.$$

Hence all the conditions of Corollary 2.5 are satisfied and hence every solution of (3.2) is oscillatory.

REFERENCES

[1]. R. P. Agrwal, Difference Equations and Inequalities; Theory, Methods and Applications, Marcel Dekker, New York, 1999.

[2]. I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford,1991.

[3]. J.W. Hooker and W.T. Patula, A second order nonlinear difference equation: Oscillation and asymptotic behavior, J.Math.Anal.Appl.,91(1983), 9-29.

[4]. T.Li, Y.V.Rogovchenko and C.Zhang, Oscillation of second –order neutral differential equations, Funkcial.Ekvac, 56(2013), 111-120.

[5]. A. Murugesan and K. Ammamuthu, Conditions for oscillation and convergence of second order solutions to neutral delay difference equations with variable coefficients, Malaya Journal of Mathematik (Communicated).

[6]. R.N.Rath. S.Padhi and B.L.S.Barik, Oscillatory and asymptotic behavior of a homogeneous neutral delay difference equation of second order, Bull. Inst. Math., Acad. Sin. (N.S) 3(3)(2008), 453-467.

[7]. A. Sternal and B.Szmanda; Asymptotic and oscillatory behaviour of certain difference equations, LE MATHEMATICHE, Vol.LI (1996)- Fasc. I, 77-86.

[8]. B. Szmanda, Note on the behaviour of solutions of a second order nonlinear difference equation, Atti.Acad.Naz.Lincei, Rend. Sci. Fiz. Mat., 69(1980), 120-125.

[9]. E. Thandapani, Asymptotic and oscillatory behavior of solution nonlinear second order difference equations Indian J. Pure Appl. Math., 24(1993), 365-372.

[10]. E. Thandapani, Asymptotic and oscillatory behavior of solutions of a second order nonlinear neutral delay difference equation, Riv. Mat. Univ. Parma, (5) 1 (1992),105-113.