Birth and Death Processes_ General Case

M.EL OUSSATI¹, A.ARBAT²

¹(Department of Mathematics, Probability laboratory and statistics, University AbdelmalekEssaâdi, Morocco)
²(Department of Mathematics, Probability laboratory and statistics, University AbdelmalekEssaâdi, Morocco)
Corresponding Author: M.EL OUSSATI

Abstract: Birth and Death Processes (BDPs) is a continuous-time Markov chain that counts the number of particles in a system over time, they are popular modeling tools in population evolution, used more particularly in biology, genetics, epidemiology, ecology, demography, physics, sociology, statistic...etc.
In this paper we will complete the resolution of the Chapman Kolmogorov’s equation in the general case (See [1]), by giving the general form of the law of this processes.

Keywords: Problems modeling, Reduction of the matrix, Recurrent Sequences of Order 2 and Identification of the law.

Date of Submission: 02-10-2017 Date of acceptance: 13-10-2017

I. INTRODUCTION

- The lack of theoretical progress in developing statistical tools for dealing with data from Birth and Death Processes (BDPs) has hindered their adoption by applied researchers.
- Statistical inference of the instantaneous particle birth and death rates also remains largely limited to continuously observed processes in which per particle birth and death rates are constants.
- In this article, and in order to complete the resolution of Chapman Kolmogorov's equation in the general case, we present in the first section a reminder on the BDPs model and in the second section the procedure followed in the resolution of this equation and also the solution found for this equation.

1. Presentation of the model:

Our study will be limited on the states between 1 and n, so we identify the states of this process with:

\[ P_i(t) \text{ for } i = 1, \ldots, n, \]

Knowing that:

\[ P_i(t) = P(X_t = i) \quad (\text{Where } \sum_{i=1}^{n} P(X_t = i) = 1) \]

(Where \( X_t \) is a discrete and homogeneous stochastic process \( t \in \mathbb{R}^+ \))

Let \( P(t) \) the column matrix of type \((n, 1)\) such as:

\[ P(t)^{i} = (P_1(t), P_2(t), \ldots, P_n(t)) \quad (t \in \mathbb{R}^+) \]

Let,

\[ P_{ij}(\Delta t) = P(X_{t+\Delta t} = j|X_t = i) \quad \text{(1)} \]

(The transition probability of the state \( i \) to the state \( j \))

- The Birth and Death Process is defined as follows (See [2]):

\[
P_{ij}(\Delta t) = \begin{cases} 
\Delta t + o(\Delta t) & ; \quad \text{if } j = i + 1 \\
\mu_j \Delta t + o(\Delta t) & ; \quad \text{if } j = i - 1 \\
1 - (i+\mu_j)\Delta t + o(\Delta t) & ; \quad \text{if } j = 1 \\
o(\Delta t) & ; \quad \text{if } |j-i| \geq 2
\end{cases}
\]

- We have the following proposition (See demonstration in [1]):

\[
\text{Let } P_i(t) \text{ is Birth and Death Process (} i = 1, \ldots, n \).
\text{With } \mu_k (k = 1, \ldots, n) \text{ is the birth rate and } \mu_k (k = 1, \ldots, n) \text{ is the death rate.}
\]
### Birth and Death Processes_ General Case

- Thus we have the following **system of linear differential equations** (See [3]):

$$
\begin{align*}
P_1(t) &= -(1 + \mu_1)P_1(t) + \mu_2P_2(t) \\
\vdots \\
P_j(t) &= -jP_{j-1}(t) - (j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t) \quad ; \quad j = 2, \ldots, n - 1 \tag{3} \\
\vdots \\
P_n(t) &= -nP_{n-1}(t) - (n + \mu_n)P_n(t)
\end{align*}
$$

Therefore, we obtain: \[ P(t) = A.P(t) \tag{4} \]

With \( A \in M_n(\mathbb{R}) \) and \( P \) column matrix of type \( n, 1 \):

$$
A = \begin{pmatrix}
-(1 + \mu_1) & \mu_2 & 0 \\
& \ddots & \mu_n \\
0 & \ddots & -(n + \mu_n)
\end{pmatrix}
$$

This represents the famous Chapman Kolmogorov’s equation (See [2] and [7]).

- As we saw in the previous article (See [1]), we have presented solutions of the Chapman Kolmogorov's equation in three cases, so that we will present the solution of this equation in the general case.

#### 2. Solving the equation \( P(t) = A.P(t) \) in the general case:

2.1 Procedure followed in solving this equation:

Our objective is solving this equation: \( P(t) = A.P(t) \) \( \tag{4} \)

So we have to solve the following **system of linear differential equations** (See [1]):

$$
\begin{align*}
P_1(t) &= -(1 + \mu_1)P_1(t) + \mu_2P_2(t) \\
\vdots \\
P_j(t) &= -jP_{j-1}(t) - (j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t) \quad ; \quad j = 2, \ldots, n - 1 \tag{3} \\
\vdots \\
P_n(t) &= -nP_{n-1}(t) - (n + \mu_n)P_n(t)
\end{align*}
$$

We have proved that the matrix \( A \) is **diagonalizable** (See [1]), then the **eigenvalues** of the matrix \( A \) and the **eigenvectors associated** will be searched.

So, the matrix \( S \), whose columns are the **eigenvectors associated** with the **eigenvalues** of the matrix \( A \), will be determined.

Therefore, we have: \( A = SDS^{-1} \tag{5} \)

(Dis a diagonal matrix with the proper values of the matrix \( A \) on her principal diagonal)

Next we will put the following change of variable: \( Q(t) = S^{-1}P(t) \) \( \tag{6} \)

So, \( Q(t) = S^{-1}P(t) = S^{-1}AP(t) = S^{-1}ASQ(t) = DQ(t) \)

(Because: \( P(t) = SQ(t) \) \( \tag{7} \))

Thus we solve the equation: \( Q(t) = DQ(t) \) \( \tag{8} \)

Then we conclude \( P(t) \) according to the relation: \( P(t) = SQ(t) \) \( \tag{9} \)

2.2 Resolution of the general case: any \( \lambda_i \) and \( \mu_1, \quad (i = 1, \ldots, n) \)

Thus we obtain the matrix \( A \) of the following form:

$$
A = \begin{pmatrix}
-(1 + \mu_1) & \mu_2 & 0 \\
& \ddots & \mu_n \\
0 & \ddots & -(n + \mu_n)
\end{pmatrix}
$$

We’ll look for the **eigenvalues** and the **associated eigenvectors** of the matrix \( A \).

Let \( \alpha \) be an **eigenvalue** of the matrix \( A \) and \( x \in \mathbb{R}^n - (0, \ldots, 0) \) an **associated eigenvector**, such that: \( AX = \alpha X \)
Birth and Death Processes_ General Case

Then, we have (10):
\[
\begin{cases}
  - (1 + \mu_1)x_1 + \mu_2x_2 = \alpha x_1 \\
  
  \vdots
  \\
  k-1 x_{k-1} - \left(k + \mu_k\right)x_k + \mu_1x_{k+1} = \alpha x_k; \quad k = 2, \ldots, n - 1
  \\
  \vdots
  \\
  n-1x_{n-1} - \left(n + \mu_n\right)x_n = \alpha x_n
\end{cases}
\]

So (11):
\[
\begin{cases}
  - (1 + \mu_1)x_1 + \mu_2x_2 = 0 \\
  
  \vdots
  \\
  k-1 x_{k-1} - \left(k + \mu_k\right)x_k + \mu_1x_{k+1} = 0; \quad k = 2, \ldots, n - 1
  \\
  \vdots
  \\
  n-1x_{n-1} - \left(n + \mu_n\right)x_n = 0
\end{cases}
\]

So (12):
\[
\mu_{k+1}x_{k+1} - \left(k + \mu_k + \alpha\right)x_k + \mu_1x_{k+1} = 0; \quad k = 1, \ldots, n
\]

(See [8])

We put: \( x_k = q^k \)

So we get:
\[
\mu_{k+1}q^{k+1} - \left(k + \mu_k + \alpha\right)q^k + \mu_1q^k = 0
\]

Thus for \( k=n \) we obtain:
\[
\mu_{n+1}q^{n+1} - \left(n + \mu_n + \alpha\right)q^n + \mu_1q^n = 0
\]

Dividing the last equation by: \( q^{k-1} \)

The relation becomes:
\[
\mu_{n+1}q^{n+1} - \left(n + \mu_n + \alpha\right)q^n + n-1q^{n-1} = 0
\]

By dividing the equation by: \( q^{k-1} \)

The characteristic equation becomes:
\[
\mu_{n+1}q^2 - \left(n + \mu_n + \alpha\right)q + n-1 = 0
\]

Of discriminant: \( \Delta = \left(n + \mu_n + \alpha\right)^2 - 4n-1\mu_{n+1} \)

- We will discuss the solutions according to the sign of \( \Delta \) and the values of the initials conditions:

1\textsuperscript{st} case: \( \alpha \in \left(-1 - \mu_n - n + \sqrt{n-1\mu_{n+1}}\right) - \left(-1 - \mu_n + n - \sqrt{n-1\mu_{n+1}}\right) \)

Therefore the characteristic equation admits two conjugate real solutions \( r_- \) and \( r_+ \), given by:
\[
r_{\pm} = \sqrt{\frac{n + \mu_n + \alpha}{2\mu_{n+1}}} \pm \sqrt{\frac{n + \mu_n + \alpha}{2\mu_{n+1}}} \]

Therefore, \((x_k)_{1 \leq k \leq n}\) is given by:
\[
x_k = \gamma_- r_-^k + \gamma_+ r_+^k
\]

Where the coefficients \( \gamma_- \) and \( \gamma_+ \) are provided by the following conditions:
\[
x_0 = x_{n+1} = 0
\]

Thus we obtain:
\[
\gamma_- = \gamma_+ = 0
\]

Therefore, we have: \( \gamma_- = \gamma_+ = 0 \) so, \( X = (0) \) Which is excluded.

\( \rightarrow \) Therefore, this case is empty.

2\textsuperscript{nd} case: \( \alpha = \left(n + \mu_n\right) \pm 2\sqrt{n-1\mu_{n+1}} \Rightarrow \Delta = 0 \)

Then the characteristic equation admits a double real solution, (Nominated by), such that (17):
\[
x_k = \gamma_- (k+\alpha) r_0^k \quad ; \quad k=1, \ldots, n
\]

The condition \( x_0 = x_{n+1} = 0 \) give \( \gamma_- = \gamma_+ = 0 \)

In the end: \( X = (0) \) which is excluded.

\( \rightarrow \) This second case is also empty.

3\textsuperscript{rd} case: \( \alpha \in \left[-\left(n + \mu_n\right) - 2\sqrt{n-1\mu_{n+1}}, -\left(n + \mu_n\right) + 2\sqrt{n-1\mu_{n+1}}\right] \Rightarrow \Delta < 0 \)

The solutions exist for: \( \Delta < 0 \)

Thus we put:
\[
\alpha = \left(n + \mu_n\right) + 2\sqrt{n-1\mu_{n+1}} \cos \theta \quad ; \quad \theta \in \left[0, \pi/2, \pi\right]
\]

www.ijmsi.org 3 | Page
Hence the characteristic equation becomes: \( \mu_{n+1} q^2 - 2q \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \theta + \mu_{n-1} = 0 \) (18)

Therefore, \( a_{n-1} \neq 0 \) (\( k = 1, ..., n \)), we have: \( \mu_{n+1} = \frac{a_{n+1}}{n-1} q^2 + 2q \sqrt{\frac{n-1}{n-1}} \cos \theta + 1 = 0 \)

Thus we have the following relation (19):

\[
\frac{\mu_{n+1}}{n-1} q^2 + 2 \frac{\mu_{n+1}}{n-1} \cos \theta q + 1 = \left( \frac{\mu_{n+1}}{n-1} q - e^{i\theta} \right) \left( \frac{\mu_{n+1}}{n-1} q - e^{-i\theta} \right)
\]

Which give (20): \( x_k = \rho^k (\gamma_+ \cos k\theta + \gamma_- \sin k\theta) \); \( k = 1, ..., n \)

With: \( \rho = |\omega| = \frac{n-1}{\mu_{n+1}} \) and \( \theta = \arg(\omega) \)

Thus the sequence of recurrence of order 2 admits as solution (21):

\[
x_k = \left( \begin{array}{c} \frac{n-1}{\mu_{n+1}} \\ \sin k\theta \end{array} \right)^k (\gamma_+ \cos k\theta + \gamma_- \sin k\theta) ; k = 1, ..., n
\]

Using the initials conditions: \( x_0 = x_{n+1} = 0 \)

We obtain for \( x_0 = 0; \gamma_- = 0 \)

So (22),

\[
x_k = \gamma_+ \left( \frac{n-1}{\mu_{n+1}} \right)^k \sin k\theta ; k = 1, ..., n
\]

The condition \( x_{n+1} = 0 \) gives \( \gamma_+ = 0 \) or \( \sin(n + 1)\theta = 0 \)

If \( \gamma_+ = 0 \) then \( x = (0) \rightarrow \text{Therefore, it is excluded.} \)

If \( \sin(n + 1)\theta = 0 \) then \( \theta = \frac{k\pi}{n+1} \)

Thus, the eigenvalues of the matrix \( A \) (Called \( \alpha_k; \ k = 1, ..., n \)) are of the form (23):

\[
\forall \alpha_k \in \left( -\mu_n + \mu_n \right) \left( -\mu_n + \mu_n + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \right)
\]

\[
\alpha_k = -\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{k\pi}{n+1} \right)
\]

And the eigenvectors associated with the eigenvalues of the matrix \( A \) are (24):

\[
\left( x_{k,j} \right) = \gamma_+ \left( \frac{n-1}{\mu_{n+1}} \right)^k \sin \left( \frac{jk\pi}{n+1} \right) ; 1 \leq j, k \leq n
\]

We return to our equation: \( P(t) = A \cdot P(t) \) (25)

We have \( S \) a matrix whose columns are the eigenvectors associated with the eigenvalues of the matrix \( A \), such that it is presented in the following form (26):

\[
S = \gamma_+ \left( \begin{array}{cccc} \frac{n-1}{\mu_{n+1}} \sin \left( \frac{\pi}{n+1} \right) & \cdots & \sin \left( \frac{n\pi}{n+1} \right) \\ \vdots & \ddots & \vdots \\ \frac{n-1}{\mu_{n+1}} \sin \left( \frac{n\pi}{n+1} \right) & \cdots & \sin \left( \frac{n^2\pi}{n+1} \right) \end{array} \right)
\]

We first solve the following differential equation: \( Q(t) = DQ(t) \) (27)

Then, we have:

\[
Q(t) = \left( -\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{\pi}{n+1} \right) \right) \cdots \left( -\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{n\pi}{n+1} \right) \right)
\]

\[
\vdots
\]

\[
0 \cdots \left( -\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{n\pi}{n+1} \right) \right)
\]

\[
Q_1(t) \cdots Q_n(t)
\]

So (28),

\[
Q_1(t) = \left( -\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{\pi}{n+1} \right) \right) Q_1(t)
\]

\[
\vdots
\]

\[
-\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{n\pi}{n+1} \right) Q_n(t)
\]

Thus (29), \( Q_k(t) = \delta_k e^{\left( -\left( \mu_n + \mu_n \right) + 2 \sqrt{\frac{n-1}{n-1}} \mu_{n+1} \cos \left( \frac{k\pi}{n+1} \right) \right) t} \); \( k = 1, ..., n \)

With \( \delta_k \) a constant to be determined, if we have has an initial condition.

Therefore, and according to the following relation: \( P(t) = SQ(t) \) (9)
We have (30):

\[ P(t) = \gamma_n \left( \sum_{k=1}^n \delta_k \left( \frac{n-1}{\mu_{n+1}} \sin \left( \frac{k \pi}{n+1} \right) e^{\left( -\left( a+u \right) + 2 \sqrt{u-1} \mu_{n+1} \cos \left( \frac{k}{n+1} \pi \right) \right) t} \right) \right) \]

Finally, for \( j \) between 1 and \( n \), \( P_j(t) \) is in the following form (31):

\[ P_j(t) = \gamma_n \sum_{k=1}^n \delta_k \left( \frac{n-1}{\mu_{n+1}} \sin \left( \frac{k \pi}{n+1} \right) e^{\left( -\left( a+u \right) + 2 \sqrt{u-1} \mu_{n+1} \cos \left( \frac{k}{n+1} \pi \right) \right) t} \right) ; \quad j = 1, \ldots, n \]

Note that the constant \( \gamma_n \) is obtained if an initial condition exists (Example: \( X_k(0) = \text{cst} ; k = 1, \ldots, n \))

Note also that the constant \( \delta_k \) is obtained if an initial condition exists (Example: \( \delta_k(0) = \text{cst} ; k = 1, \ldots, n \))

II. CONCLUSION:

Thanks to this law, several problems can be solved, used more particularly in biology, demography, physics, sociology, statistics ... etc.

And also to account for the changing size of any type of population and the problems related to waiting phenomena.

ACKNOWLEDGEMENTS:

I would like to express my sincere gratitude to my advisor Dr. A. Arbai for the continuous support, for his patience, motivation, and immense knowledge.

REFERENCES