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Birth and Death Processes_ General Case

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***Abstract*:*ABirth and Death Processes*(*BDPs*)** *is a continuous-time Markov chain that counts the number of particles in a system over time, they are popular modeling tools in population evolution, used more particularly in biology, genetics, epidemiology, ecology, demography, physics, sociology, statistic...etc.*
In this paper we will complete the resolution of ***the Chapman Kolmogorov's equation****in the general case (See* ***[1]****), by giving the general form of the law of this processes.*

***Keywords*:** *Problems modeling, Reduction of the matrix, Recurrent Sequences of Order 2 and Identification of the law.*

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I. INTRODUCTION

- The lack of theoretical progress in developing statistical tools for dealing with data from **Birth and Death Processes**(**BDPs**) has hindered their adoption by applied researchers.
- Statistical inference of the instantaneous particle birth and death rates also remains largely limited to continuouslyobserved processes in which perparticle birth and death rates are constants.
- In this article, and in order to complete the resolution of **Chapman Kolmogorov's equation** in the general case, we present in the first section a reminder on the **BDPs** model and in the second section the procedure followed in the resolution of this equation and also the solution found for this equation.

1. Presentation of the model:

Our study will be limited on the states between **1** and **n**, so we identify the states of this process with:

$$\mathbf{P_i(t)(i=1,\dots,n)}$$

Knowing that:$\mathbf{P_j(t) = P(X_t = j)}$; (Where $\sum_{i=1}^{n} \mathbf{P(X_t = i) = 1}$)

(Where $\mathbf{X_t}$is **adiscrete** and **homogeneousstochasticprocess**$(\mathbf{t} \in \mathbb{R}^+)$)

Let **P(t)**the column matrix of type**(n, 1)**such as: $\mathbf{P(t)^t = (P_1(t), P_2(t), \dots, P_n(t)}$ **;** $(\mathbf{t} \in \mathbb{R}^+)$

Let, $\mathbf{P_{ij}(\Delta t) = P(X_{t+\Delta t} = j / X_t = i)}$ (1)

(The transition probability of the state**i**to the state**j)**

- **The Birth and DeathProcess** is defined as follows (See [**2**]):

➢ **Definition:**

$$\mathbf{P_{ij}(\Delta t)} = \begin{cases} \mathbf{\lambda_i \Delta t + o(\Delta t)} & ; \quad \mathbf{if j = i + 1} \\ \mathbf{\mu_i \Delta t + o(\Delta t)} & ; \quad \mathbf{if j = i - 1} \\ \mathbf{1 - (\lambda_i + \mu_i)\Delta t + o(\Delta t)} & ; \quad \mathbf{if j = i} \\ \mathbf{o(\Delta t)} & ; \quad \mathbf{if |j - i| \geq 2} \end{cases} \quad (2)$$

- We have the following proposition (See demonstration in **[1]**):

➢ **Proposition:**

- Let $\mathbf{P_i(t)}$a**Birth and DeathProcess** $(\mathbf{i = 1, \dots, n})$.
With$\mathbf{\lambda_k (k = 1, \dots, n)}$ is **the birth rate** and$\mathbf{\mu_k (k = 1, \dots, n)}$ is **the death rate.**

- Thus we have the following **system of linear differential equations** (See $[\mathbf{3}]$):

$$\begin{cases} \mathbf{P_1'(t) = -(\lambda_1 + \mu_1)P_1(t) + \mu_2 P_2(t)} \\ \dots \\ \mathbf{P_j'(t) = \lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t)} \quad ; \quad \mathbf{j = 2, \dots, n-1} \\ \dots \\ \mathbf{P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t)} \end{cases} \quad (3)$$

Therefore,we obtain: $\mathbf{P'(t) = A.P(t)}$ (4)

With $\mathbf{A} \in \mathbf{M_n}(\mathbb{R})$ and $\mathbf{P}$ a column matrix of type $(\mathbf{n},\ \mathbf{1})$:

$$\mathbf{A} = \begin{pmatrix} -(\lambda_1 + \mu_1) & \mu_2 & 0 \\ \lambda_1 & \ddots & \mu_n \\ 0 & \lambda_{n-1} & -(\lambda_n + \mu_n) \end{pmatrix}$$

This represents **the famous Chapman Kolmogorov's equation** (See **[2]** and **[7]**).

- As we saw in the previous article (See **[1]**), we have presented solutions of the **Chapman Kolmogorov's equation** in three cases, so that we will present the solution of this equation in the general case.

2. Solving the equation $\mathbf{P'(t) = A.P(t)}$ (4) in the general case:

2.1 Procedure followed in solving this equation:

Our objective is solving this equation: $\mathbf{P'(t) = A.P(t)}$ (4)
So we have to solve the following **system of linear differential equations** (See $[\mathbf{1}]$):

$$\begin{cases} \mathbf{P_1'(t) = -(\lambda_1 + \mu_1)P_1(t) + \mu_2 P_2(t)} \\ \dots \\ \mathbf{P_j'(t) = \lambda_{j-1}P_{j-1}(t) - (\lambda_j + \mu_j)P_j(t) + \mu_{j+1}P_{j+1}(t)} \quad ; \quad \mathbf{j = 2, \dots, n-1} \\ \dots \\ \mathbf{P_n'(t) = \lambda_{n-1}P_{n-1}(t) - \mu_n P_n(t)} \end{cases} \quad (3)$$

We have proved that the matrix **A** is **diagonalizable** (See $[\mathbf{1}]$), then **the eigenvalues** of the matrix **A** and **the eigenvectors associated** will be searched.
So, the matrix **S**, whose columns are **the eigenvectors associated** with **the eigenvalues** of the matrix **A**, will be determined.
Therefore, we have: $\mathbf{A = SDS^{-1}}$ (5)
(**D** is **a diagonal matrix** with **the proper values** of the matrix **A** on her **principal diagonal**)

Next we will put the following change of variable: $\mathbf{Q(t) = S^{-1}P(t)}$ (6)
So, $\mathbf{Q'(t) = S^{-1}P'(t) = S^{-1}AP(t) = S^{-1}ASQ(t) = DQ(t)}$ (Because: $\mathbf{P(t) = SQ(t)}$ (7))
Thus we solve **first** the equation: $\mathbf{Q'(t) = DQ(t)}$ (8)
Then we conclude **P(t)** according to the relation**:** $\mathbf{P(t) = SQ(t)}$ (9)

2.2 Resolution of the general case: any $\boldsymbol{\lambda_i}$ and $\boldsymbol{\mu_i}$ $(\mathbf{i = 1, \dots, n})$

Thus we obtain the matrix **A** of the following form:

$$\mathbf{A} = \begin{pmatrix} -(\lambda_1 + \mu_1) & \mu_2 & 0 \\ \lambda_1 & \ddots & \mu_n \\ 0 & \lambda_{n-1} & -(\lambda_n + \mu_n) \end{pmatrix}$$

We'll look for **the eigenvalues** and **the associated eigenvectors** of the matrix **A.**
Let $\boldsymbol{\alpha}$ be **an eigenvalue** of the matrix **A** and **x** $\in \mathbb{R}^{\mathbf{n}} - (\mathbf{0}, \dots, \mathbf{0})$ **an associated eigenvector,** such that: $\mathbf{AX = \alpha X}$

Then, we have (10): $\begin{cases} -(\lambda_1+\mu_1)x_1+\mu_2x_2=\alpha x_1 \\ \dots \\ \lambda_{k-1}x_{k-1}-(\lambda_k+\mu_k)x_k+\mu_{k+1}x_{k+1}=\alpha x_k; \quad k=2,\dots,n-1 \\ \dots \\ \lambda_{n-1}x_{n-1}-(\lambda_n+\mu_n)x_n=\alpha x_n \end{cases}$

So (11), $\begin{cases} -(\lambda_1+\mu_1+\alpha)x_1+\mu_2x_2=0 \\ \dots \\ \lambda_{k-1}x_{k-1}-(\lambda_k+\mu_k+\alpha)x_k+\mu_{k+1}x_{k+1}=0 \quad ; \quad k=2,\dots,n-1 \\ \dots \\ \lambda_{n-1}x_{n-1}-(\lambda_n+\mu_n+\alpha)x_n=0 \end{cases}$

So (12), $\mu_{k+1}x_{k+1}-(\lambda_k+\mu_k+\alpha)x_k+\lambda_{k-1}x_{k-1}=0 \quad ; \quad k=1,\dots,n$(See **[8]**)

We put: $x_k=q^k$

So we get: $\mu_{k+1}q^{k+1}-(\lambda_k+\mu_k+\alpha)q^k+\lambda_{k-1}q^{k-1}=0$

Thus for **k=n** we obtain: $\mu_{n+1}q^{n+1}-(\lambda_n+\mu_n+\alpha)q^n+\lambda_{n-1}q^{n-1}=0$

Dividing the last equation by: q^{k-n}

The relation becomes: $\mu_{n+1}q^{k+1}-(\lambda_n+\mu_n+\alpha)q^k+\lambda_{n-1}q^{k-1}=0$(13)

By dividing the equation by: q^{k-1}

The **characteristic equation** becomes: $\mu_{n+1}q^2-(\lambda_n+\mu_n+\alpha)q+\lambda_{n-1}=0$(14)

Of **discriminant**: $\Delta=(\lambda_n+\mu_n+\alpha)^2-4\lambda_{n-1}\mu_{n+1}$(15)

- We will discuss the solutions according to the sign of Δ and the values of the initials conditions:
<u>1st case</u>: $\alpha\in\,]-\infty;-(\lambda_n+\mu_n)-2\sqrt{\lambda_{n-1}\mu_{n+1}}[\cup]-(\lambda_n+\mu_n)+2\sqrt{\lambda_{n-1}\mu_{n+1}};+\infty[\Rightarrow\Delta>0$

Therefore **the characteristic equation** admits two **conjugate realsolutions** r_- and r_+ given by:

$$r_\pm=\frac{\lambda_n+\mu_n+\alpha}{2\mu_{n+1}}\pm\sqrt{\left(\frac{\lambda_n+\mu_n+\alpha}{2\mu_{n+1}}\right)^2-\frac{\lambda_{n-1}}{\mu_{n+1}}}$$

Therefore, $(x_k)_{1\le k,i\le n}$ is given by: $x_k=\gamma_-r_-^k+\gamma_+r_+^k$(16)
Where the coefficients γ_- and γ_+ are provided by the following conditions: $x_0=x_{n+1}=0$
Thus we obtain: $\begin{cases} \gamma_-+\gamma_+=0 \\ \gamma_-(r_-^{n+1}-r_+^{n+1})=0 \end{cases}$
Therefore, we have: $\gamma_-=\gamma_+=0$ so, $X=(0)$ **Which is excluded.**
→ **Therefore, this case is empty.**

2nd case: $\alpha=-(\lambda_n+\mu_n)\pm2\sqrt{\lambda_{n-1}\mu_{n+1}}\Rightarrow\Delta=0$
Then **the characteristic equation** admits a **double real solution, (**Nominated r_0), such that (17):

$$x_k=(\gamma_-+k\gamma_+)r_0^k \quad ; \quad k=1,\dots,n$$

The condition: $x_0=x_{n+1}=0$ give $\gamma_-=\gamma_+=0$
In the end: $X=(0)$ **which is excluded**.
→ **This second case is also empty.**

3rd case: $\alpha\in\,]-(\lambda_n+\mu_n)-2\sqrt{\lambda_{n-1}\mu_{n+1}};-(\lambda_n+\mu_n)+2\sqrt{\lambda_{n-1}\mu_{n+1}}[\Rightarrow\Delta<0$
The solutions exist for: $\Delta<0$
Thus we put: $\alpha=-(\lambda_n+\mu_n)+2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\theta \quad ; \quad \theta\in\,]0,\pi[\cup]\pi,2\pi[$

Hence **the characteristic equation** becomes: $\mu_{n+1}q^2 - 2q\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\theta + \lambda_{n-1} = 0$ (18)

Therefore, as $\lambda_{n-1} \neq 0$ $(k = 1, \dots, n)$, we have: $\frac{\mu_{n+1}}{\lambda_{n-1}}q^2 + 2q\sqrt{\frac{\mu_{n+1}}{\lambda_{n-1}}}\cos\theta + 1 = 0$

Thus we have the following relation (19):

$$\frac{\mu_{n+1}}{\lambda_{n-1}}q^2 + 2\sqrt{\frac{\mu_{n+1}}{\lambda_{n-1}}}\cos\theta\, q + 1 = \left(\sqrt{\frac{\mu_{n+1}}{\lambda_{n-1}}}q - e^{i\theta}\right)\left(\sqrt{\frac{\mu_{n+1}}{\lambda_{n-1}}}q - e^{-i\theta}\right)$$

Which give (20): $x_k = \rho^k(\gamma_- \cos k\theta + \gamma_+ \sin k\theta)$; **k=1, …, n**

With: $\rho = |\omega| = \sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}$ and $\theta = \arg(\omega)$

Thus **the sequence of recurrence of order 2** admits as solution (21):

$$x_k = \left(\sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\right)^k (\gamma_+ \cos k\theta + \gamma_- \sin k\theta) \quad ;k=1, \dots, n$$

Using the initials conditions: $x_0 = x_{n+1} = 0$

We obtain for $x_0 = 0$: $\gamma_- = 0$

So (22), $$x_k = \gamma_+\left(\sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\right)^k \sin k\theta ;k=1, \dots, n$$

The condition $x_{n+1} = 0$ gives $\gamma_+ = 0$ or $\sin(n+1)\theta = 0$

If $\gamma_+ = 0$ than $X = (0) \rightarrow$ **Therefore, it is excluded.**

If $\sin(n+1)\theta = 0$ than $\theta = \frac{k\pi}{n+1}$

Thus, **the eigenvalues** of the matrix **A** (Called α_k; $k = 1, \dots, n$) are of the form(23):

$\forall\alpha_k \in \left]-(\lambda_n + \mu_n) - 2\sqrt{\lambda_{n-1}\mu_{n+1}}; -(\lambda_n + \mu_n) + 2\sqrt{\lambda_{n-1}\mu_{n+1}}\right[$

$$\alpha_k = -(\lambda_n + \mu_n) + 2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\left(\frac{k\pi}{n+1}\right)$$

And **the eigenvectorsassociated** with **the eigenvalues** of the matrix **A** are (24):

$$(x_k)_j = \gamma_+\left(\sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\right)^k \sin\left(j\frac{k\pi}{n+1}\right) \quad ; \quad 1 \leq j, k \leq n$$

We return to our equation: $P'(t) = A.P(t)$ (25)

We have **S** a matrix whose columns are **the eigenvectors associated** with **the eigenvalues** of the matrix **A**, such that it is presented in the following form (26):

$$S = \gamma_+\begin{pmatrix} \sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\sin\left(\frac{\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\right)^n \sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\sin\left(\frac{n\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\lambda_{n-1}}{\mu_{n+1}}}\right)^n \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}$$

We **first** solve the following **differential equation**: $Q'(t) = DQ(t)$ (27)

Then, we have:

$$Q'(t) = \begin{pmatrix} -(\lambda_n + \mu_n) + 2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\left(\frac{\pi}{n+1}\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -(\lambda_n + \mu_n) + 2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\left(\frac{n\pi}{n+1}\right) \end{pmatrix}\begin{pmatrix} Q_1(t) \\ \vdots \\ Q_n(t) \end{pmatrix}$$

So (28), $$\begin{pmatrix} Q'_1(t) \\ \dots \\ Q'_n(t) \end{pmatrix} = \begin{pmatrix} \left(-(\lambda_n + \mu_n) + 2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\left(\frac{\pi}{n+1}\right)\right)Q_1(t) \\ \vdots \\ \left(-(\lambda_n + \mu_n) + 2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\left(\frac{n\pi}{n+1}\right)\right)Q_n(t) \end{pmatrix}$$

Thus (29), $Q_k(t) = \delta_k e^{\left(-(\lambda_n+\mu_n)+2\sqrt{\lambda_{n-1}\mu_{n+1}}\cos\left(\frac{k\pi}{n+1}\right)\right)t}$; $k = 1, \dots, n$

With δ_k a constant to be determined, if we have has an initial condition.

Therefore, and according to the following relation: $P(t) = SQ(t)$ (9)

We have (30):

$$\mathbf{P(t)} = \boldsymbol{\gamma}_{+}\begin{pmatrix} \sqrt{\frac{\square_{n-1}}{\mu_{n+1}}}\,\mathbf{sin}\left(\frac{\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\square_{n-1}}{\mu_{n+1}}}\right)^{n}\mathbf{sin}\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{\square_{n-1}}{\mu_{n+1}}}\,\mathbf{sin}\left(\frac{n\pi}{n+1}\right) & \cdots & \left(\sqrt{\frac{\square_{n-1}}{\mu_{n+1}}}\right)^{n}\mathbf{sin}\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\delta}_1 e^{\left(-(\square_n+\mu_n)+2\sqrt{\square_{n-1}\mu_{n+1}}\cos\left(\frac{\pi}{n+1}\right)\right)t} \\ \vdots \\ \boldsymbol{\delta}_n e^{\left(-(\square_n+\mu_n)+2\sqrt{\square_{n-1}\mu_{n+1}}\cos\left(\frac{n\pi}{n+1}\right)\right)t} \end{pmatrix}$$

Finally, for $\mathbf{j}$ between $\mathbf{1}$ and $\mathbf{n}$, $\mathbf{P_j(t)}$ is in the following form (31):

$$\mathbf{P_j(t)} = \boldsymbol{\gamma}_{+}\sum_{k=1}^{n}\boldsymbol{\delta}_k\left(\sqrt{\frac{\square_{n-1}}{\mu_{n+1}}}\right)^{k}\mathbf{sin}\left(\frac{jk\pi}{n+1}\right)e^{\left(-(\square_n+\mu_n)+2\sqrt{\square_{n-1}\mu_{n+1}}\cos\left(\frac{k\pi}{n+1}\right)\right)t} \quad ; \quad \mathbf{j = 1, \dots, n}$$

Note that the constant $\boldsymbol{\gamma}_{+}$ is obtained if an initial condition exists (Example: $(\mathbf{X_k})_0 = \mathbf{cst}$;$\mathbf{k = 1, \dots, n}$)
Note also that the constant $\boldsymbol{\delta}_k$is obtained if an initial condition exists (Example: $\boldsymbol{\delta}_k(0) = \mathbf{cst}$; $\mathbf{k = 1, \dots, n}$)

II. CONCLUSION:

Thanks to this law, several problems can be solved, used more particularly in biology, demography, physics, sociology, statistics ... etc.
And also to account for the changing size of any type of population and the problems related to waiting phenomena.

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