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Edge Domination in Hypergraph

*D. K. Thakkar[1], V. R. Dave[2]
[1]*Department of Mathematics, Saurashtra University, Rajkot, India*
[2]*Shree M. & N. Virani Science College, Kalavad Road, Rajkot, India*
*Corresponding Author: *D. K. Thakkar*

ABSTRACT: *In this paper we define the concept of edge domination in hypergraphs. We characterize minimal edge dominating sets in hypergraphs. We also prove that a set of vertices S of a hypergraph is a dominating set iff S* is an edge dominating set in the dual hypergraph. We also prove a necessary and sufficient condition under which the edge domination number of a hypergraph increases when a vertex is removed from the hypergraph. We also give some examples of edge dominating set.*

KEYWORDS: *Hypergraph, Dual hypergraph, Dominating set in hypergraph, Edge dominating set, Edge domination number, Vertex covering set, Minimal edge dominating set, Minimum edge dominating set.*
AMS Subject Classification (2010):*05C15, 05C69, 05C65*

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I. INTRODUCTION

Edge dominating set & edge domination number have been explored by several authors[5, 6]. The concept of edge domination requires the adjacency relation among the edges of a graph. The same relation is also available in hypergraphs and therefore we have considered edge domination in hypergraphs. With every hypergraph one another hypergraph is associated called the dual hypergraph of the given hypergraph. We prove that S is a dominating set of G iff S* is an edge dominating set of G*.

II. PRELIMINARIES

Definition 2.1 Hypergraph[4] A hypergraph G is an ordered pair (V(G), E(G)) where V(G) is a non-empty finite set & E(G) is a family of non – empty subsets of V(G) such that there union = V(G).The elements of V(G) are called vertices & the members of E(G) are called edges of the hypergraph G.

We make the following assumption about the hypergraph.

(1) Any two distinct edges intersect in at most one vertex.

(2) If e_1 and e_2 are distinct edges with $|e_1|, |e_2| > 1$ then $e_1 \not\subseteq e_2$ & $e_2 \not\subseteq e_1$

Definition 2.2 Edge Degree[4] Let G be a hypergraph & $v \in V(G)$ then the edge degree of v = $d_E(v)$ = the number of edges containing the vertex v.

According to the definition of hypergraph $d_E(v) \geq 1$ for every vertex v

The minimum edge degree among all the vertices of G is denoted as $\delta_E(G)$ and the maximum edge degree is denoted as $\Delta_E(G)$.

$\therefore$ For any hypergraph G, $\delta_E(G) \geq 1$.

Definition 2.3 Dual Hypergraph[4] Let G be a hypergraph. For every $v \in V(G)$ define $\bar{v}$ as follows

$\bar{v} = \{e \in E(G) / v \in e\}$

Let $E(G^*) = \{ \bar{v} / v \in V(G) \}$ and let $V(G^*) = E(G)$. Then the dual hypergraph of the given hypergraph G is the hypergraph G* whose vertex set is V(G*) & the edge set is E(G*). We will write G* = (V(G*), E(G*)).

Example 2.4

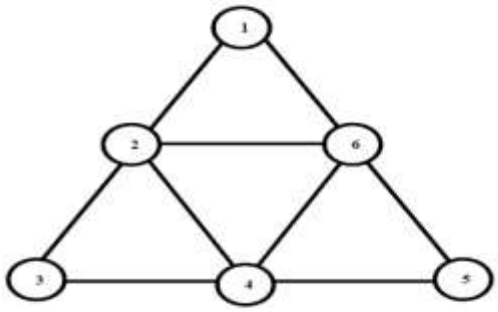

Figure 1

Consider hypergraph G whose vertices are {1, 2, 3, 4, 5, 6} and edges are e_1 = {1, 2, 3}, e_2 = {3, 4, 5}, e_3 = {5, 6, 1}, e_4 = {2, 4}, e_5 = {4, 6}, e_6 = {6, 2}
V(G*) = {$e_1, e_2, e_3, e_4, e_5, e_6$} and E(G*) = { $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, $\bar{5}$, $\bar{6}$ }
Where, $\bar{1}$ = {e_1, e_3}, $\bar{2}$ = {e_1, e_4, e_6}, $\bar{3}$ = {e_1, e_2}, $\bar{4}$ = {e_2, e_4, e_5}, $\bar{5}$ = {e_2, e_3}, $\bar{6}$= {e_3, e_5, e_6}

Remark 2.5 If G is a graph then obviously it can be regarded as hypergraph and therefore it is possible to define the dual hypergraph of the graph.

Definition 2.6 Dominating Set in Hypergraph[1] Let G be a hypergraph & S $\subseteq$V(G) then S is said to be a dominating set of G if for every v$\in$ V(G) – S there is u $\in$ S $\ni$ u and v are adjacent vertices.

A dominating set with minimum cardinality is called minimum dominating set and cardinality of such a set is called domination number of G and it is denoted as γ(G).

Definition 2.7 Vertex Covering Set[4] Let G be a hypergraph & S $\subseteq$ V(G). Then S is said to be a vertex covering set of G if for every edge e of G, e $\cap$ S $\neq \phi$.

Definition 2.8 Sub hypergraph[3] Let G be a hypergraph & v $\in$ V(G) $\ni$ {v} is not an edge of G .Consider the subset V(G) – {v} of V(G). Here the vertex set = V(G) – {v} and the edge set is equal to {e′ / e′ = e – {v} for some e $\in$ E(G)}.This hypergraph is called the sub hypergraph of G & it is denoted as G – v.

III. MAIN RESULTS

Definition 3.1 Edge Dominating Set Let G be a hypergraph & S $\subseteq$ E(G) then S is said to be an edge dominating set of G if for every e $\in$ E(G) – S there is some f in S $\ni$ e and f are adjacent edges.

An edge dominating set with minimum cardinality is called a minimum edge dominating set and cardinality of such a set is called edge domination number of G and it is denoted as γ_E(G).

Notations 3.2

- If F is a set of edges of a hypergraph G then V(F) will denote the set of all vertices which are incident with edges of F.
- Let S $\subseteq$ V(G) then S* = { $\bar{v}$ / v $\in$ S }. Obviously S* $\subseteq$ V(G*).

Example 3.3 Consider the graph in example 2.4 Let F = {e_1, e_2} then V(F) = {1, 2, 3, 4, 5}

Proposition 3.4 Let G be a hypergraph. Let F be an edge dominating set of G then V(F) is a dominating set of G.

Proof Let x $\in$ V(G) – V(F). Let e be any edge of G $\ni$ x $\in$ e. Since x $\notin$ V(F) $\Rightarrow$ e $\notin$ F
Since F is an edge dominating set of G, there is an edge f in F such that f is adjacent to e. Let y $\in$ e $\cap$ f then y $\in$ V(F) and y is adjacent to x.
Thus, V(F) is a dominating set. ■

Remark 3.5 Converse of the above statement is not true, which is mentioned in the following example.

Example 3.6 Let G be a hypergraph. V(G) = {1, 2, 3, 4, 5, 6, 7} and E(G) = {e_1, e_2, e_3, e_4}. Where e_1 = {1, 2, 3}, e_2 = {3, 4, 5}, e_3 = {7, 5}, e_4 = {1, 6, 7}. Now consider the set F = {e_1} then V(F) = {1, 2, 3} then obviously F is not an edge dominating set of G but V(F) is dominating set of G.

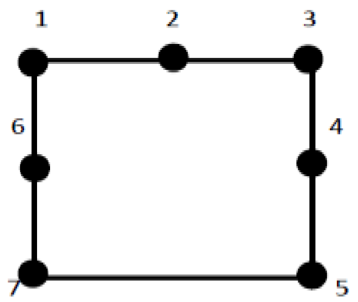

Figure 2

Proposition 3.7 Let G be a hypergraph and S $\subseteq$ E(G) then F is edge dominating set of G if and only if V(F) is a vertex covering set of G.

Proof First suppose that F is an edge dominating set of G. Let e be any edge of G if e $\in$ F then e $\subseteq$ V(F). If e $\notin$ F then e $\cap$ F $\neq \phi$, for some f in F. Let x $\in$ e $\cap$ f then x $\in$ V(F) & x $\in$ e. Thus, e intersect V(F).
$\therefore$ V(F) is a vertex covering set of G.
Conversely suppose V(F) is a vertex covering set of G. let e be any edge of G $\ni$ e $\notin$ F.
Now, e $\cap$ V(F) $\neq \phi$. Let y $\in$ e $\cap$ V(F) Since y $\in$ V(F), y $\in$ f, for some f $\in$ F. Thus, e $\cap$ f $\neq \phi$, where f $\in$ F. Thus, F is an edge dominating set of G. ■

Proposition3.8 Let G be a hypergraph & S $\subseteq$ V(G). Then S is a dominating set of G iff S* is an edge dominating set of G*.

Proof Suppose S is a dominating set of G. Let $\bar{x}$ be any edge of G* $\ni$ $\bar{x}$ $\notin$ S* then x $\notin$ S. Since S is a dominating set of G, there is a vertex y in S such that x is adjacent to y.

Let e be an edge of G such that x, y $\in$ e. Then e $\in$ $\bar{x} \cap \bar{y}$. Thus, $\bar{x}$ is adjacent to $\bar{y}$ in G* with $\bar{y}$ in S. Thus S* is an edge dominating set of G*.

Conversely suppose S* is an edge dominating set of G*. Let x $\in$ V(G) – S.

Suppose $\bar{x} = \bar{y}$ for some $\bar{y}$ $\in$ S* then there is an edge e of G such that x, y $\in$ e and y$\in$ S.

Suppose $\bar{x} \neq \bar{y}$ for any $\bar{y}$ $\in$ S* then $\bar{x}$ $\notin$ S*. Since S* is an edge dominating set of G* there is some $\bar{y}$ in S* such that $\bar{x}$ & $\bar{y}$ are adjacent edges in G*.

Let e $\in$ $\bar{x} \cap \bar{y}$ then e is an edge of G & x, y $\in$ e. Thus, x is adjacent to y in G & y $\in$ S.

Thus, S is a dominating set of G. ■

Definition 3.9 Minimal Edge Dominating Set Let G be a hypergraph & F $\subseteq$ E(G) then F is said to be a minimal edge dominating set if

(1) F is an edge dominating set

(2) No proper subset of F is an edge dominating set of G.

Obviously, an edge dominating set F is a minimal edge dominating set iff $\forall$ e $\in$ F, F – {e} is not an edge dominating set of G.

Remark 3.10 Every minimum edge dominating set is minimal edge dominating set but converse is not true.

Example 3.11

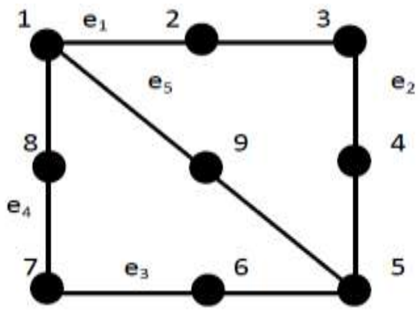

Figure 3

Consider the hypergraph shown above. This hypergraph has 9 vertices & 5 edges. In this hypergraph F = {e_1, e_2} is a minimal edge dominating set but it is not a minimum edge dominating set. Infact, edge domination number of this hypergraph = 1.

Characterization of Minimal Edge Dominating Set

Theorem 3.12 Let G be a hypergraph and F $\subseteq$ E(G) be an edge dominating set then F is a minimal edge dominating set iff $\forall$ e $\in$ F at least one of the following two conditions is satisfied

(1) e is not adjacent to any other edge of F (i.e. e is an isolate in F)

(2) There is an edge f in E(G) – F such that f is adjacent to e but f is not adjacent to any other edge of F.

Proof First suppose that F is a minimal edge dominating set. Let e $\in$ F.

Now, F – {e} is not an edge dominating set of G.

$\therefore$ There is an edge h which is not in F – {e} & h is not adjacent to any member of F – {e}

<u>Case 1</u>: h = e, then e is not adjacent to any member of F & therefore condition – 1 is satisfied.

<u>Case2</u>: h $\neq$ e, Now, h is not adjacent to any member of F – {e} & h is adjacent to some member of F ($\because$ F is an edge dominating set of G). This implies that h is adjacent to only one member of F namely e. Thus, condition – 2 is satisfied.

Conversely suppose F is an edge dominating set of G & for e $\in$ F (1) or (2) is satisfied.

Suppose (1) is satisfied. Now, let F_1 = F – {e} then e is not adjacent to any member of F_1.

$\therefore$ F_1 is not an edge dominating set of G.

Suppose (2) is satisfied. Then h is not adjacent to any member of F – {e}

$\therefore$ F – {e} is not an edge dominating set of G.

Thus, F is a minimal edge dominating set of G. ■

Theorem 3.13 Let G be a hypergraph in which every edge is adjacent to some other edge. If F is a minimal edge dominating set of G then E(G) – F is an edge dominating set of G.

Proof Let e $\in$ F since F is a minimal edge dominating set, e satisfies at least one of the conditions (1) & (2) of the above theorem. Suppose e satisfied condition (1) .Then e is not adjacent to any other member of F but by given condition e is adjacent to some edge f of E(G) then it follows that f $\in$ E(G) – F. Thus, e is adjacent to some member of E(G) – F.

Suppose e is adjacent to some member of F then condition (1) of above theorem is not satisfied. Therefore, condition (2) must be satisfied by e.

$\therefore$ There is an edge h in E(G) – F which is adjacent to e. Thus, in this case also e is adjacent to some member of E(G) – F.

$\therefore$ E(G) – F is an edge dominating set of G. ■

Remark 3.14

- Let G be a hypergraph without isolated edges. Let F be a minimal edge dominating set of G. Let m = the number of edges of the hypergraph G. Then by the above result E(G) – F is also an edge dominating set of G. Also $|F| \le \frac{m}{2}$ or $|E(G) - F| \le \frac{m}{2}$ and $\gamma_E(G) \le \min\{|F|, |E(G) - F|\}$
  $\therefore \gamma_E(G) \le \frac{m}{2}$.
- Let G be a hypergraph with $\delta_E(G) \ge 2$ & suppose v is a vertex of G such that edge degree of v = $\Delta_E(G)$ = k. Let $e_1, e_2, \ldots, e_k$ be all the edges containing the vertex v. Let T = $\{e_1, e_2, \ldots, e_k\}$ & F = V(G) – T. Then F is an edge dominating set of G.
  $\therefore \gamma_E(G) \le |F| = m - \Delta_E(G)$, where m = the number of edges of the hypergraph G.

Example 3.15 For r ≥ 2 consider the finite projective plane G which has $r^2 - r + 1$ vertices and $r^2 - r + 1$ edges which satisfy the following two properties.

(1) Any two distinct edges intersect in a single vertex.

(2) If u and v are distinct vertices then there is a unique edge e such that u, v $\in$ e.

It can be prove that every vertex v of G is contained in exactly r edges and every edge contains exactly r vertices. In this hypergraph if e is an edge then {e} is a minimum edge dominating set of G. Therefore, $\gamma_E(G)$ = 1 Also, every singleton set of vertices is a dominating set of G and therefore its domination number is 1.

In the following proposition we will consider the sub hypergraph on G – v where v $\in$V(G)

Proposition 3.16 Let G be a hypergraph & v $\in$V(G) such that {v} is not an edge of G. Then $\gamma_E(G - v) \ge \gamma_E(G)$

Proof Let F′ be a minimum edge dominating set of G – {v}.

Let F = {e $\in$ E(G) / e – {v} $\in$ F′} then | F | = | F′ |.

Now, we prove that F is an edge dominating set of G. Let g be an edge of G such that g $\notin$ F. Let g′ = g – {v} then g′ is an edge of G – {v} such that g′$\notin$ F′. Since F′ is an edge dominating set of G – {v}, G′ is adjacent to some edge h′ of F′.

Let h be the edge of F such that h – {v} = h′. Since $g' \cap h' \ne \phi \Rightarrow g \cap h \ne \phi$

Thus g is adjacent to some member of F.

$\therefore$ F is an edge dominating set of G.

$\therefore \gamma_E(G - v) \ge \gamma_E(G)$ ■

Theorem 3.17 Let G be a hypergraph and v $\in$V(G) such that {v} is not an edge of G. Then $\gamma_E(G - v) > \gamma_E(G)$ if & only if for every minimum edge dominating set F of G there is an edge g not in F $\ni g \cap h \subseteq \{v\}\ \forall\ h \in F$.

Proof Suppose $\gamma_E(G - v) > \gamma_E(G)$. Let F be a minimum edge dominating set of G. Consider F′ = {e′ / e $\in$ F} then F′cannot be an edge dominating set of G – {v} because $|F'| = |F| = \gamma_E(G) < \gamma_E(G - v)$

$\therefore$ There is an edge g′ of G – {v} $\ni$ g′$\notin$ F′& $g' \cap h' = \phi \Rightarrow g \cap h \subseteq \{v\}$

Conversely suppose the condition is satisfied. Let T be a subset of E (G - v) $\ni |T| < \gamma_E(G)$

If T is an edge dominating set of G – v then it implies that the set T_1 = {e $\in$ E(G) / e – {v} $\in$ T} is an edge dominating set of G with $|T_1| = |T| < \gamma_E(G)$. This is absurd. Thus T cannot be an edge dominating set of (G – v). Let S be a subset of E(G - v) such that $|S| = \gamma_E(G)$. If S is an edge dominating set of G – v then the set S_1 = {e / e – {v} $\in$ S} is an edge dominating set of G with $|S_1| = |S| = \gamma_E(G)$. Let g be an edge of G such that g $\notin S_1$ then g′$\notin$ S. Since S is an edge dominating set of G – {v}, $g' \cap h' = \{x\}$ for some x ≠ v.

Thus, $g \cap h$ contains a vertex different from v. This contradicts the hypothesis. Thus, any set S of edges of G – v with $|S| = \gamma_E(G)$ cannot be an edge dominating set of G – v.

$\therefore$ It cannot be a minimum edge dominating set of G – v.

Thus, any minimum edge dominating set of G – v must have the cardinality > $\gamma_E(G)$.

$\therefore \gamma_E(G - v) > \gamma_E(G)$ ■

Example 3.18 Consider the graph in example – 3.11.The edge domination number of this hypergraph = 1. Now, consider the subhypergraph G – {1}

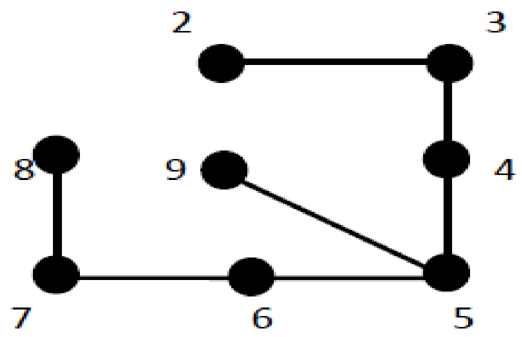

Figure 4

The edge domination number of G – {1} = 2

Thus, $\gamma_E(G - v) > \gamma_E(G)$.

Example 3.19 Consider the finite projective plan of example – 3.15

Here $\gamma_E(G - v) = \gamma_E(G)$ $\forall$ v $\in$ V(G).

IV. CONCLUSION

A minimal edge dominating set with minimum cardinality is called a minimum edge dominating set. We can consider minimal edge dominating set with maximum cardinality. If we define the upper edge domination number of a hypergraph to be the maximum cardinality of a minimal edge dominating set then it may be possible to consider the effect of the operations of vertex removal or the edge removal on the upper edge domination number.

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