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First-Passage Time moment Approximation For The Birth–Death Diffusion Process To A General moving Barrier

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ABSTRACT: *The development of a mathematical models for population growth of great importance in many fields. The growth and decline of real populations can in many cases be well approximated by the solutions of a stochastic differential equations. However, there are many solutions in which the essentially random nature of population growth should be taken into account. Thispaper focusses in approximating the moments of the first – passage time for the birth and death diffusion process to a general moving barriers. This was done by approximating the differential equations by an equivalent difference equations.*
KEYWORDS: *First Passage Time, Birth-Death Diffusion Process, Difference Equations, General Moving Barrier.*

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I. INTRODUCTION

First – passage time play an important rule in the area of applied probability theory especially in stochastic modeling. Several examples of such problems are the extinction time of a branching process, or the cycle lengths of a certain vehicle actuated traffic signals. Actually the the first – passage times to a moving barriers for diffusion and other markov processes arises in biological modeling (Cf. Ewens (1979)), in statistics (Cf. Darling and Siegert (1953) and Durbin (1971)).

Many important results related to the first – passage time have been studied from different points of view of different authors. For example, McNeil (1970) has derived the distribution of the integral functional $Wx = \int_0^{Tx} g\{X(t)\}dt$, where T_x is the first – passage time to the origin in a general birth – death process with X(0) = x and g(.) is an arbitrary function. Also, Iglehart (1965) , McNeil and Schach (1973) have been shown a number of classical birth and death processes upon taking diffusion limits to asympotically approach the Ornstein – Uhlenbeck (O.U.) .

Many properties such as a first – passage time to a barrier, absorbing or reflecting, located some distance from an initial starting point of the O.U. process and the related diffusion process and the related diffusion process such as the case of the first passage time of a Wiener process to a linear barrieris aclosed form expression for the density available is discussed in Cox and Miller (1965). Also, others such as, Karlin and Taylor (1981), Thomas (1975),Ferebee (1982), Tuckwell and Wan (1984),Al-Eideh (2004), etc.have been discussed the first passage time from different points of view.

In particular, Thomas (1975) describes some mean first – passage time approximation for the Ornstein – Uhlendeck process.Tuckwell and Wan (1984) have studied the first-passage time of a Markov process to a moving barriers as a first-exit time for a vector whose components include the process and the barrier. Also, Al-Eideh (2004), has discussed the problem of finding the moments of the first passage time distribution for the birth-death diffusion and the Wright-Fisher diffusion processes to a moving linear barriers using the method of approximating the differential equations by difference equations. In this paper, we consider the birth and death diffusion process and study the first – passage time for such a process to a general moving barrier. More specifically, the moment approximations are derived using the method of difference equations.

II. FIRST – PASSAGE TIME MOMENT APPROXIMATIONS

Consider the birth and death diffusion Process $\{X(t): t \geq 0\}$ with infinitesimal mean $bx+\varepsilon$ and variance $2ax$ starting at some $x_0 > 0$, where b and a are the drift and the diffusion coefficients respectively. Also, $\{X(t): t \geq 0\}$ is a Markov process with state space $S = [0, \infty)$ and satisfies the Ito stochastic differential equation

$$dX(t) = bX(t)dt + \sqrt{2aX(t)}dW(t) \qquad (1)$$

Where $\{W(t): t \geq 0\}$ is a standard Wiener process with zero mean and variance t. Assume that the existence and uniqueness conditions are satisfied (Cf. Gihman and Skorohod (1972)). Let $\{Y(t): t \geq 0\}$ be a general moving barrier equation such that $Y(t) = h(t)$, with $Y(0) = h(0)$. Or equivalently

$$\frac{dY(t)}{dt} = h'(t)$$

Now,denote the first – passage time of a process $X(t)$ to a general moving barrier $Y(t) = h(t)$ by the randomvariables

$$T_Y = \inf\{t \geq 0 : X(t) \geq h(t)\} \qquad (2)$$

with probability density function

$$g(t;\ x_0) = -\frac{d}{dt}\int_{-\infty}^{h(t)} p(x_0, x; t)\, dx$$

Here p (x_0 , x ; t) is the probability density function of X (t) conditional on X (0) = x_0

Let $M_n(x_0, Y; t)$; n = 1,2,3,…… , be the n-th moment of the first – passage time T_Y , i.e.

$$M_n(x_0, Y; t) = E(T_Y^n) \quad ;\ n = 1,2,3,\ldots, \qquad (3)$$

It follows from the forward Kolmogorov equation that the n-th moment of T_Y must satisfy the ordinary differential equation

$$axM_n''(x_0, Y; t) + bxM_n'(x_0, Y; t) + h'(t)M_n'(x_0, Y; t) = -nM_{n-1}(x_0, Y; t) \quad (4)$$

Or equivalently

$$M_n''(x_0, Y; t) + \frac{b}{a}M_n'(x_0, Y; t) + \frac{h'(t)}{ax}M_n'(x_0, Y; t) = -\frac{n}{ax}M_{n-1}(x_0, Y; t) \quad (5)$$

Where $M_n'(x_0, Y; t)$ and $M_n''(x_0, Y; t)$ are the first derivatives of $M_n(x_0, Y; t)$ with respect to x $(x_0 \leq x \leq Y)$, with appropriate boundary conditions for n=1,2,3,…….Note that $M_0(x_{0,} Y; t) = 1.$

Now, rewrite the equation in (5), we obtain

$$M_n''(x_0, Y; t) = -\frac{n}{ax}M_{n-1}(x_0, Y; t) - \left(\frac{b}{a} + \frac{h'(t)}{ax}\right)M_n'(x_0, Y; t) (6)$$

Let Δ be the difference operator. Then we defined the first order difference of $M_n(x_0, Y; t)$ as follows:

$$\Delta M_n(x_0, Y; t) = M_{n+1}(x_0, Y; t) - M_n(x_0, Y; t) \quad (7)$$

(Cf. Kelley and peterson (1991)).

Note that equation (6) can be approximated by

$$M_n''(x_0,Y;t)=-\frac{n}{ax}M_{n-1}(x_0,Y;t)-\left(\frac{b}{a}+\frac{h'(t)}{ax}\right)\Delta M_n(x_0,Y;t) \quad (8)$$

By applying equation (7) to equation (8) we get :

$$M_n''(x_0Y;t)=-\frac{n}{ax}M_{n-1}(x_0,Y;t)+\left(\frac{b}{a}+\frac{h'(t)}{ax}\right)M_n(x_0,Y;t) -\left(\frac{b}{a}+\frac{h'(t)}{ax}\right)M_{n+1}(x_0,Y;t) \quad (9)$$

Now, we will use the matrix theory to solve the differential equation defined in equation (9). If we let

$$\vec{M}(x_0,Y;t)=[M_1(x_0,Y;t),M_2(x_0,Y;t),\cdots]'$$

Then we get

$$\frac{d^2\vec{M}(x_0,Y;t)}{dx^2}=A\vec{M}(x_0,Y;t) \quad (10)$$

Where

$$A=\begin{bmatrix} \left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & -\left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & 0 & 0 & \cdots \\ -\frac{2}{ax} & \left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & -\left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & 0 & \cdots \\ 0 & -\frac{3}{ax} & \left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & -\left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & \cdots \\ 0 & 0 & -\frac{4}{ax} & \left(\frac{b}{a}+\frac{h'(t)}{ax}\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Now let

$$\frac{d\vec{M}(x_0,Y;t)}{dx}=\vec{R}(x_0,Y;t) \quad (11)$$

This imply

$$\frac{d^2\vec{M}(x_0,Y;t)}{dx^2}=\frac{d\vec{R}(x_0,Y;t)}{dx} \quad (12)$$

Apply to equation (10), we get

$$\frac{d}{dx}\begin{bmatrix}\vec{R}(x_0,Y;t)\\ \vec{M}(x_0,Y;t)\end{bmatrix}=\begin{bmatrix}0 & A\\ I & 0\end{bmatrix}\cdot\begin{bmatrix}\vec{R}(x_0,Y;t)\\ \vec{M}(x,Y;t_0)\end{bmatrix} \quad (13)$$

Where **I**is the identity matrix and 0 is the zero matrix.
Thus, the solution of the system of equation in (13) is then given by

$$\begin{bmatrix} \vec{R}(x_0, Y; t) \\ \vec{M}(x_0, Y; t) \end{bmatrix} = e^{\begin{bmatrix} 0 & A^* \\ D & 0 \end{bmatrix}} \cdot \begin{bmatrix} \vec{R}(x_0, Y; t) \\ \vec{M}(x, Y; t_0) \end{bmatrix} \quad (14)$$

Where $D = [d_{ij}]; \; i, j \geq 1$ is the diagonal matrix with entries

$$(15) \quad d_{ij} = \begin{cases} (h'(t) - x_0) & ; \, j = i \\ 0 & ; \text{Otherwise} \end{cases}$$

And $A^* = [a^*_{ij}]; \, i, j \geq 1$ is the matrix with entries

$$a^*_{ij} = \begin{cases} -\dfrac{i}{ax} \ln\left(\dfrac{h(t)}{x_0}\right) & ; \; j = i - 1 \\ \left(\dfrac{b}{a} + \dfrac{h'(t)}{ax}\right)(h(t) - x_0) & ; \; j = i \\ -\left(\dfrac{b}{a} + \dfrac{h'(t)}{ax}\right)(h(t) - x_0) & ; \; j = i + 1 \\ 0 & ; Otherwise \end{cases} \quad (16)$$

Note that the matrix e^{B} where $B = \begin{bmatrix} 0 & A^* \\ D & 0 \end{bmatrix}$ is defined by

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\ldots\ldots\ldots.$$

This series is convergent since it is a cauchy operator of equation (2.6) (Cf. Zeifman (1991)).

III. CONCLUSION

In conclusion the advantage of this technique is to use the difference equation to approximate the ordinary differential equation since it is the discretization of the ODE. Also, the system of the solutions in equation (14) gives an explicit solution to the first – passage time moments for the birth and death diffusion processto a general moving barriers. This increases the applicability of the diffusion process in stochastic modeling or in all area of applied probability theory.Aso, in case of a moving linear barier when $h(t) = ct + d$ and in case of a moving constant barrier when $h(t) = c$, we got the same result as in Al-Eideh (2004).

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