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A Note On Control Charts Statistics On Individual Observations

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ABSTRACT: *This article shows that control limits for the Exponentially Weighted Moving Average (EWMA) control chart and the Cumulative Sum (CUSUM) control chart can not be exceeded for a small number of process observations. An upper bound is provided for the magnitude of the EWMA statistic and the CUSUM statistic on individual observations. At least fifteen observations are recommended for construction of an EWMA control chart. It is recommended that at least seven observations be used in constructing a CUSUM control so that reliable results can be obtained. This has practical implications for control charting of short production runs or in Phase 1 statistical process control, since an out-of-control signal is not possible for a small number of observations.*

Keywords*: statistical process control, exponentially weighted moving average, cumulative sum, quality*

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I. INTRODUCTION

A number of industrial processes produce an individual unit of output at a time. Examples of such situations are the chemical and automotive industries where automated gages allow the measurement of every unit manufactured. Manufacturing processes that consist of individual measurements, $X_1, X_2, \ldots, X_n$, are often monitored by an Individuals control chart (X-chart), an Exponential Weighted Moving Average (EWMA) control chart or a Cumulative Sum (CUSUM) control chart. Montgomery (2012) suggests the use of the EWMA control chart or the CUSUM control chart for process monitoring, if small shifts in the process mean are of interest. Koshti (2011) demonstrates by simulation that the CUSUM control chart is more effective than Shewhart control chart in detecting shifts in the process mean of 1.5 sigma or less. Control limits for these charts are constructed using the average moving range, MR, or the sample standard deviation, S, as a measure of dispersion.

In this note, an achievable upper bound is provided for the magnitude of the EWMA statistic and CUSUM statistic on individual observations. Based on this upper bound, it is shown that the usual 3σ control limits can not be crossed for a small number of process observations. This has practical implications for the use of the EWMA control chart and the CUSUM control chart for short production runs or in Phase 1 statistical process control, since an out-of-control signal is impossible for a small number of observations.

II. MAXIMUM EWMA VALUES

The EWMA statistic on individual observations is given by

$$W_i = \lambda X_i + (1-\lambda)W_{i-1}, \quad i = 1, 2,..., n \qquad (1)$$

or equivalently,

$$W_i = \lambda X_i + \lambda\, \Sigma(1-\lambda)^j\, X_{i-j} + (1-\lambda)^n\, W_0, \quad i = 1, 2,..., n \qquad (2)$$

where $0 < \lambda \leq 1$ and $W_0 = \overline{X}$. The constant, λ, is the weight assigned to the present observation. Montgomery (2012) suggests values of λ in the interval $0.05 < \lambda \leq 0.25$. Smaller values of λ are used to detect smaller shifts in the process mean. Control limits are set at

$$\overline{X} \pm k\ (S/c_4)\sqrt{[\ \lambda/(2-\lambda)\ [1-(1-\lambda)^{2i}\ ]\ ]} \qquad (3)$$

where $k > 0$ (usually $k = 3$) and

$$\overline{X} = \Sigma\, X_i/n,$$
$$S = [\ \Sigma\, (X_i - \overline{X})^2 / (n-1)\ ],$$

or

$$\overline{X} \pm k\ (MR/d_2)\sqrt{[\ \lambda/(2-\lambda)\ [1-(1-\lambda)^{2i}\ ]\ ]} \qquad (4)$$

where $MR = \Sigma | X_{i+1} - X_i | / (n-1) \quad i = 2, 3, ..., n.$

The values of c_4 and d_2 are chart constants that depend on n. The chart signals if a value of W_i crosses the control limits.

It is also possible to express the EWMA statistic in its standardized form as

$$Z_i = [c_4(W_i - \bar{X}) / S] \sqrt{[\lambda/(2-\lambda)[1-(1-\lambda)^{2i}]]} \quad (5)$$

with control limits in (3) replaced by ± k. Alternatively, the standardized EWMA statistic can be stated as

$$Z_i = [d_2(W_i - \bar{X}) / MR] \sqrt{[\lambda/(2-\lambda)[1-(1-\lambda)^{2i}]]} \quad (6)$$

with control limits in (4) changed to ± k.

Let $X_1, X_2, ..., X_{n-1}$ be a set of individual observations from a production process. Without loss of generality assume that the observations are ordered and $\bar{X}_{n-1} = 0$. If another observation X_n is added to the previous observations such that $X_n > X_{n-1}$, then

$$\bar{X}_n = (\bar{X}_{n-1} + X_n)/n = X_n/n \quad (7)$$

and

$$S_n^2 = [(n-2)/(n-1)] S_{n-1}^2 + X_n^2/n \quad (8)$$

Shiffler (1998) showed that this sequence of observations results in the largest possible Z score where $Z = (X_n - \bar{X}_n) / S_n$, when S_{n-1} is minimized. For this sequence of observations equation (2) becomes

$$W_n = X_n [\lambda + 1/n (1-\lambda)^n] \quad (9)$$

By substituting (7), (8) and (9) into (5) , the largest positive EWMA is for the n values is

$$Z_n = \frac{c_4 X_n [\lambda + 1/n (1-\lambda)^n] - X_n/n}{\sqrt{\{[(n-2)/(n-1)] S_{n-1}^2 + X_n^2/n\}} \sqrt{[\lambda/(2-\lambda)[1-(1-\lambda)^{2n}]]}} \quad (10)$$

This Z_n is maximized when S_n is minimized. This occurs when $S_{n-1} = 0$ resulting in

$$Z_n = \frac{(c_4/\sqrt{n}) [n\lambda + (1-\lambda)^n - 1]}{\sqrt{[\lambda/(2-\lambda)[1-(1-\lambda)^{2n}]]}} \quad (11)$$

The maximum EWMA values based on standard deviations are given in Table I for different sample sizes and λ values. Table I indicates that a minimum of 70, 35 and 27 observations are necessary to obtain an out-of-control signal when λ is 0.10, 0.20 and 0.25 respectively and control limits are set to k=3 .

The maximum values for the Z-score in (6) based on moving ranges are not as simple to calculate as they are functions of the value of n, and the time order location j of the Z within the series of observations. Woodall (1992) showed that sequences with the format $X_1 = ... = X_j = 0$, $X_{j+1} = X$, $X_{j+2} = ... = X_n = cX$ where $0 \le c \le 1$, $j \ge 1$ and $j \ge n-j-1$, result in the maximum Z possible score for the Individuals control chart when c=1. For such sequences, equation (6) becomes

$$Z_{j+1} = \frac{d_2 \{ X [\lambda + 1/n(1-\lambda)^n(n-j-1) cX] - [X+(n-j-1) cX]/n \}}{MR \sqrt{[\lambda/(2-\lambda)[1-(1-\lambda)^{2n}]]}} \quad (12)$$

The average moving range becomes

$$MR = \Sigma | X_{i+1} - X_i | / (n-1) = X + (X-cX) / (n-1) = X(2-c) / (n-1). \quad (13)$$

Substituting (13) into (12) yields a maximum Z score of

$$Z_{j+1} = \frac{d_2\ (n-1)\ \{n\lambda + [(1-\lambda)^n - 1]\ (n-j)\}}{n\ \sqrt{[\ \lambda/(2-\lambda)\ [1-(1-\lambda)^{2n}]\ ]}} \qquad (14)$$

when c = 1.

Table I. Maximum EWMA Values based on Standard Deviations

λ	n	Z_n
0.10	50	2.460
0.10	70	3.110
0.20	34	2.985
0.20	35	3.043
0.25	28	2.928
0.25	27	3.000

Table II displays the maximum EWMA values based on moving averages for different values of n and λ. Table II shows that a minimum of 15, 10 and 9 observations are necessary to obtain an out-of-control signal when λ equals 0.10, 0.20 and 0.25 respectively, if the control limits are set to k=3.

Table II. Maximum EWMA Values based on Moving Ranges

λ	n	j	Z
.10	14	13	2.75
.10	15	14	3.31
.20	9	8	2.84
.20	10	9	3.39
.25	8	7	2.89
.25	9	8	3.53

III. MAXIMUM CUSUM VALUES

The CUSUM statistic can be represented by a V-mask or equivalently by the use of two one-sided cumulative sums. For individual observations, the latter representation requires calculating:

$$S_i = \max(0, Z_i - k + S_{i-1}), \quad i = 1, 2,..., n \qquad (15)$$

$$T_i = \min(0, Z_i + k + T_{i-1}), \quad i = 1, 2,..., n \qquad (16)$$

where

$$Z_i = c_4(X_i - \overline{X}) / S \qquad (17)$$

and $\overline{X} = \Sigma X_i/n$,

$S = [\Sigma (X_i - \overline{X})^2 / (n-1)]$,

or

$$Z_i = d_2(X_i - \overline{X}) / MR, \qquad (18)$$

where $MR = \Sigma |X_{i+1} - X_i| / (n-1), \quad i = 2, 3,..., n.$

The values of c_4 and d_2 are chart constants that depend on n and k is the smallest size shift in the mean considered important enough to be detected quickly. Commonly used values of k are 0.5 and 0.75. The chart signals if $S_i > h$ or $T_i < -h$, where h is a critical value. Typical values of h are 4 or 5.

Suppose $X_1, X_2, ..., X_{n-1}$ are individual observations from a process. Without loss of generality assume that the observations are ordered and $\overline{X}_{n-1} = 0$. If another observation X_n is added to the previous observations such that $X_n > X_{n-1}$, then

$$\overline{X}_n = (\overline{X}_{n-1} + X_n)/n = X_n/n \qquad (7)$$

and $S_n^2 = [(n-2)/(n-1)]\ S_{n-1}^2 + X_n^2/n$.

(8)

Shiffler (1988) showed that this sequence of observations results in the largest possible Z score where Z = (X_n - $\overline{X}_n$) / S_n, when S_{n-1} is minimized. For this sequence of observations, Woodall (1992) showed that the maximum Z-score in (17) for the Individuals control chart is

$$Z_n = c_4(n-1) /\sqrt{n} \quad . \qquad (19)$$

By substituting (19) into (15) recursively, the maximum positive CUSUM value can be shown to be

$$S_n = [\Sigma\, c_4(n-1) / \sqrt{n}] - (n-1)k \; . \qquad (20)$$

The maximum values based on standard deviations are given in Table I for different sample sizes. Table I indicates that a minimum of six observations are necessary to obtain an out-of-control signal if h = 4 and at least seven are required if h = 5.

T**able I.** Maximum CUSUM Values based on Standard Deviations with k = 0.5

n	S_n
2	0.06
3	0.59
4	1.47
5	2.65
6	4.10
7	5.77
8	7.66

The maximum value for the Z-score in (18) based on moving ranges, is also given by Woodall (1992). He showed that data sets with the format $X_1 = ... = X_j = 0$, $X_{j+1} = X$, $X_{j+2} = ... = X_n = cX$ where $0 \le c \le 1$, $j \ge 1$ and $j \ge n-j-1$, where j is the time order location of the Z-score within the sequence of observations, result in the maximum possible Z score for the Individuals control chart when c=1. For these sequences, the maximum Z-score is given by

$$Z_{j+1} = d_2\, j(n-1) / n \; , \qquad j = 2, 3,..., n. \qquad (21)$$

Substituting (21) into (15) recursively leads to a maximum positive CUSUM value of

$$S_{j+1} = [\Sigma\, d_2\, j(j-1) /nj] - (n-1)k \; , \qquad j = 2, 3,..., n. \qquad (22)$$

The maximum CUSUM values based on moving ranges are given in Table II for different sample sizes.

Table II. Maximum CUSUM Values based on Moving Ranges with k = 0.5

Time Order position (j)	n = Sample Size				
	2	**3**	**4**	**5**	**6**
1	0	0	0	0	0
2	0.06	0.32	0.66	1.06	1.50
3		1.63	2.82	4.13	5.51
4			4.99	7.19	9.51
5				10.26	13.52
6					17.52

Table II shows that at least four observations are necessary to obtain a signal when the control limit, h = 4 and minimum of five observations when h = 5.

IV. CONCLUSIONS

This article provides an achievable upper bound for the magnitude of the EWMA and CUSUM control chart statistics on manufacturing processes that consist of individual observations. Caution should be exercised when the EWMA control chart or the CUSUM control chart are used to monitor processes that consist of short production runs or in Phase 1 statistical process control. An out-of-control signal may not be possible for small data sets because the control limits cannot be crossed. A minimum of fifteen observations is recommended for construction of an EWMA control chart for monitoring short production runs. At least seven observations are recommended for the construction of a CUSUM control chart.

It is also suggested that EWMA control chart limits and CUSUM control chart limits be based on moving ranges and not the standard deviation as a measure of dispersion, since both charts can signal with less observations with limits based on moving ranges.

REFERENCES

[1]. Koshti, V. V. (2011). Cumulative Sum Control Chart, International Journal of Physics and Mathematical Sciences, **1** (1), 28-32.
[2]. Montgomery, D. C. (2012). Introduction to Statistical Quality Control (John Wiley and Sons, New York, 7th ed.).
[3]. Shiffler, R. E. (1988). Maximum Z-Scores and Outliers, American Statistican, **42**, 79-80.
[4]. Woodall, W. H. (1992). A Note on Maximum Z-Scores for Control Charts for Individuals, Communications in Statistics-Theory and Methods, **21** (11), 3211-3217.

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