International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321- 4767 P-ISSN: 2321-4759
www.ijmsi.org Volume 8 Issue 1 || Jan, 2020 || PP-25-29

Number of Equivalent and Non-Equivalent Homeomorphic Topological Spaces for $n \leq 4$

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ABSTRACT: *In this paper, the number of equivalent and non-equivalent homeomorphic topologies are obtained for $1 \leq n \leq 4$. Let X be a finite set having $n \leq 4$ elements, the results for equivalent and homeomorphisms and non-equivalent homeomorphisms are considered from linear recurrence relations $h_{k+2} = (k+3)h_{k+1} - (h+2)h_k$, and extended to the sum of a falling factorial. Meanwhile, non-equivalent homeomorphisms were considered to also follow the difference between number of topology defined on a set and the number of non-equivalent homeomorphisms defined on that set.*

KEYWORDS: *Topological Space, Continuity, Equivalent Homeomorphism, Non-Equivalent homeomorphism*

Date of Submission: 25-02-2020 Date Of Acceptance: 11-03-2020

I. INTRODUCTION

Topology is one of the branches of mathematics that has applications in the practical life. The purpose of topology is to look qualitatively and quantitatively to the properties of geometric objects which do not depend on the exact shape of an object, but more on how the object is put together. It is important to know what it means to say that two different topological spaces are really the same; this thought gave birth to homeomorphisms. The common and classic example in topology suggests that a doughnut and coffee cup are indistinguishable to a topologist. This is because one of the geometric objects can be stretched and bent continuously from the other. There are many concepts related to topologies and homeomorphisms, and however it is interesting to determine how many different homeomorphisms and non-homeomorphisms topologies can be formed on n-points. Nevertheless, progress have been recorded, several research have carried out and many are on the move both on homeomorphisms, topological spaces and their related area some are; Benoumhani in [6], computed the number of topologies having k open sets T(n,k) on the finite set X, having n elements for $2 \leq k \leq 12$. Ern´e and Stege [7] provided the best methods, and gave the number of topologies on an n element set up to n = 14 Kolli [8] and Sharp [10], enumerated the topologies on a finite set. Stanley studied the number of open set of finite topologies [9].The number of chain topologies on X, having k open sets take the form $C(n,k) = \sum_{r-1}^{n-1} \binom{n}{r} C(r, k-1) = (k-1)!\, S(n, k-1)$ with $S(n,k) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^r \binom{k}{r}(k-r)^n$ as proved by Stephen in his work [11]. Kamel in [12] studied the concept of partial chain topology on any finite set with respect to the given subset, and study some properties with respect to some concepts of topological spaces. They also, find the number of all partial chain topologies with respect to the given subset A which help in obtaining the values of number of all chain topologies on it. Francis and Adeniji [1], obtain the number of elements defined on each topological space for open and clopen topologies on $n \leq 4$, which was called k-element. And also discuss the graphical relationship between topologies for open an-d clopen set for $n \leq 3$. Francis and Adeniji [2], obtain the cardinality of cofinite and non-cofinite topologies on set for $n \leq 4$. And, also study the graphical relationship between topologies which are defined on cofinite and non-cofinite topologies on set with $n \leq 3$. many authors introduced various types of homeomorphisms in topological spaces. In [5], Maki, Devi and Balachandran [4] introduced the concepts of semi-generalized homeomorphisms and generalized semi-homeomorphisms and studied some semi topological properties. Devi and Balachandran introduced a generalization of α-homeomorphism in 2001. Renzo's [3], gives a well detailed example on homeomorphism

In this paper our consideration is only on finite topological spaces; i.e space having only finite number of points, and our findings is on $n \leq 4$.

II. PRELIMINARIES

Throughout this paper (X, ρ) and (Y, τ) shall means topological spaces on which no separation are assumed unless otherwise stated. Let T(n) denote the number of topologies on X with $T(n,k)$ as the number of all topologies having k-elements that can be defined on topologies on X . There are many concepts related to topological space. But we consider cardinality of equivalent homeomorphisms topologies and non-equivalent homeomorphisms topological spaces in this paper. These are vital elements, relevant to the work. However, throughout this paper equivalent homeomorphisms topologies and non-equivalent homeomorphisms topologies are denoted by the notations: $T_{eh}(n)$ and $T_{neh}(n)$ respectively, the following examples and definitions are important to sequel.

Definition 2.1[2]: A topology ρ on a set X is a collection of subsets of X such that

I. $\emptyset, X \in \rho$
II. If $M_i \in \rho$ for each $i \in I$, then $\bigcup_{i \in I} M_i \in \rho$
III. If $M_1, \dots, M_n \in \rho$, then $M_1 \cap \dots \cap M_n \in \rho$

A set $M \subseteq X$ is called open if $M \epsilon \rho$, The pair (X, ρ) is called a topological space.

Definition 2.2
Let (X, ρ) and (Y, τ) be topological spaces. Then they are said to be homeomorphic if there exists a function $f: X \rightarrow, Y$ which has the following properties
(i) f is one-to-one
(ii) f is onto
(iii) for each $p \in \tau, f^{-1}(p) \in \rho$
(iv) for each $q \in \rho, f^{-1}(q) \in \tau$

Homeomorphism is the notion of equality in topology and it is a somewhat relaxed notion of equality. But, a common problem in topology is to decide whether two topological spaces are homeomorphic or not. To prove that two spaces are homeomorphic, we shall take a look at equivalent and also establish some basic theorems.
The map f is said to be a homeomorphism between (X, ρ) and (Y, τ).

Example 2.3
Considering topological spaces (X, ρ_x) and (Y, ρ_y)
If $X = \{a, b, c\}$ and $Y = \{a, b, c\}$
We define a mapping such that $f: X \rightarrow Y$ which implies that
$f: \{a, b, c\} \rightarrow \{a, b, c\}$
Where $\rho_x = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\rho_y = \{\emptyset, \{a\}, \{a, c\}, X\}$
Are two topologies define on the two set of points. the mapping above is such that
$f(\{a\}) = \{a\}, f(\{b\}) = \{c\}$
The sets $\{a, b, c\}$ and $\{a, b, c\}$ we shall establish a schematic diagram Figure 3.7 for this class. There is a one-to-one correspondence between the points, and the fact that $f(ab) = \{ac\}$ shows that a one-to-one correspondence exist between the open sets as well. Taking the inverse function, so as to establish continuity
We have
$f^{-1}: Y \rightarrow X$ that is; $f^{-1}: \{a, b, c\} \rightarrow \{a, b, c\}$
Since f^{-1} is continuous, $(f^{-1})^{-1}(P)$ is open in Y for every open set P in X. However, since homeomorphisms are bijective, $(f^{-1})^{-1}(P) = f(H)$ for all $H \subseteq X$. Since f is continuous, $f^{-1}(Q)$ is open in X for every open set $Q \subset Y$. Thus, a homeomorphism f not matches every open set $P \subseteq X$ to a unique open set $Q \subseteq Y$. As shown above, and in the schematic below, $f: \{a, b, c\} \rightarrow \{a, b, c\}$ is homeomorphic

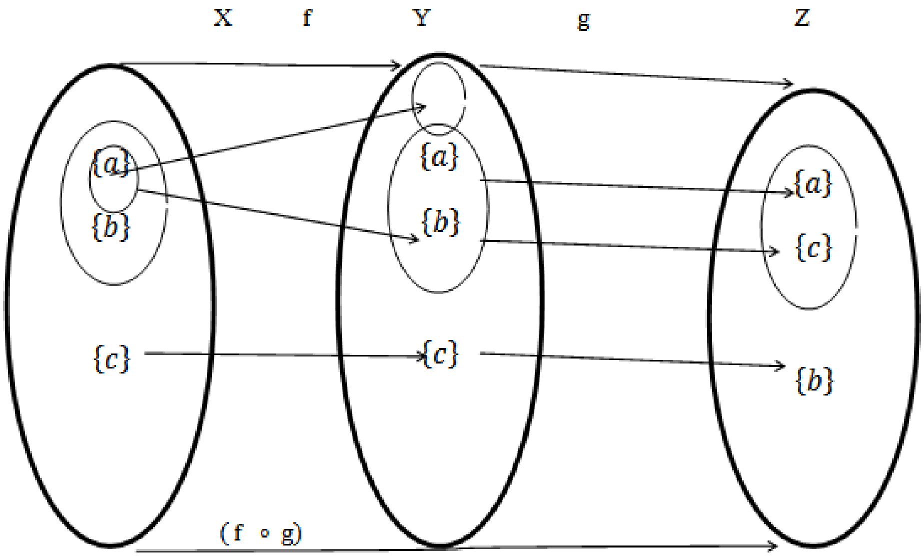

Figure 2.1: Homeomorphism Mapping

Definition 2.4
Let (X, ρ) and (Y, τ) be topological spaces. Then they are said to be homeomorphic if there exists a function $f: X \rightarrow, Y$ which has the following properties
(v) f is one-to-one
(vi) f is onto
(vii) for each $p \in \tau, f^{-1}(p) \in \rho$
(viii) for each $q \in \rho, f^{-1}(q) \in \tau$

III. EQUIVALENCE AND NON-EQUIVALENCE HOMEOMORPHISMS ON TOPOLOGICAL SPACE

Proposition 3.1
Let f be a function from a set X into a set Y. Then
(i) The function f has an inverse if and only if f is bijective
(ii) Let p_1 and p_2 be functions from Y into X. If p_1 and p_2 are both inverse functions of f, then $p_1 = p_2$; that is $p_1(y) = p_2(y)$, from all $y \in Y$
(iii) Let g be a function from Y into X. Then p is an inverse function of f if and only is an inverse function of p.

Theorem 3.2
$$T_{neh}(n) = \sum_{k=1}^{n} K!$$
Proof
Recurrence relation was considered for the proof
Let $h_{k+2} = (k+3)h_{k+1} - (h+2)h_k$ be a recurrence relation, with
$h_1 = 1$ and $h_2 = 3$ for all positive k
If $n = 1,2,3,4, \dots k$. Since, $h_1 = 1$ and $h_2 = 3$ the relation yield the following results
for $n = 3$: $h_3 = 4h_2 - 3h_1 = 9$
for $n = 4$: $h_4 = 5h_3 - 4h_2 = 33$
for $n = 5$: $h_5 = 6h_4 - 5h_3 = 153$
Hence, the series yield the following results in terms of factorials
for $n = 1$: $h_1 = 1 = 1!$
for $n = 2$: $h_2 = 3 = 1! + 2!$
for $n = 3$: $h_3 = 9 = 1! + 2! + 3!$
for $n = 4$: $h_4 = 33 = 1! + 2! + 3! + 4!$
for $n = 5$: $h_5 = 153 = 1! + 2! + 3! + 4! + 5!$
$- -$
for $n = K$: $h_k = 1! + 2! + 3! + 4! + \cdots + k!$
for $n = (K-1)$: $h_{k-1} = 1! + 2! + 3! + 4! + \dots + (k-1)!$

We generalize that the sum of first k factorials satisfies the recurrence relation and gives the non-homeomorphism topologies for any given n, hence

$$T_{neh}(n) = \sum_{k=1}^{n} K!$$

Theorem 3.3

$$T_{eh}(n) = \sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n} K!$$

Proof

We then, proceed with the proof of the next equation for which $T_{eh}(n) = \sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n} K!$. It is important to note that $T(n) = \sum_{k=1}^{2^n} T(n,k)$ which gives the topology for small values of n, have been proof in theorem 4.1. Therefore, the proof that $T_{eh}(n) = \sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n} K!$ gives an Equivalent homeomorphism of a topology for a given point, is done by taken the difference between theorem 4.1 and theorem 4.5.1 for any given n. And the proof is completed as follows:
If $n = 1$. The equivalent homeomorphism is obtains as follows:

$$T_{eh}(1) = \{(T-(1,1) + T(1,2)) - (1!)\} = \{(0+1) - (1)\} = 1 - 1 = 0$$

$$T_{eh}(1) = \sum_{k=1}^{2} T(1,k) - \sum_{k=1}^{2} K! = \sum_{k=1}^{2^1} \mathrm{T}(2,\mathrm{k}) - \sum_{k=1}^{2} \mathrm{K}! = 0$$

The equivalent homeomorphism for $n = 2$, is

$$T_{eh}(2) = \{(T(2,1) + T(2,2) + T(2,3) + T(2,4)) - (1! + 2!)\} = \{(0+1+2+1) - (1+2)\} = 4 - 2 = 2$$

$$T_{eh}(2) = \sum_{k=1}^{2^2} T(2,k) - \sum_{k=1}^{2} K! = \sum_{k=1}^{4} T(2,k) - \sum_{k=1}^{2} K! = 2$$

Similarly the equivalent homeomorphism for $n = 3$, is given by

$$T_{eh}(3) = \{(T(3,1) + T(3,2) + T(3,3) + T(3,4) + T(3,5) + T(3,6) + T(3,7) + T(3,8)) - (1! + 2! + 3!)\}$$

$$T_{eh}(3) = \{(0+1+6+9+6+6+0+1) - (1+2+9)\} = 29 - 9 = 20$$

$$T_{eh}(3) = \sum_{k=1}^{8} T(3,k) - \sum_{k=1}^{3} K! = \sum_{k=1}^{2^3} T(3,k) - \sum_{k=1}^{3} K! = 20$$

Hence, the equivalent homeomorphism for n

$$T_{eh}(n) = \{((T(n,1) + T(n,2) + T(n,3) + \cdots . + T(n,k)) - (1! + 2! + 3! + \cdots + k!))\}$$

therefore

$$T_{eh}(n) = \sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n} K!$$

These satisfy and complete the required proof.

IV. RESULTS

Below is the table of values for homeomorphisms and non-homeomorphisms topologies for some values of n

N	Equivalent Homeomorphism	Non- equivalent homeomorphism	Homeomorphisms
1	0	1	1
2	1	3	4
3	20	9	29
4	322	33	355

V. CONCLUSION

There are lots of computations on formulas for the numbers of topological spaces in finite set. In this paper, we formulated special case for computing the number of equivalent and non-equivalent homeomorphic topological spaces for small value of n. Our results may be refined for $n \geq 4$.

ACKNOWLEDGMENTS

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

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Francis Moses Obinna "Number of Equivalent and Non-Equivalent Homeomorphic Topological Spaces for n≤4" International Journal of Mathematics and Statistics Invention (IJMSI), vol. 08(01), 2020, pp.25-29.