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On Convex Optimization in Hilbert Spaces

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ABSTRACT:*In this paper,we employ the techniques for convex optimization problems in infinite dimensional real Hilbert spaces. We review the necessary theorems and present concise proofs of relevant results.*
KEYWORDS:*Convexity, Hilbert spaces & Optimality Conditions.*

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I. INTRODUCTION

The study of optimization problems in infinite-dimensional spaces began in the 17th century: the development of the calculus of variations, motivated by physical problems, focused on the development of necessary and sufficient optimality conditions and finding closed-form solutions. Much later, the advent of computers in the mid-20th century led to the consideration of finite-dimensional optimization from an algorithmic point of view, with linear and nonlinear programming. Finally, a general theory of optimization in normed spaces began to appear in the 70's [10]. Infinite-dimensional optimization problems arise in many research fields, including minimal surfaces, elliptic PDEs, image processing, semiconductor design, structural optimization, optimal control, and shape optimization or topology optimization. ([2], [8], [10]). These problems were often solved approximately by applying discretization techniques; the original infinite-dimensional problem is replaced by a finite-dimensional approximation that can then be tackled using standard optimization techniques. However, the resulting discretized optimization problems may comprise a large number of optimization variables, which grows unbounded as the accuracy of the approximation is refined [10].

Existing theory and algorithms that directly analyze and exploit the infinite-dimensional nature of an optimization problem are mainly found in the field of convex optimization. These algorithms rely mostly on duality in convex optimization in order to construct upper and lower bounds on the optimal solution value, although establishing strong duality in infinite-dimensional problems can prove difficult ([10], [11], [13]). Convex optimization plays important roles on many areas of mathematics, applied sciences, and practical applications [3]. It is a constituent of three major disciplines: optimization, convex analysis and numerical computation [6]. In recent years, convex optimization became a computational tool of central importance in engineering due to its ability to solve very large, practical engineering problems reliably and efficiently [7]. [5] presented convexity as a simple and natural notion which can be traced back to Archimedes (circa 250 B.C.), in connection with his famous estimate of the value of π (using inscribed and circumscribed regular polygons). He noticed the important fact that the perimeter of a convex figure is smaller than the perimeter of any other convex figure, surrounding it.

In infinite-dimensional spaces, the fundamental problem of convex optimization is that, unlike finite-dimension spaces, being closed and bounded does not imply that a set is compact. In reflexive Banach spaces, this problem is mitigated by working in weak topologies and using the result that the closed unit ball is weakly compact. This in turn enables the mimicking of similar concepts in finite dimensional spaces when working on unconstrained optimization problems. In real Hilbert spaces, closed and convex set is also weakly sequentially closed and any bounded sequence has a weakly convergent subsequence. Moreover, in Banach spaces, the Eberlein-Šmulian Theorem states that weak compactness and sequential weak compactness are equivalent ([1], 9]).

In this paper, we investigate the optimization of convex functionals in infinite-dimensional real Hilbert spaces and review relevant theory and results.

II. PRELIMINARIES

Definition 2.1. A set $D \subseteq \mathbb{R}^n$ is said to be bounded if there exists a constant $M > 0$ such that $\|x\| < M$, for all $x \in D$.

Definition 2.2. The set D is said to be compact if it is closed and bounded.

Definition 2.3. A normed space X is called a Banach space if it is complete, i.e., if every Cauchy sequence in X converges to an element of X.

Definition 2.4. An inner product space H is called a Hilbert space if it is complete with respect to the induced norm.

Remark. A *Hilbert space* is an inner product space H that is complete with respect to the induced norm [4]. That is, for $x \in H \|x\| = \sqrt{\langle x, x\rangle}$.

Definition 2.5. A sequence $\{x_n\}$ in a Banach space B is said to converge to $x \in B$ if $lim_{n\to\infty} x_n = x$. Also as sequence x_n in a Hilbert space H converges weekly to x if, $lim_{n\to\infty}\langle x_n, u\rangle = \langle x, u\rangle$, $\forall u \in H$. We use the notation $x_n \rightharpoonup x$ to mean that x_n converges weekly to x.

Definition 2.6. A real valued function f on a Banach space is lower semi-continuous (lsc) if

$$f(x) \leq \lim_{n\to\infty} \inf f(x_n)$$

for all sequence $\{x_n\}$ in X such that $x_n \to x$ (strongly) and weakly sequentially lower continuous (weakly LSC) if $f(x) \leq \lim_{n\to\infty} \inf f(x_n)$

for all sequence $\{x_n\}$ in X such that $x_n \rightharpoonup x$.

Definition 2.7. Let M be a closed subspace of a Hilbert space H.

(a) Given $x \in H$, the unique vector $p \in M$ that is closest to x is called the orthogonal projection of x onto M.

(b) The function $p: H \to H$ defined by $Px = p$, where p is the orthogonal projection of x onto M, is called the orthogonal projection of H onto M.

Definition 2.8. A sub-set C of H is said to be convex if for all for $\alpha \in [0,1]$ and for all $x, y \in C$, $\alpha x + (1-\alpha)y \in C$.

Definition 2.9. Let C be a non-empty convex subset of H. A function $f: C \to \mathbb{R}$ is convex if for all $\alpha \in [0,1]$ and for all $x, y \in C f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$.

Remark. The function f in the above definition is said to be strictly convex if the inequality in the above definition is strict for $x \neq y$ and $\alpha \in (0,1)$.

Remark. An optimization problem is convex if both the objective function and the feasible set is convex.

Definition 2.10. The epigraph of a function $epi(f) = f(x,\lambda) \in dom(f) \times \mathbb{R}: f(x) \leq \lambda$.

Lemma 2.1. A function $f: C \to \mathbb{R}$ is convex if convex if and only if the epigraph is convex.

Definition 2.11. Let $\mathbb{R}^n$ be an n-dimensional real space and $K \subseteq \mathbb{R}^n$. We say that $\bar{x} \in \mathbb{R}^n$ is a global minimizer of the optimization problem $min_{x\in K} f(x)$, if there exist Definition 2.7 Let $\mathbb{R}^n$ be an n-dimensional real space and $K \subseteq \mathbb{R}^n$. We say that $\bar{x} \in \mathbb{R}^n$ is a global minimizer of the optimization problem $min_{x\in K} f(x)$, if $\bar{x} \in K$ and $f(\bar{x}) \leq f(x)$, for all $x \in K$.

Definition 2.12. Let $\mathbb{R}^n$ be an n-dimensional real space and $K \subseteq \mathbb{R}^n$. We say that $\bar{x} \in \mathbb{R}^n$ is a local minimizer of the optimization problem $min_{x\in K} f(x)$, if $\bar{x} \in K$ and $f(\bar{x}) \leq f(x)$, for all $x \in K$ if there exists $\varepsilon > 0$ such that $f(\bar{x}) \leq f(x)$, for all $x \in K$ which satisfies $\|x - \bar{x}\| \leq \varepsilon$.

Remark. Any local minimizer of a convex optimization is a global minimizer [6].

Theorem 2.2 Every continuous function on a compact set attains its extreme values on that set.

Definition 2.14. A real valued function f on a Banach space X is said to be coercive if

$\lim_{\|x\|\to\infty} f(x) = \infty$.

III. LOWER SEMICONTINUOUS FUNCTIONS

Proposition 3.1. Let X be a Banach space and $f: X \to \mathbb{R}$. Then the following are equivalent [12].

(a) f is (weakly sequentially) LSC.

(b) epi(f), is (weakly sequentially) closed.

Lemma 3.2. Let $C \subseteq H$ be a (strongly) closed and convex set. Then, C is weakly sequentially closed.

Proof. Let $\{x_n\}$ be a sequence in C and suppose $x_n \rightharpoonup \bar{x}$. We show $x \in C$ by showing $\bar{x} = \varphi_C(\bar{x})$ where $\varphi_C(\bar{x})$ denotes the projection of $\bar{x}$ into the closed convex set C. But that the projection $\varphi_C(\bar{x})$ satisfies the variational inequality, $\langle \bar{x} - \varphi_C(\bar{x}), y - \varphi_C(\bar{x})\rangle \leq 0$ for all $y \in C$. Therefore,

$$\langle \bar{x} - \varphi_C(\bar{x}), x_n - \varphi_C(\bar{x})\rangle \leq 0, \text{ for all } n \in \mathbb{N} \tag{3.1}$$

Since $x_n \rightharpoonup \bar{x}$,

$$\|\bar{x} - \varphi_C(\bar{x})\|^2 = \langle \bar{x} - \varphi_C(\bar{x}), \bar{x} - \varphi_C(\bar{x})\rangle$$
$$\lim_{n\to\infty} \langle \bar{x} - \varphi_C(\bar{x}), x_n - \varphi_C(\bar{x})\rangle$$

Thus, by equation (3.1) we have, $\|\bar{x} - \varphi_C(\bar{x})\| = 0$, that $\bar{x} = \varphi_C(\bar{x})$. ■

Lemma 3.3. $f: H \to \mathbb{R}$ be a LSC convex function. Then, f is weakly LSC.

Proof. Since f is convex, $epi(f)$ is convex. Since f is strongly (LSC), $epi(f)$ is strongly closed. By Lemma 3.2. $epi(f)$ is weakly sequentially closed, which implies that f is weakly LSC. ■

IV. OPTIMALITY CONDITIONS

Theorem 4.2 Let C be a weakly sequentially closed and bounded subset of H. Let $f: C \to \mathbb{R}$ be weakly sequentially LSC. Then f is bounded from below and has a minimizer on C.

Proof. Firstly, we prove that f is bounded from below. Suppose to the contrary that f is not bounded from below. Then there exist a sequence $\{x_n\} \in C$ such that $f(x_n) < -n$ for all n. Since C is bounded $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\}$, $x_{n_k} \rightharpoonup \bar{x}$. Furthermore, C is weakly sequentially closed and hence $\bar{x} \in C$. Then, since f is weakly sequentially LSC we have $f(\bar{x}) \leq lim \inf f(x_{n_k}) = -\infty$ which is a contradiction. Hence, f is bounded from below.

Next, we prove the existence of a minimizer. Let $\{x_n\} \in C$ be a minimizing sequence for f; that is $f(x_n) \to \inf_C f(x)$. Let $\lambda = \inf_C f(x)$. Since C is bounded and weakly sequentially closed, it follows that $\{x_n\}$ has a weakly convergent subsequence $x_{n_k} \rightharpoonup \bar{x} \in C$. Next, since f is weakly sequentially LSC we have

$$\lambda \leq f(\bar{x}) \leq lim \inf f(x_{n_k}) = lim\, f(x_{n_k}) = \lambda.$$

Hence, $f(\bar{x}) = \lambda$. ■

Theorem 4.2 Let C be a convex, strongly closed, and bounded subset of H. Let $f: C \to \mathbb{R}$ be a strongly LSC and convex function. Then f is bounded from below and attains a minimizer on C.

Proof. We want to show that the hypotheses of Theorem 4.1holds. Since C is strongly closed and convex, then it is also weakly sequentially closed by Lemma 3.2. Moreover, since f is strongly LSC and convex, it is also weakly LSC by Corollary 3.3. Thus, we have $f: C \to \mathbb{R}$ weakly LSC and C a weakly closed and bounded set in H. By **Theorem 2.2,** we conclude that f is bounded from below and attains a minimizer on C.
■

Corollary 4.3 Let $f: H \to \mathbb{R}$ be a strongly lsc, convex, and coercive function. Then f is bounded from below and attains a minimizer.

Proof. [1].

V. CONCLUSION

This paper has presented the techniques for convex optimization problems in infinite dimensional real Hilbert spaces. It reviewed the necessary theorems and presented concise proofs of relevant results.

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