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An Inventive Approach to Beal's Conjecture

Joseph L. Ramirez
Alumnus, California State University, Fresno, California – 93740

ABSTRACT*: A "user–friendly" model of Beal's Conjecture demonstrates the essential role of a common factor. Proof of the conjecture follows from abstract algebra and binomial expansion.*
KEYWORDS*: Basis, Comparative model, Modification, Summative identity.*

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Beal's Conjecture: If $A^x+B^y = C^z$, where A, B, C, x, y and z are positive integers and x, y and z are all greater than 2, then A, B and C must have a common prime factor.

"I see nobody on the road," said Alice. "I only wish I had such eyes," the King remarked in a fretful tone, "To be able to see Nobody!"
— *Lewis Carroll, Alice's Adventures in Wonderland & Through the Looking-Glass*

I. INTRODUCTION

In 1993, D. Andrew Beal discovered the equation $A^x+B^y = C^z$ is impossible for co–prime bases[1]. To date, the problem known as Beal's conjecture remains unsolved despite Beal's offer of a reward for a solution.

Using the basic rule of problem solving, "state the problem," we single out the obscurity of the equation $A^x+B^y = C^z$ as the reason Beal's conjecture remains unsolved. Unfortunately, "seeing obscurity" does not shield us from the circular arguments encountered trying to "circumvent obscurity." Knee–deep in oxymora, it comes to mind the task of side–stepping "what is not easily distinguished" is commonplace in the real world courtesy of the process of *imaging* (x–ray, ultrasound, etc.). In particular, the *comparative images* produced are routinely used to perform inspections when direct observation is out of the question. Inspired by this, we embark on the task of building our own *comparative model* of $A^x+B^y = C^z$ with the help of several useful principles derived from the 2nd degree and later extended to powers greater than 2.

For example, the equation $4^2+3^2 = 5^2$ is rewritten as $2^4+3^2 = 5^2$ under the *modification* $4^2 = (2^2)^2 = 2^4$, which follows from the properties of exponents. Because this only requires a suitable base, the term A^x becomes the easiest piece of the puzzle to solve.

The largest piece of the puzzle is A, B and C must have a common factor. Since the common factor in the equation $(ak)^2+(bk)^2 = (ck)^2$ requires the equation $a^2+b^2 = c^2$ as a framework or *basis*, the common factor alleged to be in A, B and C would likewise require a basis comparable in function to $a^2+b^2 = c^2$. To find such a basis, we call on the *summative identity* (shown below) as a means of investigating $a^2+b^2 = c^2$ in detail.

$$1+3+5+7+9\ldots+2a-1 = a^2 \qquad (1)$$

The relationship between the odd numbers and powers of 2 is often used to demonstrate the Principle of Mathematical Induction[2] and therefore has been proven multiple times; hence, working hand in hand with the Pythagorean theorem, the summative identity provides an established platform for the principles we put forth in pursuit of a model of $A^x+B^y = C^z$ ideally as transparent as the perfect example $(ak)^2+(bk)^2 = (ck)^2$.

Our investment in the 2nd degree proves worthwhile as a preliminary basis for our comparative model emerges. Things move quickly once we extend the preliminary design to powers greater than 2 and determine a general solution. True to our underlying goal, the transparency in the design showcases the common factor, whose role incidentally is apparent even in the preliminary stage. After verifying the conjecture has substance, we address the issue of discounting the possibility of a solution for co–prime bases. This is made possible by the connection we make between the Binomial Theorem[2] and the summative identity.

II. NOTATION

BWOC	:	By way of contradiction.
Conjugate	:	Having common features but opposite or inverse in some particular.
gcd	:	Greatest common divisor.
WLOG	:	Without loss of generality.
Z[x]	:	The set of polynomials with integer coefficients.
Z^+	:	The set of positive integers.

III. FORMAL PRINCIPLES

We begin by refashioning the *summative identity* (Figure 1) using the inductive extension:

$$(1+3+5\ldots+2a-1)+2a+1=(a+1)^2 \qquad (2)$$

Assume non–negative integer values for the variable a. By substituting from Figure 1 into Figure 2, we obtain: $a^2+2a+1=(a+1)^2$

The abbreviation is useful: $(1+3+5+7+9+11+13+15+17+19)+21=(10+1)^2 \Leftrightarrow 10^2+21=11^2$. The expression a^2+2a+1indicates a^2 is followed by exactly one odd term, which by definition is neatly indicated by $(a+1)^2$. For $k\in Z^+$, $(a+k)^2$indicates a^2 is followed by k terms. The k terms are formally identified by $2ak+k^2$.

$2ak+k^2=(2a_1+2a_2+2a_3\ldots+2a_k)+(1+3+5\ldots+2k-1)=2a+1+2a+3+2a+5\ldots+2a+2k-1$

Principle 1: Because a^2 and $(a+k)^2$ are *automatically defined* by the progression of odd sums, the value of the distal sum $2ak+k^2$ is the deciding factor between $a^2+b^2=c^2$ and the *conjugate* $a^2+b=c^2$. Example: $(1+3+5)+(7+9)=(3+2)^2 \Leftrightarrow 3^2+4^2=5^2$

To verify, it is sufficient to show for k =1 and k =2, $2ak+k^2$ determines the relatively co–prime triples.

(a) $2a+1=(2n+1)^2 \Leftrightarrow a=2n^2+2n$ (3, 4, 5), (5, 12, 13), (7, 24, 25), … $(2n+1,\ 2n^2+2n,\ 2n^2+2n+1)$

((b) $4a+4=(2n)^2 \Leftrightarrow a=n^2-1$ (2·4, 15, 17), (2·6, 35 ,37), (2·8, 63, 65), … $(2n,\ n^2-1,\ n^2+1)$

Each relatively co–prime triple serves as a basis for composite triples, which is where *common factors* come in.

Operations are necessarily associative on the summative sequence. For example:

$4^2+3^2=(1+3+5+7)+9=(1+3+5)+(7+9)=3^2+4^2$

Principle 2: A *common factor* expands the summative sequence at each term.

The distributive property operates as an associative expansion of the summative sequence. To illustrate:

$2^2(3^2)=4(1+3+5)=4+12+20=(1+3)+(5+7)+(9+11)=6^{\,2}$

Each of the terms in (1+3+5) expands under $k^2(2a'+1)=2(ka')k+k^2$. The significance is, this occurs apart from the basis $a^2+b^2=c^2$ and thus independently of the closure rule $x^a(y^a)=(xy)^a$.

Principle 3: The identity $a^2+2a+1=(a+1)^2$ determines the argument of summation $2a+1=(a+1)^2-a^2$.

$1+3+5+7\ldots+2a+1 \Leftrightarrow (1^2-0^2)+(2^2-1^2)+(3^2-2^2)+(4^2-3^2)\ldots+(a+1)^2-a^2$

Observations: By applying our principles, we adapt the Pythagorean model as follows:

$a^2+b=c^2 \Rightarrow (ab)^2+b^3=(bc)^2$

The composite $(ak)^2+bk^2=(ck)^2$ exhibits the modification b^3 for the case k=b. The adaptation is thus a special case of $a^2+b=c^2 \Rightarrow (ak)^2+bk^2=(ck)^2$, where in addition to Princlple1, we find the condition:

$b^3=2ak+k^2 \Rightarrow a^2+b^3=c^2$ **provided gcd(a,b,c) = b**.

With our preliminary basis in hand, we are prepared to let Principle 3 provide the means with which we extend our design beyond the 2nd degree.

IV. THE BEAL BASIS

Proposition: Every non–trivial positive integer power n has a unique summative identity.

Proof: Assume a is non–negative and $n\in Z^+$. Adding zero to $(a+1)^n$ obtains:

$$a(0)+(a+1)^n=(1-1)+(2^n-2^n)+(3^n-3^n)\ldots+(a^n-a^n)+(a+1)^n$$

$$(a+1)^n=1+(2^n-1)+(3^n-2^n)+(4^n-3^n)\ldots+(a^n-(a-1)^n)+(a+1)^n-a^n$$

- As consecutive integers, the minuend and subtrahend in each binomial difference are of opposite parity and therefore each respective difference is odd.
- The argument of summation is the unique polynomial form defined by the Binomial Theorem.

$$(a+1)^n-a^n=m_1a^{n-1}+m_2a^{n-2}+m_3a^{n-3}\ldots+m_{n-2}a^2+m_{n-1}a+1$$

A comprehensive example in the 3rd degree is given by:

$$1+7+19+37+61\ldots+3a^2+3a+1=(a+1)^3 \Leftrightarrow a^3+3a^2+3a+1=(a+1)^3 \Leftrightarrow 3a^2+3a+1=(a+1)^3-a^3$$

Application: To see a specific solution in the 3rd degree, consider:

$$16^3+817=(17)^3 \Rightarrow (16\cdot817)^3+817^4=(17\cdot817)^3 \Leftrightarrow A^z+B^y=C^z$$

For the finishing touch, we take the modifications $16^3=(2^4)^3$ and $817^{12}=(817^4)^3$, with which we obtain:

$$A^z+B^y=C^z \Leftrightarrow (2^4\cdot817^4)^3+817^{13}=(17\cdot817^4)^3 \Leftrightarrow (2^3\cdot817^3)^4+817^{13}=(17\cdot817^4)^3 \Leftrightarrow A^x+B^y=C^z$$

Generalization: Let a, c and z≥3 be positive integers. WLOG, $a^z<c^z \Rightarrow 0<c^z-a^z=b\in Z^+$. Then $b=c^z-a^z$ determines the **basis** $a^z+b=c^z$. By multiplying both sides of the equation by b^z, we obtain the partial solution:

$$(ab)^z+b^{z+1}=(bc)^z \Leftrightarrow A^z+B^y=C^z$$

Let $(a^x)^z$ represent a modifiable case. Then a full solution is modeled by:

$$(a^x)^z+b=c^z \Rightarrow (a^xb^x)^z+b^{xz+1}=(cb^x)^z \Leftrightarrow (a^zb^z)^x+b^{xz+1}=(cb^x)^z \Leftrightarrow A^x+B^y=C^z$$

In particular: $b^y=(a+k)^z-a^z \Rightarrow a^z+b^y=c^z$ **provided gcd(a,b,c) = b**.

Concluding observations: The key principle is, B^y is a multiple of $b=(a+k)^z-a^z$; therefore, by the property of equality, a **common factor** is inevitable as long as we adhere to the rule $x^a(y^a)=(xy)^a$.

To put this in perspective, consider that – in the 2nd degree – a common factor determines the fixed ratio between corresponding parts (leg, base, and hypotenuse) of *similar* right triangles in the plane. To be creative, we imagine a one–dimensional "triangle" whose leg and base form a 180° angle; hence, the hypotenuse coincides with the combined length of leg and base. A common factor therefore determines similar "triangles."

$\vert\leftarrow\ 3^2\ \rightarrow\vert$	$\vert\leftarrow\ 15^2\ \rightarrow\vert$	$\vert\leftarrow\ 17^3\ \rightarrow\vert$	$\vert\leftarrow\ (17\cdot817^4)^3\ \rightarrow\vert$
●--------●---------●	●---------●-----------●	●-----------●-------------●	●------------●--------------------●
2^2 5	10^2 5^3	16^3 817	$(8\cdot817^3)^4$ 817^{13}

V. PROVING THE CONJECTURE

Assume a, b, k, x, y and z are positive integers. Set $y\geq3$ and $z\geq 3$. Let p, f, g denote polynomials. It is sufficient to show $a^z+b^y=(a+k)^z$ is impossible for co-prime values of a, b and a+k.

BWOC, assume $a^z+b^y=(a+k)^z$ for co-prime values of a, b and a+k. By subtraction, $b^y=(a+k)^z-a^z$. Furthermore, by the Binomial Theorem, where $m_i\in Z^+$ for each $i\geq1$:

$$b^y = m_1a^{z-1}k+m_2a^{z-2}k^2\ldots+m_{z-1}ak^{z-1}+k^z \qquad (3)$$

Let p denote the polynomial on the right–hand side of the equation in Figure 3. By premise, p contains positive integer coefficients. Since $Z^+\subset Z$, $\mathbf{p\in Z[x]}$.

It is evident k is the gcd of the coefficients of p and therefore $k^{-1}p\in Z[x]$.[3] In particular:

$$k^{-1}p = [(m_1a^{z-1}k)k^{-1} + (m_2a^{z-2}k^2)k^{-1}\ \ldots +(a^0k^z)k^{-1}] \quad\Rightarrow\quad p = k[(m_1a^{z-1} + m_2a^{z-2}k\ \ldots +a^0k^{z-1}]$$

Given $y\geq3$, $bb^{y-1}=p$ implies b divides p. It is immediate $b^{-1}p\in Z[x]$.

$b^{-1}p\in Z[x]$ and $k^{-1}p\in Z[x]$ implies $(bk)^{-1}p\in Z[x]$, then by construction, $fig\in Z[x]$ where:

$$p = fig = (bk)[(m_1a^{z-1})b^{-1} + (m_2a^{z-2}k)b^{-1}\ \ldots +(m_{z-1}ak^{z-2})b^{-1} +(a^0k^{z-1})\, b^{-1}]$$

Observe $(m_1a^{z-1})b^{-1}$ and $(a^0k^{z-1})b^{-1}$ imply b divides a and k; otherwise, fg contains rational coefficients.

It must be the case $a=(ba')$, $k=(bk')$ and $(a+k)=(ba'+bk')$ – a contradiction.

This shows $a^z+b^y=c^z$ is impossible for co–prime values of a, b and c = a+k. In addition, since a was arbitrary, the conclusion holds for the case $(a^x)^z+b^y=c^z$. Comparatively, the equation $A^x+B^y=C^z$ is impossible for co–prime values of the bases A, B and C.

VI. ADDITIONAL CONSIDERATIONS

1. *Wording of the conjecture*: Instances such as $(2\cdot152^3)^6+152^{19}=(6\cdot152^6)^3$ exhibit non–prime common factors; therefore, the requirement for a "common prime factor" in the wording is superfluous.
2. *Modifications*: Other modifications of z include $(a^n)^z=a^{nz}$ and $a^z=(a^n)^x$ for $z=nx$. Composite powers in particular provide a larger selection of possible modifications.
3. *Short cut*: Since A^x is determined relative to a^z, we may use $b'=b\pm b_1$ such that $b^y=(a+k)^z-a^z\pm b^z(b_1)$. Example: $2^3+(8+11)=3^3 \Rightarrow 2^4+11=3^3$. Hence, $a^x<c^z \Rightarrow 0<c^z-a^x=b\in Z^+$ is a useful extension.

VII. CONCLUSION

We show for values of x, y and z greater than 2, $A^x+B^y=C^z$ provided A, B and C have a common factor.

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