Contra Pairwise Continuity in 
Bitopological Spaces 

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ABSTRACT: Erdal Ekici in a paper [6] has been introduced a new class of functions, called Contra R-map and proved various results related with Contra R-maps and nearly compact Spaces, S-closed spaces, contra R-graphs, connected spaces, hyper connected spaces etc. By introducing the notion of contra pairwise continuity in bitopological spaces we obtain generalization of various results of Erdal Ekici. 

KEY WORDS AND PHRASES : Bitopological space, contra pairwise continuity, pairwise super continuity, contra pairwise R-map, pairwise almost s-continuous functions. 

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1. INTRODUCTION: 
Noiri [10] introduced the concepts of \( \delta \)-continuity and strong \( \theta \)-continuity in topological spaces. “Recall a function \( f:(X, \tau) \rightarrow (Y, r) \) is \( \delta \)-continuous if for each \( x \in X \) and \( V \in r \) containing \( f(x) \) then there is a \( U \in \tau \) containing \( x \) such that \( f(\text{int}(CIU)) \subseteq \text{int}(Cl(V)) \). Munsi and Bassan introduced a strong form of continuity, called super continuous mappings, which implies \( \delta \)-continuity and is implied by strong \( \theta \)-continuity. In (1983a) Reilly and Vamanamurtty [14] have investigated super continuous functions by relating these functions with continuous functions. Continuing work in the same direction Arya and Gupta [1] introduced the class of completely continuous
functions and obtained several preservation results for topological properties with respect to these functions.

Velicko [17] introduced the notion of $\delta$-open and $\delta$-closed sets in a space $(X, \tau)$.

In a topological space $(X, \tau)$ a set $A$ is called regular open if $A = \text{int}(\text{Cl}(A))$ and regular closed if $A = \text{Cl}(\text{int}(A))$. Since the intersection of two regular open sets is regular open, the family of regular open sets forms a base for a smaller topology $\tau_s$ on $X$, called the semi regularization of $\tau$. The space $(X, \tau)$ is said to be semi-regular if $\tau_s = \tau$. Any semi regular space is semi regular, but the converse is not true.

**Remark 2.3** of [11] is: for a continuous function $f: X \to Y$ gives the following implications:

$\delta$-Closedness $\rightarrow$ Star-Closedness $\rightarrow$ almost-closedness and

**Remark 2.6** of [10] is: for a function $f: X \to Y$ gives the following implications :-

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Continuity

SC  \leftarrow \delta\text{-Continuity} \rightarrow \text{almost continuity}
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"Recall a function $f: X \to Y$ is said to be super continuous (Munshi and Bassan 1982) (resp. $\delta$-continuous (Noiri 1980)), almost continuous (Singal and Singal 1968) if for each $x \in X$ and each open nbhd $V$ of $f(x)$, there exists an open nbhd $U$ of $x$ such that $f(\text{int}(\text{Cl}(U))) \subset V$ [resp. $f(\text{Int}(\text{Cl}(U))) \subset \text{Int}(\text{Cl}(V))$, $f(U) \subset \text{Int}(\text{Cl}(V))$ "Super Continuous" will be briefly denoted by “SC”."
“Recall a function \( f: X \rightarrow Y \) is called a R-map (Carnhan 1973) if \( f^{-1}(V) \) is regular open in \( X \) for each regular open set \( V \) of \( Y \).”

Example: Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\} \). Let \( f: (X, \tau) \rightarrow (X, \tau) \) be the identity function. Then \( f \) is super continuous (12) but it is not strongly \( \theta \)-continuous by proposition (3) of Reilly and Vamanmurty (14). Moreover, \( f \) is an R-map, but it is not completely continuous since \( \{a, b\} \) is not regular open in \( (X, \tau) \).

Remark: By theorem 4.11 of T. Noiri [11] gives the following implications:

Strong Continuity \( \rightarrow \) Perfect Continuity \( \rightarrow \) Complete Continuity \( \rightarrow \) R-map \( \rightarrow \delta \)-Continuity.

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\begin{align*}
\text{CoC} & \rightarrow \text{S.Q.C.} & \text{SC} & \rightarrow \text{C} \\
\text{SC} & \rightarrow \text{PC} & \text{C} & \rightarrow \text{R} & \delta \text{-C} & \rightarrow \text{aC}
\end{align*}
\]

where CoC = Cl-open Continuous, S\( \theta \)C = Strongly \( \theta \)-Continuous, C = Continuous, SC = Strongly Continuous, PC=Perfectly Continuous, CC=Completely Continuous, R = R-map, \( \delta \text{C} = \delta \)-Continuous ac = almost continuous

Now it is clear that there are many types of continuities introduced by several authors. A new class of function, called contra R-map has been defined and studied in[6]. It is shown that contra R-map is strictly weaker than regular set connectedness. Erdal Ekici [6] investigated the relationships among contra R-map and nearly compact spaces, S-closed spaces, contra R-graphs connected spaces hyper connected spaces etc. and discuss various properties of such spaces. In this paper we introduced a new class of functions called pairwise contra R-map in bitopological space and have got many theorems giving a generalization to Erdal's theorems by using the tools of pairwise perfectly continuous functions, pairwise almost S-continuous functions and pairwise regular set-connected functions etc.
“A bitopological space is a triple \((X, \tau_1, \tau_2)\), where \(X\) endowed with two topologies \(\tau_1\) and \(\tau_2\). The study of bitopological was initiated by J.C. Kelly and was further pursued by various authors including T. Birsan, H.B. Hoyle, C.W. Patty, W.J. Patty, I.L. Reilly, J. Swart etc.”

Throughout by a space we mean a bitopological space. Symbols \(X\) and \(Y\) are used for spaces and \(f\) is used for maps between bitopological spaces. For terms and notation not explained here we refer the reader to [6, 7, 8, 12, 15].

II. PRELIMINARIES:

Throughout this paper \((X, \tau_1, \tau_2)\) and \((Y, \tau_r, \tau_2)\) represent bitopological spaces. Let \(A\) be a subset of a space \((X, \tau_1, \tau_2)\), then

\(\tau_1\)-Cl\((A)\) represents \(\tau_1\)-closure of \(A\)

\(\tau_2\)-Cl\((A)\) represents \(\tau_2\)-closure of \(A\)

\(\tau_1\)-Int\((A)\) represents \(\tau_1\)-interior of \(A\)

\(\tau_2\)-Int\((A)\) represents \(\tau_2\)-interior of \(A\)

\(A \subseteq \tau_1\text{cl}\ (\tau_1\text{ int}(A))\) represents \(\tau_1\)-semi open set

\(A \subseteq \tau_2\text{cl}\ (\tau_2\text{ int}(A))\) represents \(\tau_2\)-semi open set

“The complement of a \(\tau_1\)-semi open set (respectively \(\tau_2\)-semi open) is called \(\tau_1\)-semi closed (respectively \(\tau_2\)-semi closed).”

“The intersection of all \(\tau_1\)-semi closed sets (respectively \(\tau_2\)-semi closed) containing \(A\) is called the \(\tau_1\)-semi-closure (respectively \(\tau_2\)-semi closure) of \(A\) and is denoted by \(\tau_{1s}\text{-Cl}(A)\) (respectively \(s-\tau_2\text{Cl}(A)\)).”
“The \(\tau_1\)-semi interior (respectively \(\tau_2\)-semi interior) of \(A\) is defined by the union of all \(\tau_1\)-semi open sets (respect. \(\tau_2\)-semi open) contained in \(A\) and is denoted by \((s-\tau_1\text{-int}(A))\) (\(s-\tau_2\text{-int}(A))\).”

“A subset \(A\) is said to be \(\tau_1\)-regular open (resp. \(\tau_2\)-regular open) if \(A = (\tau_1\text{-int}(\tau_1\text{Cl}(A)), \) resp. \(A = (\tau_2\text{-int}(\tau_2\text{Cl}(A))).”

“A is said to be \(\tau_1\)-regular closed (resp. \(\tau_2\)-regular closed) is \(A = \tau_1\text{-Cl}(\tau_1\text{-int}(A)). \) (resp. \(A = \tau_2\text{-Cl}(\tau_2\text{-int}(A)).”

“The \(\tau_1\)-\(\delta\)-interior (resp. \(\tau_2\)-\(\delta\)-interior) of subset \(A\) is the union of all regular \(\tau_1\)-open sets (resp. pairwise regular \(\tau_2\)-open sets) of \((X, \tau_1)\) (respect \((X, \tau_2)\)) contained in \(A\) is denoted by \(\tau_1\)-\(\delta\)-int(A) (resp. \(\tau_2\)-\(\delta\)-int(A)).

“A subset \(A\) is said to be \(\tau_1\)-\(\delta\)-open (resp. \(\tau_2\)-\(\delta\)-open if \(A = \tau_1\)-\(\delta\)-int(A) (resp. \(A = \tau_2\)-\(\delta\)-int(A)) i.e. a set is \(\tau_1\)-\(\delta\)-open (resp. \(\tau_2\)-\(\delta\)-open) if it is the union of pairwise regular \(\tau_1\)-open (resp. pairwise regular \(\tau_2\)-open) sets. The complement of \(\tau_1\)-\(\delta\)-open (resp. \(\tau_2\)-\(\delta\)-open) set is called \(\tau_1\)-\(\delta\)-closed (res. \(\tau_2\)-\(\delta\)-closed).

Alternatively, a set \(A\) of \((X, \tau_1, \tau_2)\) is said to be \(\tau_1\)-\(\delta\)-closed if \(A = \tau_1\)-\(\delta\)-Cl(A), where \(\tau_1\)-\(\delta\)-Cl(A) = \(\{x \in (X, \tau_1):A \cap \tau_1\text{-int}(\tau_1\text{Cl}(U)) \neq \phi, \ U \in \tau_1 \) and \(x \in U\) \) or \(\tau_2\)-\(\delta\)-closed if \(A = \tau_2\)-\(\delta\)-Cl(A), where \(\tau_2\)-\(\delta\)-Cl(A) = \(\{x \in (X, \tau_2):A \cap \tau_2\text{-int}(\tau_2\text{Cl}(V)) \neq \phi, \ V \in \tau_2 \) and \(x \in V\}.

III. CONTRA-PAIRWISE-CONTINUITY

3.1 Definition: A function \(f: (X, \tau_1, \tau_2) \to (Y, r_1, r_2)\) is said to be \textbf{pairwise super continuous} if \(f^{-1}(V_1)\) is \(\tau_1\)-\(\delta\)-open and \(f^{-1}(V_2)\) is \(\tau_2\)-\(\delta\)-open in \((X, \tau_1)\) and \((X, \tau_2)\) respectively, for every \(\tau_1\)-open set \(V_1\) of \((Y, r_1)\) and \(\tau_2\)-open set \(V_2\) of \((Y, r_2)\).
3.2 Definition: A function \( f: (X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) is said to be **pairwise \( \delta \)-continuous** if \( f^{-1}(V_1) \) is \( \tau_1 \)-\( \delta \)-open in \((X, \tau_1)\) and \( f^{-1}(V_2) \) is \( \tau_2 \)-\( \delta \)-open in \((X, \tau_2)\) for every \( V_1 \in PRO(Y, r_1) \) and \( V_2 \in PRO(Y, r_2) \).

The family of all pairwise regular open (resp. pairwise regular closed, pairwise semi open) set of a space \((X, \tau_1)\) and \((X, \tau_2)\) is denoted by \( PRO(X, \tau_1) \) and \( PRO(X, \tau_2) \) respectively (resp. \( PRC(X, \tau_1) \) and \( PRC(X, \tau_2) \), \( PSO(X, \tau_1) \) and \( PSO(X, \tau_2) \)). The family of pairwise regular open (resp. pairwise regular closed, pairwise semi open) of \((X, \tau_1)\) and \((X, \tau_2)\) containing \( x \in (X, \tau_1, \tau_2) \) is denoted by \( PRO(X, \tau_1, x) \) and \( PRO(X, \tau_2, x) \) respectively (resp. \( PRC(X, \tau_1, x) \) and \( PRC(X, \tau_2, x) \), \( PSO(X, \tau_1, x) \) and \( PSO(X, \tau_2, x) \)).

3.3 Definition: **Pairwise Perfectly Continuous**: A function \( f: (X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) is said to be pairwise perfectly continuous if \( f^{-1}(V_1) \) is \( \tau_1 \)-clopen in \((X, \tau_1)\) and \( f^{-1}(V_2) \) is \( \tau_2 \)-clopen in \((X, \tau_2)\) for every \( r_1 \)-open set \( V_1 \) of \((Y, r_1)\) and \( r_2 \)-open set \( V_2 \) of \((Y, r_2)\) respectively.

3.4 Definition: A function \( f: (X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) is said to be pairwise **regular set connected** if \( f^{-1}(V_1) \) is \( \tau_1 \)-clopen in \((X, \tau_1)\) and \( f^{-1}(V_2) \) is \( \tau_2 \)-clopen in \((X, \tau_2)\) for every \( V_1 \in PRO(Y, r_1) \) and \( V_2 \in PRO(Y, r_2) \).

3.5 Definition: A function \( f: (X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) is said to be pairwise almost \( s \)-continuous if for each \( x \in (X, \tau_1, \tau_2) \), each \( V_1 \in PSO(Y, f(x), r_1) \) and \( V_2 \in PSO(Y, f(x), r_2) \) then there exists an \( \tau_1 \)-open set \( U_1 \) in \((X, \tau_1)\) and \( \tau_2 \)-open set \( U_2 \) in \((X, \tau_2)\) respectively containing \( x \) such that \( f(U_1) \subset s-\tau_1 Cl(V_1) \) and \( f(U_2) \subset s-\tau_2 Cl(V_2) \).

3.6 Definition: A function \( f: (X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) is said to be **contra pairwise \( R \)-map** if \( f^{-1}(V_1) \) is pairwise regular \( \tau_1 \)-closed in
(X, τ₁) and f⁻¹(V₂) is pairwise regular τ₂-closed in (X, τ₂) for every pairwise regular r₁-open set V₁ of (Y, r₁) and pairwise regular r₂-open set V₂ of (Y, r₂) respectively.

3.7 Remark: For a function f:(X, τ₁, τ₂) → (Y, r₁, r₂) we have the following implications:

Pairwise Perfectly Continuous

↓

Pairwise regular set connected → Contra Pairwise R-map

↑

Pairwise almost s-Continuous

3.8 Theorem: For a function f:(X, τ₁, τ₂) → (Y, r₁, r₂), following are equivalent:

(1) f is contra pairwise R-map

(2) the inverse image of a pairwise regular r₁-closed set of (Y, r₁) and pairwise regular r₂-closed set of (Y, r₂) is pairwise regular τ₁-open and τ₂-open respectively.

(3) f⁻¹(τ₁-int(τ₁-cl(G₁))) is pairwise regular τ₁-closed for every r₁-open subset G₁ of (Y, r₁) and f⁻¹(τ₂-int(τ₂-cl(G₂))) is pairwise regular τ₂-closed for every r₂-open subset G₂ of (Y, r₂).

(4) f⁻¹{τ₁-cl(τ₁-int(F₁))} is pairwise regular τ₁-open for every r₁-closed subset F₁ of (Y, r₁) and f⁻¹{τ₂-cl(τ₂-int(F₂))} is pairwise regular τ₂-open for every r₂-closed subset F₂ of (Y, r₂).

Proof: 1 ⇔ 2: Let f:(X, τ₁, τ₂) → (Y, r₁, r₂) be a function. Let F₁ be any pairwise regular r₁-closed set of (Y, r₁) and F₂ be pairwise regular r₂-closed set of (Y, r₂). Then (Y, r₁)\F₁∈PRO (Y, r₁) and (Y, r₂)\F₂∈PRO (Y, r₂). By (1) f⁻¹{(Y, r₁)\F₁} = (X, τ₁)/f⁻¹(F₁) ∈PRC(Y, r₁).
and we have $f^1(F_1) \in \text{PRO}(Y, r_1)$ and $f_1(Y, r_2) \setminus F_2 = (X, \tau_2) \setminus f^1(F_2) \in \text{PRC}(Y, r_2)$, we have $f^1(F_2) \in \text{PRO}(Y, r_2)$.

(1) $\Leftrightarrow$ (3) Let $G_1$ be an $r_1$-open set $(Y, r_1)$ and $G_2$ be $r_2$-open set $(Y, r_2)$. Since $r_1-\text{Int}\{r_1\text{Cl}(G_1)\}$ and $r_2-\text{Int}\{r_2\text{Cl}(G_2)\}$ are pairwise regular $r_1$-open and pairwise regular $r_2$-open, then by (1), it follows that $f^1\{\tau_1-\text{int}(\tau_1\text{Cl}(G_1))\}$ is pairwise regular $\tau_1$-closed in $(X, \tau_1)$ and $f^1\{\tau_2-\text{int}(\tau_2\text{Cl}(G_2))\}$ is pairwise regular $\tau_2$-closed in $(X, \tau_2)$.

(2) $\Leftrightarrow$ (4) : it can be obtained similar as (1) $\Leftrightarrow$ (3).

For two topological space $X$ and $Y$ and any function $f:X \to Y$, the subset $\{x, f(x)\}$ of the product space $[X \times Y]$ is called the graph of $f$.

For topological spaces Long and Herrington [8] defined a function $f:X \to Y$ to have a strongly closed graph is for each $(x, y) \notin G(f)$ there exists open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $[U \times V] \cap G(f) = \emptyset$.

According to Cammaroto and Noiri this graph $G(f)$ is $\gamma$-closed with respect to $Y$ is for every $(x, y) \notin G(f)$ there exists $U \in u_x$ and $V \in u_y$ such that $[U \times V] \cap G(f) = \emptyset$.

In [16] we investigated the conditions for $G(f)$ to be closed when the concerning spaces are bitopological spaces. We take the product $X \times Y$ as the bitopological space $(X \times Y, \tau_1 \times r_2, \tau_2 \times r_1)$ and define a function $f:(X, \tau_1, \tau_2) \to (Y, r_1, r_2)$ to have a pairwise $\theta$-closed graph if $G(f)$ is either $\theta$-closed in $(X \times Y, \tau_1 \times r_2)$ or $\theta$-closed in $(X \times Y, \tau_2 \times r_1)$ i.e. for each $(x, y) \notin G(f)$. Either there exists $\tau_1$-open set $U$ containing $x$ and $r_2$-open set $V$ containing $y$ such that $\{\tau_1\text{Cl}(U) \times r_2\text{Cl}(V) \cap G(f)\} = \emptyset$ or there exists $\tau_2$-open set $S$ containing $x$ and $r_1$-open set $T$ containing $y$ such that $\tau_2\text{Cl}(S) \times r_1\text{Cl}(T) \cap G(f) = \emptyset.$
3.9 Definition: A bitopological space \((X, \tau_1, \tau_2)\) is said to be **pairwise weakly** \(T_2\) if for each element of \((X, \tau_1, \tau_2)\) is an intersection of pairwise regular closed sets.

3.10 Definition: A pairwise graph \(G(f)\) of function \(f: (X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)\) is said to be contra pairwise \(R\)-graph is for each \((x, y) \in \{X, \tau_1, \tau_2\} \times \{Y, r_1, r_2\}\) \(\setminus G(f)\), there exists a pairwise regular \(\tau_1\)-open set \(U_1\) in \((X, \tau_1)\) and pairwise regular \(\tau_2\)-open set \(U_2\) in \((X, \tau_2)\) both containing \(x\), there exists a pairwise regular \(r_1\)-closed set \(F_1\) and pairwise regular \(r_2\)-closed set \(F_2\), both containing \(y\) such that \(U_1 \times F_2 \setminus G(f) = \emptyset\) and \(U_2 \times F_2 \setminus G(f) = \emptyset\).

3.11 Theorem: The following properties are equivalent for a pairwise graph \(G(f)\) of a function \(f: (X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)\).

(1) \(G(f)\) is contra pairwise \(R\)-graph,

(2) for each \((x, y) \in \{X \times Y, \tau_1 \times r_2, \tau_2 \times r_1\} \setminus G(f)\), there exists a pairwise regular \(\tau_1\)-open set \(U_1\) and pairwise regular \(\tau_2\)-open set \(U_2\); both containing \(x\) and a pairwise regular \(r_1\)-closed set \(F_1\) and pairwise regular \(r_2\)-closed set \(F_2\) both containing \(y\) such that \(f(U_1) \cap F_2 = \emptyset\) and \(f(U_2) \cap F_2 = \emptyset\).

**Proof:** The straight forward proof follows from definition itself is omitted.

3.12 Theorem: If \(f: (X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)\) is Contra pairwise \(R\)-map and \((Y, r_1, r_2)\) is pairwise Urysohn, \(G(f)\) is Contra pairwise \(R\)-graph in \(f: (X, \tau_1, \tau_2) \times (Y, r_1, r_2)\).

**Proof:** Suppose that \((Y, r_1, r_2)\) is pairwise Urysohn. Let \((x, y) \in \{X, \tau_1, \tau_2\} \times \{Y, r_1, r_2\} \setminus G(f)\). It follows that \(f(x) \neq y\). Since \(Y\) is pairwise
Urysohn, there exists \( r_1 \)-open sets \( V \) and \( r_2 \)-open set \( W \) containing \( f(x) \) and \( y \) respectively, such that \( r_2 \text{cl}(V) \cap r_1 \text{Cl}(W) = \emptyset \). Since \( f \) is contra pairwise \( R \)-map, there exists a pairwise \( \tau_1 \)-open set \( U_1 \) in \((X, \tau_1)\) and \( \tau_2 \)-open set \( U_2 \) in \((X, \tau_2)\) both containing \( x \) such that \( f(U_1) \subset r_1 \text{Cl}(W) \) and \( f(U_2) \subset r_2 \text{cl}(V) \).

Therefore, \( f(U_1) \cap r_1 \text{Cl}(W) = \emptyset \) and \( f(U_2) \cap r_2 \text{cl}(V) = \emptyset \) and \( \text{G}(f) \) is contra pairwise \( R \)-graph in \((X, \tau_1, \tau_2) \times (Y, r_1, r_2)\).

3.13 Theorem: Let \( f:(X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) have a contra pairwise \( R \)-graph. If \( f \) is injective then \((X, \tau_1, \tau_2)\) is pairwise \( T_1 \).

Proof: Let \( x \) and \( y \) be any two distinct points of \((X, \tau_1, \tau_2)\). Then we have \((x, f(y)) \in \{(X, \tau_1, \tau_2) \times (Y, r_1, r_2)\} \setminus \text{G}(f)\). Then there exists \( U_1 \in \text{PRO}(X, \tau_1, x), U_2 \in \text{PRO}(X, \tau_2, x) \) and \( F_1 \in \text{PRC}(Y, r_1, f(y)), F_2 \in \text{PRC} \((Y, r_2, f(y))\) such that \( f(U_1) \cap F_2 = \emptyset \) and \( f(U_2) \cap F_1 = \emptyset \). Therefore, we get \( y \notin U_1 \cup U_2 \). This implies that \((X, \tau_1, \tau_2)\) is pairwise \( T_1 \).

3.14 Theorem: Let \( f:(X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) have a contra pairwise \( R \)-graph. If \( f \) is surjective, then \((Y, r_1, r_2)\) is pairwise weakly \( T_2 \).

Proof: Let \( x \) and \( y \) be two distinct points of \((X, \tau_1, \tau_2)\). Since \( f \) is a contra pairwise \( R \)-graph, then we have \((x, f(y)) \in (x \times y, \tau_1 \times r_2, \tau_2 \times r_1) \setminus \text{G}(f)\). Then there exists \( U_1 \in \text{PRO} \((X, \tau_1, x)\), \( U_2 \in \text{PRO} \((X, \tau_2, x)\) and \( F_1 \in \text{PRC} \((Y, r_1, f(y))\), \( F_2 \in \text{PRC} \((Y, r_2, f(y))\) such that \( f(U_1) \cap F_2 = \emptyset \) and \( f(U_2) \cap F_1 = \emptyset \); therefore \( U_1 \cap f^{-1}(F_2) = \emptyset \) and \( U_2 \cap f^{-1}(F_1) = \emptyset \); then we have \( y \notin U_1 \cup U_2 \). This shows that \((X, \tau_1, \tau_2)\) is pairwise \( T_1 \).

3.15 Theorem: Let \( f:(X, \tau_1, \tau_2) \to (Y, r_1, r_2) \) have contra pairwise \( R \)-graph. If \( f \) is surjective, then \((Y, r_1, r_2)\) is pairwise weakly \( T_2 \).
Proof: Let $y_1$ and $y_2$ be any two distinct points of $(Y, r_1, r_2)$. Since $f$ is surjective, $f(x) = y_1$ for some $x \in (X, \tau_1, \tau_2)$ and $(x, y_2) \in (X \times Y, \tau_1 \times r_2, \tau_2 \times r_1) \setminus G(f)$. Then there exists $U_1 \in \text{PRO}(X, \tau_1, x)$, $U_2 \in \text{PRO}(X, \tau_2, x)$ and $F_1 \in \text{PRC}(Y, r_1, y_2)$, $F_2 \in \text{PRC}(Y, r_2, y_2)$ such that:

$$f_2(U_1) \cap F_2 = \emptyset \quad \text{and} \quad f(U_2) \cap F_1 = \emptyset;$$

(since $f$ is contra pairwise R-graph) hence $y_1 \notin F_1$ and $F_2$. Then it is clear that $(Y, r_1, r_2)$ is pairwise weakly $T_2$.

### 3.16 Theorem:

If $f: (X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)$ and $g: (X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)$ are contra pairwise R-map and $(Y, r_1, r_2)$ is pairwise Urysohn, then $E_1 = \{x \in (X, \tau_1) : f(x) = g(x)\}$ and $E_2 = \{x \in (X, \tau_2) : f(x) = g(x)\}$ are $\tau_1$-closed in $(X, \tau_1)$ and $\tau_2$-closed in $(X, \tau_2)$ respectively.

**Proof:** Suppose $x \in (X, \tau_1) \setminus E_1$ and $x \in (X, \tau_2) \setminus E_2$ then it follows that $f(x) \neq g(x)$. Since $Y$ is pairwise Urysohn, there exists $r_1$-open sets $V_1$ and $r_2$-open set $V_2$ containing $f(x)$ and $g(x)$, respectively, such that $r_2\text{Cl}(V_1) \cap r_1\text{Cl}(V_2) = \emptyset$. Since $f$ and $g$ are contra pairwise R-map there exists a pairwise regular $\tau_1$-open set $U_1$ and pairwise regular $\tau_2$-open set $U_2$ containing $x$ such that $f(U_1) \subset r_1\text{Cl}(V_2)$ and $g(U_2) \subset r_2\text{Cl}(V_1)$. Set $\alpha = U_1 \cap U_2$ then, $\alpha$ is pairwise regular $\tau_1$-open in $(X, \tau_1)$ and $\tau_2$-open in $(X, \tau_2)$. Hence $f(\alpha) \cap g(\alpha) = \emptyset$ and it follows that $x \notin \delta-\tau_1\text{Cl}(E_2)$ and $\delta-\tau_2\text{Cl}(E_1)$. This implies that $E_1$ is $\tau_1$-$\delta$-closed in $(X, \tau_1)$ and $E_2$ is $\tau_2$-$\delta$-closed in $(X, \tau_2)$.

### 3.17 Theorem:

If $f$ is contra pairwise R-map injection and $(Y, r_1, r_2)$ is pairwise Urysohn then $(X, \tau_1, \tau_2)$ is pairwise Hausdorff.

**Proof:** Suppose that $(Y, r_1, r_2)$ is pairwise Urysohn, by the injectivity of $f$, it follows that $f(x) \neq f(y)$ for any distinct points $x$ and $y$ in $(X, \tau_1, \tau_2)$. Since $(Y, r_1, r_2)$ is pairwise Urysohn, there exists $r_1$-open sets $V_1$ and $r_2$-open set $V_2$ containing $f(x)$ and $f(y)$, respectively.
such that $r_2\text{Cl}(V_1) \cap r_1\text{Cl}(V_2) = \phi$. Since $f$ is contra pairwise R-map, there exists pairwise regular $\tau_1$-open sets $U_1$ in $(X, \tau_1)$ and pairwise regular $\tau_2$-open sets $U_2$ in $(X, \tau_2)$ containing $x$ and $y$ respectively. Such that $f(U_1) \subset r_1\text{Cl}(V_2)$ and $f(U_2) \subset r_2\text{Cl}(V_1)$.

Hence $U_1 \cap U_2 = \phi$. This shows that $(X, \tau_1, \tau_2)$ is pairwise Hausdorff.

3.18 Theorem: If $f$ is a contra pairwise R-map injection and $(Y, r_1, r_2)$ is pairwise weakly $T_2$, then $(X, \tau_1, \tau_2)$ is pairwise $T_1$.

Proof: Suppose that $(Y, r_1, r_2)$ is pairwise weakly $T_2$. For any distinct points $x$ and $y$ in $(X, \tau_1, \tau_2)$, there exists $V_1 \in \text{PRC} (Y, r_1)$ and $V_2 \in \text{PRC} (Y, r_2)$, such that $f(x) \in V_1$, $f(y) \notin V_1$ $f(x) \notin V_2$ and $f(y) \in V_2$. Since $f$ is contra pairwise R-map $f^{-1}(V_1)$ is pairwise regular $\tau_1$-open subset in $(X, \tau_1)$ and $f^{-1}(V_2)$ is pairwise regular $\tau_2$-open subset in $(X, \tau_2)$ such that $x \in f^{-1}(V_1)$, $y \in f^{-1}(V_2)$, $y \notin f^{-1}(V_1)$ and $x \notin f^{-1}(V_2)$. This shows that $(X, \tau_1, \tau_2)$ is pairwise $T_1$.

3.19 Definitions: A space $(X, \tau_1, \tau_2)$ is said to be

(1) Pairwise S-closed if every pairwise regular, pairwise closed cover of $(X, \tau_1, \tau_2)$ has a finite sub cover.

(2) Pairwise countably S-closed if every pairwise countable cover of $(X, \tau_1, \tau_2)$ by pairwise regular closed sets (w.r.t. $\tau_1, \tau_2$) has a finite subcover.

(3) Pairwise S-Lindelof if every pairwise cover of $(X, \tau_1, \tau_2)$ by pairwise regular closed sets (w.r.t. $\tau_1, \tau_2$) has a finite pairwise countable sub cover.

(4) Nearly pairwise compact if every pairwise regular open cover of $(X, \tau_1, \tau_2)$ has a finite subcover.
(5) Pairwise countably nearly pairwise compact if every pairwise countable cover of \((X, \tau_1, \tau_2)\) by pairwise regular open sets (w.r.t. \(\tau_1, \tau_2\)) has a finite subcover.

(6) Nearly pairwise Lindelof if every pairwise regular open cover of \((X, \tau_1, \tau_2)\) has a pairwise countable subcover.

3.20 Theorem: Let \(f: (X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)\) be a contra pairwise R-map surjection. Then the following statements are true:

(1) if \((X, \tau_1, \tau_2)\) is nearly pairwise compact, then \((Y, r_1, r_2)\) is pairwise S-closed.

(2) if \((X, \tau_1, \tau_2)\) is nearly pairwise Lindelof; then \((Y, r_1, r_2)\) is pairwise S-Lindelof.

(3) if \((X, \tau_1, \tau_2)\) is pairwise countably nearly pairwise compact, then \((Y, r_1, r_2)\) is pairwise countably S-closed.

Proof: Let \(\{V_i\} \; \text{and} \; \{S_i : i \in I\}\) be any pairwise regular closed cover of \((Y, r_1)\) and \((Y, r_2)\) respectively. Since \(f\) is contra pairwise R-map, then \(\{f^{-1}(V_i) : i \in I\}\) is a pairwise regular open cover of \((X, \tau_1)\) and \(\{f^{-1}(S_i) : i \in I\}\) is a pairwise regular open cover of \((X, \tau_2)\). Hence there exists a finite subset \(I_0\) of \(I\) such that \((X, \tau_1) = \bigcup \{f^{-1}(V_i) : i \in I_0\}\) and \((X, \tau_2) = \bigcup \{f^{-1}(S_i) : i \in I_0\}\). Then, we have

\[
(Y, r_1) = \bigcup \{V_i : i \in I_0\} \quad \text{and} \quad (Y, r_2) = \bigcup \{S_i : i \in I_0\} \quad \text{and} \quad (Y, r_1, r_2) \text{ is pairwise S-closed.}
\]

The proofs of (2) and (3) is left for the reader.

3.21 Corollary: Let \(f:(X, \tau_1, \tau_2) \rightarrow (Y, r_1, r_2)\) be a contra pairwise R-map surjection. Then the following are true:
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(1) if \((X, \tau_1, \tau_2)\) is pairwise S-Closed then \((Y, r_1, r_2)\) is nearly pairwise compact. (2)

(2) if \((X, \tau_1, \tau_2)\) is pairwise S-Lindelof, then \((Y, r_1, r_2)\) is nearly pairwise Lindelof.

(3) if \((X, \tau_1, \tau_2)\) is pairwise countably S-Closed, then \((Y, r_1, r_2)\) is pairwise countably nearly pairwise compact.

**REFERENCE**


