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A New Type of Generalized Difference Sequence Spaces of Fuzzy Numbers Defined By Modulas Function

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ABSTRACT: *In this article we introduce and study the sequence space $w^F(\Delta_r^s, f, p)$, $\Delta_r^s -$ summable sequence of fuzzy numbers, $\Delta_r^s -$ statistical convergent and also $\Delta_r^s -$ pre-Cauchy sequences of fuzzy numbers by using modulas function . Further we show that $w^F(\Delta_r^s, f, p)$ is a complete metric space .*

KEYWORDS: *Sequence of fuzzy numbers; Difference sequence; modulas fuction , statistically convergent; pre-Cauchy sequences of fuzzy numbers , modulas function.*

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [25] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [7] where it was shown that every convergent sequence is bounded. Nanda [9] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces. In [13] Savaş studied the space $m(\Delta)$, which we call the space of Δ-bounded sequence of fuzzy numbers and showed that this is a complete metric space.

Let D denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line R. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define $d(X, Y)$ by

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|).$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R *i.e.* a mapping $X : R \to L(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The α- level set $[X]^\alpha$ set of a fuzzy real number X for $0 < \alpha \le 1$, defined as

$$X^\alpha = \{ t \in R : X(t) \ge \alpha \}.$$

A fuzzy real number X is called *convex*, if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.
If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semi- continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in L$ is open in the usual topology of R.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.
The absolute value $|X|$ of $X \in L(R)$ is defined as (see for instance Kaleva and Seikkala [2])

$$|X|(t) = \max\{X(t), X(-t)\}, \quad \text{if } t > 0$$
$$= 0, \quad \text{if } t < 0.$$

Let $\bar{d} : L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ defines a metric on $L(R)$.
For $X, Y \in L(R)$ define

$$X \le Y \text{ iff } X^\alpha \le Y^\alpha \text{ for any } \alpha \in [0, 1].$$

A subset E of $L(R)$ is said to be bounded above if there exists a fuzzy number M, called an upper bound of E, such that $X \le M$ for every $X \in E$. M is called the least upper bound or supremum of E if M is an upper bound and M is the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly. E is said to be bounded if it is both bounded above and bounded below.

II. DEFINITIONS AND BACKGROUND

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of all positive integers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the k-th term or general term of the sequence.

Definition2.1: A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\bar{d}\,(X_k, X_0) < \varepsilon \quad \text{for } k > n_0$$

Definition2.2: The set of convergent sequences is denoted by c^F. $X = (X_k)$ of fuzzy numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\bar{d}\,(X_k, X_l) < \varepsilon \quad \text{for } k, l > n_0$$

Definition2.3: A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded and the set of bounded sequences is denoted by ℓ_∞^F.

The notion of difference sequence of complex terms was introduced by Kizmaz [6]. This notion war further generalized by Et and Colak [2], Tripathy and Esi [16], Tripathy, Esi and Tripathy [17] and many others.

The idea of the statistical convergence of sequence was introduced by Fast [3] and Schoenberg [12] independently in order to extend the notion of convergence of sequences. It is also found in Zygmund [26]. Later on it was linked with summability by Fridy and Orhan [4], Maddox [8], Rath and Tripathy [11] and many others. In [10] Nuray and Savaş extended the idea to sequences of fuzzy numbers and discussed the concept of statistically Cauchy sequences of fuzzy numbers. In this article we extend these notions to difference sequences of fuzzy numbers.

The natural density of a set K of positive integers is denoted by $\delta(K)$ and defined by

$$\delta(K) = \lim_n \frac{1}{n} \operatorname{card}\{k \le n : k \in K\}$$

Definition2.4: If a sequence $X = (X_k)$ satisfies a property P for almost all k except a set of natural density zero , then we say that X_k satisfies P for almost all k and we write a.a.k.

Definition2.5: A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$, $\lim_n \frac{1}{n} \operatorname{card}\{k \le n : d(X_k, X_0) \ge \varepsilon\} = 0$. We write st-lim $X_k = X_0$.

Throughout the article we denote by ${}_2w^F$ the set of all sequences of fuzzy numbers.

Definition2.6: A sequence (X_k) of fuzzy numbers is said to be double Δ_r^s- convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ there exist $k_0 \in N$ such that,

$$\bar{d}\,(\Delta_r^s X_k, X_0) > \varepsilon \quad for\ all\ \ k \ge k_0 .$$

We write , $\lim_{k\to\infty} \Delta_r^s X_k = X_0$, where $r\ and\ s\ two\ non\ negative\ integers$ .and

$\Delta_r^s X_k = \Delta_r^{s-1} X_k - \Delta_r^{s-1} X_{k+r}$ and $\Delta_r^0 X_k = X_k$ for all $k \in N$, which is equivalent to the following binomial representation :

$$\Delta_r^s X_k = \sum_{i=0}^{s} (-1)^i \binom{s}{i} X_{k+ir}$$

We recall that a modulas function f is a function from $[0,\infty)$ to $[0,\infty)$ such that :

(i) $f(x) = 0\ iff\ x = 0$

(ii) $f(x+y) \le f(x) + f(y)$ for all $x, y \ge 0.$

(iii) f is increasing.

(iv) f is continous from the right at 0 .

It follows that f must be continous everywhere on $[0,\infty)$ and a modulas function may be bounded or not bounded . Ruckle [24], Maddox[18] , Srivastava and Mohanta [15], used modulas function f to construct some sequence spaces. subsequently many authors.

A metric $\bar{d}$ on L(R) is said to be translation invariant if $\bar{d}(X+Y, Y+Z) = \bar{d}(X,Y)$ for all fuzzy numbers X , Y ,Z .

Let $(E_k, \bar{d}_k)$ be a sequence of fuzzy linear metric spaces under the translation invariant metrices $\bar{d}_k^{'}s$ such that $E_{k+1} \subseteq E_k$ for ,each $k \in N$ where $X_k = \left((X_{k,l})_{l=1}^{\infty} \right) \in E_k$. Define $W(E) = \{X = (X_k): X_k \in E_k \text{ for each } k \in N\}$. $W(E)$ is a linear space of fuzzy numbers under coordinatewise addition and scalar multiplication . (see for instance[15])

Let f be a modulas function and $p = (p_k)$ be a bounded sequence of positive real numbers .Also r and s be two non negative integers ; we present the following new sequence space

$$w^F(\Delta_r^s, f, p) = \left\{ X = (X_k) \in W(E): \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \left(f(sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k)) \right)^{p_k} \text{ where } L_k \in E_k \right\}$$

where $X_{k,l} = \Delta_r^{s-1} X_{k,l} - \Delta_r^{s-1} X_{k+r,l}$ and $\Delta_r^0 X_{k,l} = X_{k,l}$ for all $\in N$, which is equivalent to the following binomial representation

$$\Delta_r^s X_{k,l} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} X_{k+ir,l} \, .$$

Definition2.7: A sequence $X = \left(\left((X_{k,l})_{l=1}^{\infty} \right)_k \right)$ of fuzzy numbers is said to be Δ_r^s –staistically convergent to a fuzzy number $L_k \in E_k$, $k \in N$ if for each ε > 0 such that

$$\lim_{n\to\infty} \frac{1}{n} card\{l \le n : \; sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k) \ge \varepsilon\} = 0$$

The set of all Δ_r^s –staistically convergent is denoted by $S^F(\Delta_r^s)$.

Definition2.8: A sequence $X = \left(\left((X_{k,l})_{l=1}^{\infty} \right)_k \right)$ of fuzzy numbers is said to be Δ_r^s –staistically Cauchy sequence , if for each $\varepsilon > 0$, there exists a possitive integer l_0 such that

$$\lim_{n\to\infty} \frac{1}{n} card\{l \le n : \; sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, \Delta_r^s X_{k,l_0}) \ge \varepsilon\} = 0$$

Definition2.9: A sequence $X = \left(\left((X_{k,l})_{l=1}^{\infty} \right)_k \right)$ of fuzzy numbers is said to be Δ_r^s –staistically pre-Cauchy sequence , if for all $\varepsilon > 0$ '

$$\lim_{n\to\infty} \frac{1}{n^2} card\{(x,y) : x, y \le n, sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y}) \ge \varepsilon\} = 0$$

Lemma2.1: If $\bar{d}$ is translation invariant then

(a) $\bar{d}(\Delta_r^s X_{k,l} + \Delta_r^s Y_{k,l}, \bar{0}) \le \bar{d}(\Delta_r^s X_{k,l}, 0) + \bar{d}(\Delta_r^s Y_{k,l}, \bar{0})$

(b) $\bar{d}(\alpha \Delta_r^s X_{k,l}, \bar{0}) \le |\alpha| \bar{d}(\Delta_r^s X_{k,l}, \bar{0})$, $|\alpha| > 1$.

Lemma2.2: Let (α_k) and (β_k) be sequences of real or complex numbers and (p_k) be a bounded sequence of positive real numbers , then

$$|\alpha_k + \beta_k|^{p_k} \le C(|\alpha_k|^{p_k} + |\beta_k|^{p_k})$$

and

$$|\lambda|^{p_k} \le max(1, |\lambda|^G)$$

where C = $max(1, |\lambda|^{G-1}), G = supp_k$, λ is any real or complex number

III. MAIN RESULTS

Theorem3.1: If f be a modulas function and $0 < h = inf\, p_k \le p_k \le sup\, p_k = H$, then

$$w^F(\Delta_r^s, f, p) \subset S^F(\Delta_r^s) \, .$$

Proof : Let $\varepsilon > 0$ be given and $X = \left(\left((X_{k,l})_{l=1}^{\infty} \right)_k \right) \in w^F(\Delta_r^s, f, p)$. Then

$$\frac{1}{n}\sum_{l=1}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k}$$
$$= \frac{1}{n}\sum_{\substack{l=1\\ sup_k\bar{d}_k(\Delta_r^s X_{k,l},L_k)\geq\varepsilon}}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k}$$
$$+\frac{1}{n}\sum_{\substack{l=1\\ sup_k\bar{d}_k(\Delta_r^s X_{k,l},L_k)<\varepsilon}}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k} \geq \frac{1}{n}\sum_{l=1}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k}$$
$$\geq \frac{M}{n} card\{l \leq n : sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k) \geq \varepsilon\}$$

where $M = min\{f(\varepsilon)^h, f(\varepsilon)^H\}$. This follows that $X \in S^F(\Delta_r^s)$ and hence completes the proof.

Theorem3.2: If f is bounded modulas function and is $X = \left(\left((X_{k,l})_{l=1}^{\infty}\right)_k\right)$ −staistically convergent , then $X \in w^F(\Delta_r^s, f, p)$.

Proof: Since f is bounded modulas function, therefore there exists an integer K such that $f(x) < K$. let $\varepsilon > 0$ be given .Consider

$$\frac{1}{n}\sum_{l=1}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k}$$
$$= \frac{1}{n}\sum_{\substack{l=1\\ sup_k\bar{d}_k(\Delta_r^s X_{k,l},L_k)\geq\varepsilon}}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k}$$
$$+\frac{1}{n}\sum_{\substack{l=1\\ sup_k\bar{d}_k(\Delta_r^s X_{k,l},L_k)<\varepsilon}}^{n}\left(f\left(sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k)\right)\right)^{p_k}$$
$$\leq \max(K^h, K^H)\frac{1}{n} card\{l \leq n : sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k) \geq \varepsilon\} + max\{f(\varepsilon)^h, f(\varepsilon)^H\} \to 0 \; as \; n \to \infty$$

Thus $X \in w^F(\Delta_r^s, f, p)$. This completes the proof.

Theorem3.3: If a sequence $X = \left(\left((X_{k,l})_{l=1}^{\infty}\right)_k\right)$ is Δ_r^s − staistically convergent , then it is Δ_r^s −statistically Cauchy sequence.

Proof: Since X is Δ_r^s − staistically convergent, so we have for each $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{n} card\{l \leq n : sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k) \geq \varepsilon\} = 0$$

i.e

$$sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k) < \varepsilon \quad a.a.l.$$

We can choose $l_1 \in N$ such that

$$sup_k\bar{d}_k(\Delta_r^s X_{k,l_1}, L_k) < \varepsilon$$

Now,

$$sup_k\bar{d}_k(\Delta_r^s X_{k,l}, \Delta_r^s X_{k,l_1}) \leq sup_k\bar{d}_k(\Delta_r^s X_{k,l}, L_k) + sup_k\bar{d}_k(\Delta_r^s X_{k,l_1}, L_k)$$
$$< \varepsilon + \varepsilon = 2\varepsilon \quad a.a.l$$

This implies that X is Δ_r^s −statistically Cauchy sequence.

Theorem3.4: If a sequence $X = \left(\left(\left(X_{k,l}\right)_{l=1}^{\infty}\right)_k\right)$ is $\Delta_r^s -$ staistically convergent , then it is Δ_r^s −statistically bounded sequence.

Proof: Since X is $\Delta_r^s -$ staistically convergent, so we have for each > 0 ,

$$\lim_{n\to\infty} \frac{1}{n} card\{l \le n :\ sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k) \ge \varepsilon\} = 0$$

i.e

$$sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k) < \varepsilon \quad .$$

One can find, $sup_k\, \bar{d}_k(L_k, \bar{0}) < N(say)$. Then we have

$$\begin{aligned} sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, \bar{0}) &\le sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k) + sup_k\, \bar{d}_k(L_k, \bar{0}) \\ &< \varepsilon + N \qquad a.a.k. \end{aligned}$$

Hence X is Δ_r^s −statistically bounded sequence.

Remark : The converse of the above theorem is not true . To justify it, we consider the following example .

Example 3.1 : Take $f(x) = x\,, r = s = 1\,, p_k = 1\ for\ each\ k \in N\ and\ E_k = L(R)$.
Define the sequence (X_k) as follows :
When $k = 10^n$

$$X_k(t) = \begin{cases} 1 + tk^2 & if\ -\dfrac{1}{k^2} \le t \le 0 \\ 1 - tk^2 & if \quad 0 \le t \le \dfrac{1}{k^2} \\ 0 & otherwise \end{cases}$$

When $k \ne 10^n$ and k is odd,

$$X_k(t) = \begin{cases} t + 5 & if\ -5 \le t \le -4 \\ -t - 3 & if\ -4 \le t \le -3 \\ 0 & otherwise \end{cases}$$

When $k \ne 10^n$ and k is even,

$$X_k(t) = \begin{cases} t - 3 & if\ 3 \le t \le 4 \\ -t + 5 & if\ 4 \le t \le 5 \\ 0 & otherwise \end{cases}$$

Then,

$$[X_k]^\alpha = \begin{cases} \left[\dfrac{\alpha - 1}{k^2}, \dfrac{1-\alpha}{k^2}\right] & when\ k = 10^n \\ [-5+\alpha, -3-\alpha] & when\ k \ne 10^n \text{ and } k \text{ is odd} \\ [3+\alpha, 5-\alpha] & when\ k \ne 10^n \text{ and } k \text{ is even} \end{cases}$$

Therefore,

$$[\Delta X_k]^\alpha = \begin{cases} \left[\dfrac{\alpha - 1 + \alpha k^2 + 3k^2}{k^2}, \dfrac{1 - \alpha + 5k^2 - \alpha k^2}{k^2}\right] & when\ k = 10^n \\ [-10+2\alpha, -6-2\alpha] & when\ k \ne 10^n \text{ and } k \text{ is odd} \\ [6+2\alpha, 10-2\alpha] & when\ k \ne 10^n \text{ and } k \text{ is even} \end{cases}$$

It follows that X is $\Delta_r^s -$ staistically bounded but not X is $\Delta_r^s -$ staistically convergent sequence .

Theorem3.5 : If $X = \left(\left(\left(X_{k,l}\right)_{l=1}^{\infty}\right)_k\right)$ is a sequence for which there exists a $\Delta_r^s -$ staistically convergent sequence If $Y = \left(\left(\left(Y_{k,l}\right)_{l=1}^{\infty}\right)_k\right)$ such that $\Delta_r^s X_{k,l} = \Delta_r^s Y_{k,l}\ a.a.l.$ Then X is also $\Delta_r^s -$ staistically convergent.

Proof : Given that, $\Delta_r^s X_{k,l} = \Delta_r^s Y_{k,l}\ a.a.l.$ and Y is $\Delta_r^s -$ staistically convergent sequence. Then for each $\varepsilon > 0$ and each n , we have ,

$$\{l \leq n : sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k) \geq \varepsilon\}$$
$$\subset \{ l \leq n :\ sup_k \bar{d}_k(\Delta_r^s Y_{k,l}, L_k) \geq \varepsilon\} \cup \{l \leq n :\ \Delta_r^s X_{k,l} \neq \Delta_r^s Y_{k,l}\ \}$$

Y is $\Delta_r^s -$ staistically convergent sequence, therefore the set $\{l \leq n :\ sup_k\, \bar{d}_k(\Delta_r^s Y_{k,l}, L_k) \geq \varepsilon\}$ contains a fixed number $l_0 = l_0(\varepsilon)$. Then

$$\frac{1}{n} card\{l \leq n :\ sup_k\, \bar{d}_k(\Delta_r^s X_{k,l}, L_k) \geq \varepsilon\} \leq \frac{l_0}{n} + \frac{1}{n} card\{l \leq n :\ \Delta_r^s X_{k,l} \neq \Delta_r^s Y_{k,l}\ \} \to 0 \ \ as \ \ n \to \infty$$

This implies that X is $\Delta_r^s -$ staistically convergent .

Theorem3.6: Let $X = (X_k)$ be a sequence of fuzzy number and $\Delta_r^s -$ bounded . Then X is $\Delta_r^s -$staistically pre-Cauchy sequence if and only if

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{x,y \leq n} f(\, sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) = 0$$

where f is bounded modulas function.

Proof: Let us first assume that,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{x,y \leq n} f(\, sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) = 0.$$

Given $\varepsilon > 0$ and for $n \in N$, we have ,

$$\begin{aligned}
&\frac{1}{n^2} \sum_{x,y \leq n} f(\, sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) \\
&= \frac{1}{n^2} \sum_{\substack{x,y \leq n \\ sup_k \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y}) \geq \varepsilon}} f(\, sup_k \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) \\
&+ \frac{1}{n^2} \sum_{\substack{x,y \leq n \\ sup_k \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y}) < \varepsilon}} f(\, sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) \\
&\geq \frac{1}{n^2} \sum_{\substack{x,y \leq n \\ sup_k \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y}) \geq \varepsilon}} f(\, sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) \\
&\geq f(\varepsilon) \frac{1}{n^2} card\{(x,y) : x, y \leq n, sup_k \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y}) \geq \varepsilon\}
\end{aligned}$$

and hence X is $\Delta_r^s -$statistically pre-Cauchy.

Conversely let, X is $\Delta_r^s -$statistically pre-Cauchy and $\varepsilon > 0$ be given. Choose $\lambda > 0$ such that $f(\lambda) < \frac{\varepsilon}{2}$. Since f is bounded modulas function , therefore there exists an integer M such that

$$f(sup_k\, \bar{d}_k(\Delta_r^s X_{k,x}, \Delta_r^s X_{k,y})) < M.$$

Now,

$$\frac{1}{n^2}\sum_{x,y\le n} f(\sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y}))$$
$$= \frac{1}{n^2}\sum_{\substack{x,y\le n\\ \sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})\ge\lambda}} f(\sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y}))$$
$$+ \frac{1}{n^2}\sum_{\substack{x,y\le n\\ \sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})<\lambda}} f(\sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y}))$$
$$\le f(\lambda) + \frac{M}{n^2} card\{(x,y): x,y\le n, \sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})\ge\lambda\}$$
$$\le \frac{\varepsilon}{2} + \frac{M}{n^2} card\{(x,y): x,y\le n, \sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})\ge\lambda\}$$

By our assumption,

$$\lim_{n\to\infty}\frac{1}{n^2} card\{(x,y): x,y\le n, \sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})\ge\lambda\} = 0$$

i.e there exists a positive integer n_0 such that ,

$$\frac{1}{n^2} card\{(x,y): x,y\le n, \sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})\ge\lambda\} < \frac{\varepsilon}{2M} \quad \forall\ n\ge n_0$$

i.e.

$$\frac{1}{n^2}\sum_{x,y\le n} f(\sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})) \le \varepsilon \qquad \forall\ n\ge n_0$$

Thus we have

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{x,y\le n} f(\sup_k \bar{d}_k(\Delta_r^s X_{k,x},\Delta_r^s X_{k,y})) = 0$$

This completes the proof.

Remark : A sequence X is $\Delta_r^s -$ staistically pre Cauchy but not $\Delta_r^s -$ staistically convergent. To justify it, we consider the following example .

Example 3.2 :Take $f(x) = x, r = 1, p_k = 1$ *for each* $k \in N$ *and* $E_k = L(R)$. Consider the sequence X = (X_k) given as follows :
When k is even ,

$$X_k(t) = \begin{cases} t-3 & if\ 3\le t\le 4\\ -t+5 & if\ 4\le t\le 5\\ 0 & otherwise\end{cases}$$

When k is odd ,

$$X_k(t) = \begin{cases} t+5 & if\ -5\le t\le -4\\ -t-3 & if\ -4\le t\le -3\\ 0 & otherwise\end{cases}$$

Then

$$[\Delta_r^s X_k]^\alpha = \begin{cases} [2^s(3+\alpha), 2^s(5-\alpha)] & if\ k\ is\ even\\ [2^s(-5+\alpha), 2^s(-3-\alpha)] & if\ k\ is\ odd\end{cases}$$

This implies that X is $\Delta_r^s -$ staistically pre Cauchy but not $\Delta_r^s -$ staistically convergent.

Theorem3.7: If (p_k) be a bounded sequence of positive real numbers . Then the space $w^F(\Delta_r^s, f, p)$. is a linear space over the real field R .
Proof : The proof is easy , so omitted .

Theorem3.8: Let $(E_k, \bar{d}_k)$ be a sequence of complete metric spaces and (p_k) be a bounded sequence of positive real numbers such that $\inf p_k > 0$. Then the space $w^F(\Delta_r^s, f, p)$ is complete metric space under the metric g defined by-

$$g(X,Y) = sup_n \left(\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k (\Delta_r^s X_{k,l}, \Delta_r^s Y_{k,l}) \right) \right)^{p_k} \right)$$

Proof : It is easy to see that g is a metric on $w^F(\Delta_r^s, f, p)$. We just prove completeness. Let $(X^{(i)})$ be a Cauchy sequence in $w^F(\Delta_r^s, f, p)$, where $X^{(i)} = \left(\left(X_{k,l}^{(i)} \right)_{l=1}^{\infty} \right)_{k=1}^{\infty} \in w^F(\Delta_r^s, f, p) \quad \forall\ i \in N$. Then we have ,

$$g\left(X^{(i)}, X^{(j)}\right) \to 0 \ as\ i, j \to \infty$$

This implies

$$(3.1) \qquad sup_n \left(\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l}{}^{(j)} \right) \right) \right)^{p_k} \right) \to 0 \ as\ i, j \to \infty$$

Since f is modulas function we have –

$$sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l}{}^{(j)} \right) \to 0 \ as\ i, j \to \infty \text{ and for each } l = 1,2,3, \dots, n$$

This follows that ,

$$\bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l}{}^{(j)} \right) \to 0 \ as\ i, j \to \infty \text{ and for each } l = 1,2,3, \dots, n$$

i.e. $(\Delta_r^s X_{k,l}{}^{(i)})$ is a Cauchy sequence in E_k. Since E_k is complete so $\Delta_r^s X_{k,l}{}^{(i)}$ is convergent in E_k. For simplicity let

$$\lim_{i\to\infty} \Delta_r^s X_{k,l}{}^{(i)} = \sum_{u=0}^{s} (-1)^u \binom{s}{u} X_{k+ur,l} = N_{k,l}\ , say\ for\ each\ k \geq 1.$$

Considering $k = 1,2,3, \dots, rs, \dots$ and $l = 1,2,3, \dots, s.$ we can easily conclude that

$$\lim_{i\to\infty} X_{k,l}{}^{(i)} = X_{k,l} \ \ for\ l = 1,2,3, \dots, s.$$

Taking limit as $j \to \infty$ in (3.1) , we have ,

$$(3.2) \qquad \lim_{i\to\infty} sup_n \left(\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l} \right) \right) \right)^{p_k} \right) = 0$$

i.e.

$$\lim_{i\to\infty} g\left(X^{(i)}, X\right) = 0$$

Now it remains to show $X \in w^F(\Delta_r^s, f, p)$. From (3.2) we get ,

$$\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l} \right) \right) \right)^{p_k} \to 0 \ as\ i \to \infty \quad \forall\, n \in N$$

Therefore for any $\varepsilon > 0$ there exists a positive integer i_0 such that,

$$\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l} \right) \right) \right)^{p_k} < \frac{\varepsilon}{3} \quad \forall\, i \geq i_0 \ and\ n \in N.$$

Now one can find for each $n_0\ ,\ n_1 \in N$ such that,

$$\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, L_k^{(i)} \right) \right) \right)^{p_k} < \frac{\varepsilon}{3} \qquad \forall\ n \geq n_0 \ and\ L_k^{(i)} \in E_k.$$

and

$$\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(j)}, L_k^{(j)} \right) \right) \right)^{p_k} < \frac{\varepsilon}{3} \qquad \forall\ n \geq n_1 \ and\ L_k^{(j)} \in E_k$$

Take $i, j \geq i_0$ and $n_2 = \max\,(n_0, n_1)$. Then ,

$$\frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(L_k^{(i)}, L_k^{(j)} \right) \right) \right)^{p_k}$$

$$\leq C \frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, L_k^{(i)} \right) \right) \right)^{p_k} + C \frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(j)}, L_k^{(j)} \right) \right) \right)^{p_k}$$

$$+ C \frac{1}{n} \sum_{l=1}^{n} \left(f\left(sup_k \bar{d}_k \left(\Delta_r^s X_{k,l}{}^{(i)}, \Delta_r^s X_{k,l} \right) \right) \right)^{p_k} < C\varepsilon \quad \forall\ i, j \geq n_2$$

Since f is monotone function , we have

$$\bar{d}_k \left(L_k^{(i)}, L_k^{(j)} \right) < \varepsilon_1 \qquad \forall\ i, j \geq n_2$$

i.e. $(L_k^{(i)})$ is Cauchy sequence in E_k which is complete. So let $L_k^{(i)} \to L_k$ as $i \to \infty$.
Therefore,

$$\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(L_k^{(i)}, L_k\ ))\right)^{p_k} < C\varepsilon \quad \forall\ i \geq n_2$$

Thus we get,

$$\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}, L_k\ ))\right)^{p_k}$$

$$\leq C\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}{}^{(i_0)}, \Delta_r^s X_{k,l}))\right)^{p_k}$$

$$+ C\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}{}^{(i_0)}, L_k^{(i_0)}))\right)^{p_k} + C\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(L_k^{(i_0)}, L_k\ ))\right)^{p_k}$$

$$< \frac{2C\varepsilon}{3} + C\varepsilon \quad \forall\ n \geq n_2$$

Which implies that $X \in w^F(\Delta_r^s, f, p)$. This completes the proof.

Theorem3.9: Let (p_k) and (q_k) be two sequences of positive real numbers such that $0 < p_k \leq q_k$ and the sequence $\left(\frac{q_k}{p_k}\right)$ is bounded. Then $w^F(\Delta_r^s, f, q) \subset w^F(\Delta_r^s, f, p)$.

.**Proof :** Let $X \in w^F(\Delta_r^s, f, q)$. therefore ,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}, L_k\ ))\right)^{q_k} = 0$$

Take , $\alpha_k = \left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}, L_k\ ))\right)^{q_k}$ *and* $\gamma_k = \frac{q_k}{p_k}$ $s.t\ 0 < \gamma \leq \gamma_k \leq 1$.

Define ,

$$a_k = \begin{cases} \alpha_k & if\ \ \alpha_k \geq 1 \\ 0 & if\ \ \alpha_k < 1 \end{cases} \quad \text{and} \quad b_k = \begin{cases} 0 & if\ \ \alpha_k \geq 1 \\ \alpha_k & if\ \ \alpha_k < 1 \end{cases}$$

The , $\alpha_k = a_k + b_k$ *and* $\alpha_k{}^{\gamma_k} = a_k{}^{\gamma_k} + b_k{}^{\gamma_k}$, this implies that $a_k{}^{\gamma_k} \leq a_k \leq \alpha_k$ *and* $b_k{}^{\gamma_k} \leq b_k{}^{\gamma}$
Therefore,

$$\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}, L_k\ ))\right)^{p_k} \leq \frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}, L_k\ ))\right)^{q_k} + \frac{1}{n}\sum_{l=1}^{n} b_k{}^{\gamma} \to 0 \quad as \quad n \to \infty$$

i.e.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\left(f(sup_k \bar{d}_k(\Delta_r^s X_{k,l}, L_k\ ))\right)^{p_k} = 0$$

It follows that $X \in w^F(\Delta_r^s, f, p)$. This completes the proof.

REFERENCES

[1] Diamond P and Kloeden P. Metric spaces of fuzzy sets, Fuzzy Sets and Systems 35. (1990), 241-249.
[2] Et. M. and Colak R. On generalized difference sequence spaces, Soochow Jour. Math. 21(1995), 377-386.
[3] Fast H. Sur la, convergence statistique, Colloq. Math. (1951) 241-244.
[4] Fridy. J.A. and Orhan C. (1997), Statistical limit superior and limit inferior, Proc. Amer. Math. Soc. 125(12), (1997), 3625-3631.
[5] Freedman .A.R., Sember .J.J. and Raphael .M. Some Cesàro-type summability spaces, Proc. Lond. Math. Soc. 37(3) , (1978),508-520.
[6] Kizmaz. H. On certain sequence spaces, Canad. Math. Bull. 24(2) , (1981), 168-176.
[7] Matloka. M. Sequences of fuzzy numbers, BUSEFAL 28 ; (1986), 28-37.
[8] Maddox. I.J. A Tauberian condition for statistical convergence, Math. Proc. Camb. Phil. Soc. 106(1989), 277-280.
[9] Nanda .S. On sequence of fuzzy numbers, Fuzzy Sets and Systems 33 ; (1989), 28-37.
[10] Nuray. F. and Savaş .E. ,Statistical convergence of sequences of fuzzy numbers, Math. Slovaca 45(3), (1995). 269-273.
[11] Rath. D. and Tripathy. B.C. Matrix maps on sequence spaces associated with sets of integers, Indian J. Pure & Appl. Math. 27(2), (1996), 197-206.
[12] Schoenberg. I.J. The integrability of certain functions and related summability methods, Amer. Math. Monthly 66,(1959), 3621-4375.
[13] Savaş. E . A note on sequence of fuzzy numbers, Inform. Sciences 124 . (2000), 297-300.
[14] Savaş. E. On strongly λ-summable sequences of fuzzy numbers, Inform. Sciences 125, (2000),181-186.
[15] Srivasyava .P.D and Mohanta .S. (2013),TAIWAN JOURNAL OF MATHEMATICS, 17 . (2013),1659-1676.
[16] Tripathy. B.C. and Esi .A. Esi, A new type of difference sequence spaces, Int. Jour. of Sci. & Tech. 1 (2006), 11-14.

[17] Tripathy. B .C , Esi A. and Tripathy. B. K, On a new type of generalized difference Cesàro Sequence spaces, Soochow J. Math. 31(3), (2005), 333-340.

[18] Maddox. I.J , Sequence spaces defined by a modulas ,Mat. Proc. Camb. Phil.Soc.100 (1986), 161-166.

[19] Raj K. , Sharma A.K, Kumar A. and Sharma S.K , Double multiplier fuzzy sequence spaces defined by a sequence of Orlicz function, J.Appl. Functional Analysis,7 (2012), 241-247.

[20] Ahmad H.A.Bataineh , On a generalized difference sequence spaces defined by a modulas function and statistical convergence , Commun. Korean Math. Soc.21. (2006), 261-272.

[21] Tripathy B .C. and Sarma B . Sequence spaces of fuzzy real numbers defined by Orlicz functions , Math. Slovaca,58. (2008),621-628.

[22] Connor J., On strong summability with respect to a modulas and statistical convergence, Can Math. Bull ,32(2). (1989),194-198.

[23] Connor J. and Fridy J . and Kline J. Statistically pre-Cauchy sequence , Analysis, 14(4). (1994) ,311-317.

[24] Ruckle W.H , FK spaces in which the sequence of coordinate vectors is bounded,Canada J Math.25(1973), 973-978.

[25] Zadeh L.A. Zadeh, (1965) , Fuzzy sets, Inform and Control 8 ,(1965),338-353.

[26] Zygmund A. Trigonometric series, Vol. 2, Cambridge (1993),.