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On Finding B-Algebras Generated By Modulo Integer Groups Z_n

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Abstract
This paper explores the construction and properties of B-Algebras derived from Modulo Integer Groups, specifically focusing on the set of residue classes modulo n, denoted as Z_n.The algebraic structure is built by incorporating modulo addition and a binary operation, leading to a B-Algebra over Z_n. The paper also establishes the framework for investigating the unique properties and characteristics of B-Algebras constructed from modulo integer groups, presenting a foundation for further exploration within the realm of abstract algebra
Keywords: *B-Algebras, Modulo Integer Groups, Residue Classes, Algebraic Structure, Modulo Addition, Binary Operation, Mathematical Structures, Abstract Algebra.*

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I. INTRODUCTION

B-Algebras represent a class of mathematical structures characterized by two binary operations, often addition and multiplication, satisfying specific algebraic properties. Understanding and exploring B-Algebras contribute to the broader field of abstract algebra. The set Z_n comprises residue classes modulo n, offering a rich framework for exploring modular arithmetic. The modulo addition operation on Z_n provides an intriguing avenue for constructing algebraic structures.

Algebraic structures are non-empty sets equipped with one or more binary operations that fulfill certain axioms or properties. Some examples of algebraic structures are the rings, semigroups, e.t.c. There are other algebraic structures such as K-algebras, B-algebra, e.t.c. [1] has introduced an algebraic structure, namely K-algebra. It is known that K-algebras is divided into two classes based on its building group, namely *BCH/BCL/BCK*-algebra which is built on a commutative group, and B-algebra which is built on groups. Here, we discussed the class of K-algebra that is also B-algebra. [2] has introduced B-algebra as a non-empty set of *A* with a constant 0 completed by a binary operation and satisfy certain axioms. The algebraic structure of B-algebra is an algebraic structure that can be generated by a group with 0 identity elements [3].Homomorphism is a mapping between two algebraic structures of the same type that preserves the binary operation of the algebraic structure. The concept of homomorphisms can be found in groups called group homomorphisms. Because B-algebra is an algebraic structure that can be built on groups, B-algebra also has the same concept as groups, namely homomorphism, in B-algebra the concept of a homomorphism is called B-homomorphism. The set of all modulo n integers that are completed by the addition operation modulo is a group. The motivation behind this study lies in the intersection of B-Algebras and modulo integer groups. By examining the construction of B-Algebras from Z_n , we aim to unravel unique algebraic properties and relationships that emerge in this specific context.

II. PRELIMINARIES

Definition 2.1. A B-Algebra is an algebraic structure (A, $\oplus$,$\odot$), where A is a set, and $\oplus$ and $\odot$ are binary operations on A satisfying the following properties:

1. Closure: For all $x, y \in A$, $x \oplus y$ and $x \odot y$ are in A.
2. Associativity: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \odot y) \odot z = x \odot (y \odot z)$ for all $x, y, z \in A$.
3. Identity Elements: There exist elements 0 and 1 in A such that $x \oplus 0 = x$ and $x \odot 1 =$ for all $x \in A$.
4. Distributivity: $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in A$.

Definition 2.2. Let $(A, \oplus_A, \oplus_A)$ and $(B, \oplus_B, \oplus_B)$ be two B-Algebras. A function ϕ: $A \to B$ is a B-homomorphism if for all $x, y \in A$:

1. $\emptyset(x \oplus_A y) = \emptyset(x) \oplus_B \emptyset(y)$
2. $\emptyset(x \odot_A y) = \emptyset(x) \odot_B \emptyset(y)$

Proposition 2.3. If ϕ: $A \to B$ is a B-homomorphism, then $\emptyset(0_A) = 0_B$ and $\emptyset(1_A) = 1_B$

Proof. For any $x \in A$: $\emptyset(x \oplus_A 0_A) = \emptyset(x) \oplus_B \emptyset(0_A)$. Since $x \oplus_A 0_A = x$ it follows that $\emptyset(x) \oplus_B \emptyset(0_A) = \emptyset(x)$. This implies that $\emptyset(0_A) = 0_B$. Similarly, $\emptyset(1_A) = 1_B$.

Proposition 2.4. If ϕ: $A \to B$ is a B-homomorphism and x has an inverse x' in A, then $\phi(x')$ is the inverse of $\phi(x)$ in B

Proof. For any $x \in A$: $\emptyset(x \oplus_A x') = \emptyset(0_A) = 0_B$ Since $x \oplus_A x' = 0_A$, it follows that that $\emptyset(x \oplus_B x') = 0_B$ indicating that $\phi(x')$ is the inverse of $\phi(x)$ in B

Illustration 2.5. Consider B-Algebras $(A, \oplus_A, \odot_A)$ and $(B, \oplus_B, \odot_B)$ with $A = \{0,1\}$ and $B = \{0,1,2\}$ equipped with modulo 3 addition and multiplication:

$\oplus_A$	0	1
0	0	1
1	1	0
$\odot_A$	0	1
0	0	0
1	0	1

$\oplus_B$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1
$\odot_B$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Now, let ϕ: $A \to B$ be defined by $\phi(0) = 0$ and $\phi(1) = 1$.

Example 2.5.1. Preservation of Identity Elements

$\emptyset(0_A) = 0_B$

$\emptyset(1_A) = 1_B$

Example 2.5.2. Preservation of Inverses

Let $x = 1$ in A, and $x' = 1$ is the inverse.

$\emptyset(x') = \emptyset(1) = 1_B$

$\emptyset(x) = \emptyset(1) = 1_B$

III. CENTRAL IDEA

Theorem 3.1. Let $(Z_n + n)$ be a group, defined the binary operations "$*$" in Z_n with $[x]_n * [y]_n = [x]_n -_n [y]_n$, for every $[x]_n, [y]_n \in Z_n$ then $(z_n{:}*, [0]_n)$ is B-algebra

Proof. $(z_n{:}*, [0]_n)$ forms a B-Algebra, where the group $(Z_n + n)$ is defined with modulo *n* addition, and the binary operation $*$ is defined as $[x]_n -_n [y]_n$, for all $[x]_n, [y]_n \in Z_n$

1. Closure under $*$

For any $[x]_n, [y]_n \in Z_n$, $[x]_n * [y]_n = [x - y]_n$ which is in Z_n since it represents the residue class modulo *n*.

2. Associativity of $*$

For any $[x]_n, [y]_n, [z]_n \in Z_n$:

$([x]_n * [y]_n) * [z]_n = ([x - y]_n * [z]_n = [(x - y) - z]_n$

$[x]_n * ([y]_n * [z]_n) = [x]_n * [y - z]_n = [x - (y - z)_n$

Both expressions are equivalent modulo *n*, demonstrating associativity.

3. Existence of Identity Element $[0]_n$

For any $[x]_n \in Z_n$:

$[x]_n * [0]_n = [x - 0]_n = [x]_n$

$[0]_n * [x]_n = [0 - x]_n = [-x]_n = [x]_n$

Thus, $[0]_n$ serves as the identity element for $*$.
4. Existence of Inverses
For any $[x]_n \in Z_n$
$[x]_n * [-x]_n = [x-(-x)]_n = [2x]_n = [0]_n$
$[-x]_n * [x]_n = [-x-x]_n = [-2x]_n = [0]_n$
Thus, the inverse of $[x]_n$ under $*$ is $[-x]_n$

Example 3.1.1.
- Let n = 5 and consider $Z5$.

$[2]_5 * [2]_4 = [2-3]_5 = [4]_5$
- Let n = 4 and consider $Z4$.
- $[3]_4 * [2]_4 = [3-2]_4 = [1]_4$
- Let n = 6 and consider $Z6$.
- $[4]_6 * [5]_6 = [4-5]_6 = [5]_6$
- Let n = 3 and consider $Z3$.
- $[1]_3 * [2]_3 = [1-2]_3 = [2]_3$
- Let n = 7 and consider $Z7$.
- $[4]_7 * [6]_7 = [4-6]_7 = [5]_7$

Construction of B-Algebra from Modulo Integer Groups 3.2.

1. **Set Definition**
- Let Z_n be the set of residue classes modulo n, where n is a positive integer. The elements of Zn are [0], [1], [2],…, [n−1].

2. **Modulo Addition Operation**
- Define ⊕ as the modulo addition operation: [a] ⊕ [b] = [a + b mod n].

3. **B-Algebra Operations**
- Define ⊙ as a binary operation: [a] ⊙ [b] = [$a \cdot b$ mod n].

4. **Verify B-Algebra Properties**
- **Closure under ⊕ and ⊙:** For any [a], [b] ∈ Z_n, [a] ⊕ [b] and [a] ⊙ [b] are in Z_n.
- **Associativity:**

([a] ⊕ [b]) ⊕ [c] = [a] ⊕ ([b] ⊕ [c]) and
([a] ⊙ [b]) ⊙ [c] = [a] ⊙ ([b] ⊙ [c]).
- **Existence of Additive Identity:** There exists $I \in Z_n$ such that for any [a],[a] ⊕ I =[a].
- **Existence of Inverses:** For any [a], there exists [a'] such that

[a] ⊕ [a'] = I.

Examples and Illustrations 3.21.
- N = 5 and consider Z5.

[2] ⊕ [3] = [2 + 3 mod 5] = [0]
[2] ⊙ [3] = [2 · 3 mod 5] = [1]
- Let n = 4 and consider Z4.

[3] ⊕ [2]= [3 + 2 mod 4] = [1]
[3] ⊙ [2] = [3 · 2 mod 4] = [2]
- Let n = 6 and consider Z6.

[4] ⊕ [5] = [4 +5 mod 6] = [3]
[4] ⊙ [5] = [4 · 5 mod 6] = [4]
- Let n = 3 and consider Z3.

[1] ⊕ [2] = [1 + 2 mod 3] = [0]
[1] ⊙ [2] = [1 · 2 mod 3] = [2]
- Let n = 7 and consider Z7.

[4] ⊕ [6] = [4 + 6 mod 7] = [3]
[4] ⊙ [6] = [4 · 6 mod 7] = [3]

IV. CONCLUSION

The construction of a B-Algebra from modulo integer groups provides a rich and insightful exploration of algebraic structures. The defined operations within Z_n exhibit interesting properties and behaviors, emphasizing the interplay between modular arithmetic and abstract algebra. The examples and proofs serve as a foundation for further studies in the realm of B-Algebras and their applications within different mathematical frameworks. This construction not only contributes to theoretical developments but also highlights the practical implications of abstract algebra in solving problems involving modular structures.

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