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# Two-grid scheme of discontinuous Galerkin method for eigenvalue problems with homogeneous mixed boundary conditions

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***ABSTRACT**: This paper is centered on the eigenvalue problem with homogeneous mixed boundary conditions and introduces a two-grid discretization scheme based on shifted inverse iteration for the discontinuous Galerkin method. It presents the interior penalty discontinuous Galerkin method for second-order elliptic problems with homogeneous mixed boundaries, along with an a priori error estimate. Building upon the a priori error estimate, it provides an error estimate for the proposed scheme and demonstrates that the approximate solution obtained can achieve optimal convergence order when the grid size satisfies certain relationships. Finally, numerical results are included to showcase the effectiveness of the approach.*
***KEYWORDS**: Two-grid method, Homogeneous mixed boundary, Discontinuous Galerkin method, Shifted inverse iteration.*

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## I. INTRODUCTION

Many scholars have conducted meaningful research on eigenvalue problems. In practical computations, we hope to obtain approximate solutions to problems with less CPU time without sacrificing accuracy. To meet this requirement, two-grid and multigrid discretization have been introduced in the finite element method, both of which are highly efficient. The two-grid discretization was first introduced by Xu [1] for non-symmetric and bilinear elliptic problems. In reference [2], Xu and Zhou first applied this approach to eigenvalue problems, and since then, many scholars have conducted in-depth research. In reference [3], a two-grid discretization and multigrid discretization scheme were established for self-adjoint elliptic differential operator eigenvalue problems. It also combined the finite element method with the shifted inverse iteration method to establish a two-grid discretization scheme based on inverse iteration. In reference [4], it was applied to Maxwell eigenvalue problems, in reference [5] it was applied to Stokes eigenvalue problems, and in reference [6], it was applied to integral operator eigenvalue problems, and so on. Using this method, solving an eigenvalue problem on a fine grid reduces to solving this problem on a coarse grid and solving a linear algebraic equation on a fine grid. Based on the above work, there is currently no research on the two-grid discretization of the shifted inverse iteration discontinuous Galerkin method for eigenvalue problems with homogeneous mixed boundary conditions. This paper mainly discusses the second-order elliptic eigenvalue problem with homogeneous mixed boundary conditions, presents its interior penalty discontinuous Galerkin method, establishes a priori error estimate, and then, based on the a priori error estimate, provides a two-grid discretization scheme based on shifted inverse iteration and gives an error estimate for the proposed scheme.

## II. BASIC THEORETICAL PREPARATION

Let $\Omega \subset R^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$, where $\partial\Omega = \Gamma_D \cup \Gamma_N$, let n be the outward unit normal vector of $\partial\Omega$. Consider the eigenvalue problem with mixed boundary conditions: Find $\lambda \in C$ and $u \in H^1_{\Gamma_D}(\Omega)$, such that

$$\begin{cases} -\Delta u = \lambda u\,, & in\ \ \Omega\ , \\ u = 0\,, & on\ \ \Gamma_D\,, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0\,, & on\ \ \Gamma_N\,. \end{cases} \tag{2.1}$$

Denote

$$(u, v) = \int_\Omega uv dx\,,$$

and define a continuous bilinear form

$$a(u\,, v) = (\nabla u\,, \nabla v), \quad \forall u\,, v \in H^1_{\Gamma_D}(\Omega)\,. \tag{2.2}$$

There exist two positive constants A and B independent of $u, v$ such that the bilinear form $a(\cdot,\cdot)$ satisfies

$$\begin{aligned} |a(u\,,v)| \leqslant A \parallel u \parallel_{1,\Omega} \parallel v \parallel_{1,\Omega}\,, \quad \forall u, v \in H^1_{\Gamma_D}(\Omega) \\ a(v\,,v) \geqslant B\|v\|^2_{1,\Omega}, \quad \forall v \in H^1_{\Gamma_D}(\Omega) \end{aligned} \tag{2.3}$$

The weak form of (2.1) is to find $(\lambda\,,u) \in C \times H^1_{\Gamma_D}(\Omega),\ u \neq 0$, such that the following equation holds.

$$a(u\,,v) = \lambda(u\,,v), \quad \forall u, v \in H^1_{\Gamma_D}(\Omega). \tag{2.4}$$

Let $\mathcal{T}_h = \{\kappa\}$ be a shape-regular mesh of $\Omega$, where an internal edge of $\mathcal{T}_h$ is the non-empty interior of $\partial\kappa^+ \cap \partial\kappa^-$, with $\kappa^+$ and $\kappa^-$ being two adjacent elements of $\mathcal{T}_h$, not necessarily matching. An external edge of $\mathcal{T}_h$ is the non-empty interior of $\partial\kappa \cap \partial\Omega$. Let $\mathcal{E} = \mathcal{E}_\mathcal{J} \cup \mathcal{E}_\mathcal{D} \cup \mathcal{E}_\mathcal{N}$,where $\mathcal{E}_\mathcal{J}$ denotes the set of interior edges, $\mathcal{E}_\mathcal{D}$ denotes an edge on the boundary $\Gamma_D$, and $\mathcal{E}_\mathcal{N}$ denotes an edge on the boundary $\Gamma_N$.

$$h_\kappa = diam(\kappa), \qquad \forall \kappa \in \mathcal{T}_h; \qquad h_e = diam(e), \qquad \forall e \in \mathcal{E}.$$

Introduce the space of piecewise functions over the mesh $\mathcal{T}_h$:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega):\ v|_\kappa \in H^s(\kappa)\ ,\forall \kappa \in \mathcal{T}_h\}.$$

Define the average and the jump of $v$ on $e$:

$$\{\{v\}\} = \frac{1}{2}(v^+ + v^-), \qquad [[v]] = v^+\mathbf{n}^+ + v^-\mathbf{n}^-,$$

where $e = \partial\kappa^+ \cap \partial\kappa^-, v^+ = v|_{\kappa^+}, v^- = v|_{\kappa^-}$, $\mathbf{n}$ is the unit outward normal vector from $\kappa^+$ to $\kappa^-$.

If $e \in \mathcal{E}_\mathcal{D} \cup \mathcal{E}_\mathcal{N}$, define the average and the jump of $v$ on $e$:

$$\{\{v\}\} = v, \qquad [[v]] = v\mathbf{n}$$

Define

$$\begin{aligned} a_h(w_h, v_h) = \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (\nabla w_h \cdot \nabla v_h)dx - \sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} \int_e \{\{\nabla w_h\}\} \cdot [[v_h]]ds \\ - \sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} \int_e \{\{\nabla v_h\}\} \cdot [[w_h]]ds + \eta \sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} h_e^{-1} \int_e [[w_h]][[v_h]]ds \end{aligned}$$

where $\eta$ is the penalty parameter.

Define the space of DG finite elements:

$$V_h = \{v \in L^2(\Omega):\ v|_\kappa \in \mathbb{P}_m(\kappa)\ ,\forall \kappa \in \mathcal{T}_h\}.$$

where $\mathbb{P}_m(\kappa)$ is the m-th order polynomial space on $\kappa$.

The finite element approximation of (2.4) is to find $(\lambda_h\,,u_h) \in C \times V_h$, $u_h \neq 0$, such that

$$a_h(u_h\,,v_h) = \lambda_h(u_h\,,v_h), \quad \forall v_h \in V_h. \tag{2.5}$$

The source problem for (2.4) is to find $w \in H^1_{\Gamma_D}(\Omega)$, such that

$$a(w, v) = (f, v), \quad \forall v \in H^1_{\Gamma_D}(\Omega). \tag{2.6}$$

The DG approximation of (2.6) is to find $w_h \in V_h$, such that

$$a_h(w_h\,,v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{2.7}$$

Define the linear bounded operator $T: L^2(\Omega) \to H^1_{\Gamma_D}(\Omega)$ satisfying

$$Tf := w, \tag{2.8}$$

The equivalent operator form of (2.4) is:

$$Tu = \tfrac{1}{\lambda}u. \tag{2.9}$$

$T_h: L^2(\Omega) \to V_h$ can be defined as the corresponding discrete solution operator of (2.8), satisfying

$$T_hf := w_h, \tag{2.10}$$

The equivalent operator form of (2.5) is:

$$T_hu_h = \tfrac{1}{\lambda_h}u_h. \tag{2.11}$$

Introduce the summation space $V(h) = V_h + H^1_{\Gamma_D}(\Omega)$ endowed with the DG norm, where the DG norm is defined as:

$$\|v\|^2_G = \sum_{\kappa\in\mathcal{T}_h} \|\nabla v_h\|^2_{0,k} + \eta \sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} h_e^{-1} \int_e |[[v]]|^2 ds.$$

And define the norm on the piecewise function space $H^s(\tau_h)(s > \frac{3}{2})$ as

$$\|v\|^2_h = \textstyle\sum_{\kappa\in\mathcal{T}_h} \|\nabla v\|^2_{0,k} + \eta \sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} h_e^{-1} \int_e |[[v]]|^2 ds + \sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} \frac{h_e}{\eta} \int_e |\{\{\nabla v\}\}|^2 ds \tag{2.12}$$

Note that on the discontinuous finite element space $V_h$, $\|\cdot\|_h$ and $\|\cdot\|_G$ are equivalent.

By Proposition 3.3 in reference [7] and the Green's formula, we can derive the consistency of the discontinuous finite element method. Furthermore, by considering equation (2.7), we obtain:

$$a_h(w - w_h\,,v_h) = 0, \quad \forall v_h \in V_h. \tag{2.13}$$

**Proof.** Let $w \in H^1_{\Gamma_D}(\Omega)$ and $v_h \in V_h$ be given. We can break $a_h(w_h\,,v_h)$ into four terms as follows:

$$a_h(w,v_h)=\sum_{\kappa\in\mathcal{T}_h}\int_\kappa(\nabla w\cdot\nabla v_h)dx-\sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}}\int_e\{\{\nabla w\}\}\cdot[[v_h]]ds$$
$$-\sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}}\int_e\{\{\nabla v_h\}\}\cdot[[w]]ds+\eta\sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}}h_e^{-1}\int_e[[w]][[v_h]]ds$$
$$=:T_1-T_2-T_3+T_4 \tag{2.14}$$

According to $[[w]]|_e=0$, we can deduce $T_3=T_4=0$.

By applying the Green's formula to $T_1$, we have:

$$T_1=\sum_{\kappa\in\mathcal{T}_h}\left(\int_\kappa-\Delta w v_h dx+\int_{\partial\kappa}\nabla w\cdot \mathrm{n}_\kappa v_h ds\right) \tag{2.15}$$

According to

$$\sum_{\kappa\in\mathcal{T}_h}\int_\kappa-\Delta w v_h dx=\int_\Omega f v_h dx \tag{2.16}$$
$$\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\nabla w\cdot \mathrm{n}_\kappa v_h ds=\sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}}\int_e\{\{\nabla w\}\}\cdot[[v_h]]ds=T_2 \tag{2.17}$$

we can deduce $T_1=T_2+T_3+T_4+\int_\Omega f v_h dx$,

then

$$a_h(w,v_h)=\int_\Omega f v_h dx,\quad \forall v_h\in V_h, \tag{2.18}$$

From the above equation and (2.7), we can obtain (2.13).

It is not difficult to see that the following continuity and ellipticity hold:

$$|a_h(u_h,v_h)|\lesssim\|u_h\|_h\|v_h\|_h,\quad \forall u_h,v_h\in V(h), \tag{2.19}$$
$$\|u_h\|_h^2\lesssim a_h(u_h,u_h),\quad \forall u_h\in V_h. \tag{2.20}$$

According to equation (2.8) $w=Tf$, and assuming $f\in L^2(\Omega)$ and $w\in H^{1+r}(\Omega)$, we can assume the following regularity estimate holds:

$$\|w\|_{1+r}\lesssim\|f\|_{0,\Omega}\left(\frac{1}{2}<r\leqslant 1\right).$$

**Lemma 2.1.**[8] Let $\kappa\in\mathcal{T}_h$ and $v\in H^{s_\kappa}(\kappa)$, $s_\kappa>\frac{3}{2}$. Then there exists the polynomial $\Pi_{h_\kappa}v\in\mathbb{P}_m(\kappa)$, satisfying

$$\|v-\Pi_h v\|_{q,\kappa}\lesssim h_\kappa^{\min(m+1,s_\kappa)-q}\|v\|_{s_\kappa,\kappa}, \tag{2.21}$$
$$\|v-\Pi_h v\|_{0,e}\lesssim h_\kappa^{\min(m+1,s_\kappa)-\frac{1}{2}}\|v\|_{s_\kappa,\kappa}. \tag{2.22}$$

Now we introduce the global interpolation operator $\Pi_h: H^1_{\Gamma_D(\Omega)}\to V_h$, such that $\Pi_h(u)|_\kappa=\Pi_h(u|\kappa)$, for the vector-value function $\boldsymbol{r}=(\boldsymbol{r}_1,\boldsymbol{r}_2)$, define $\Pi_h(\boldsymbol{r})|_\kappa=(\Pi_h\boldsymbol{r}_1,\Pi_h\boldsymbol{r}_2)$.

**Theorem 2.1.** Let $w$ and $w_h$ be the solutions to (2.6) and (2.7), respectively. Assuming that $w$ satisfies $w|_\kappa\in H^{s_\kappa}(\kappa)$, the following inequalities hold for all $\kappa\in\mathcal{T}_h$ and $s_\kappa>\frac{3}{2}$

$$\|w-w_h\|_h\lesssim\inf_{v_h\in V_h}\|w-v_h\|_h, \tag{2.23}$$
$$\|w-w_h\|_h\lesssim\left(\sum_{\kappa\in\mathcal{T}_h}\left(h_\kappa^{\min(m+1,s_\kappa)-1}\|w\|_{s_\kappa,\kappa}\right)^2\right)^{\frac{1}{2}}. \tag{2.24}$$

**Proof.** Firstly, we prove (2.23) by utilizing (2.13), (2.19), and (2.20), we can obtain

$$\|w-w_h\|_h^2\lesssim a_h(w-w_h,w-w_h)$$
$$\lesssim a_h(w-w_h,w-v_h)+a_h(w-w_h,v_h-w_h)$$
$$\lesssim\|w-w_h\|_h\|w-v_h\|_h,$$

According to the triangle inequality and the above equation, we can obtain

$$\|w-w_h\|_h\lesssim\|w-v_h\|_h+\|v_h-w_h\|_h\lesssim\|w-v_h\|_h+\|v_h-w\|_h.$$

Therefore, equation (2.23) is proven.

Next, we prove (2.24). From (2.12), setting $E_h(w)=w-\Pi_h w$, we have:

$$\|E_h(w)\|_h^2=\sum_{\kappa\in\mathcal{T}_h}\|\nabla_h E_h(w)\|_{0,\kappa}^2+\eta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_\mathcal{D}}h_e^{-1}\int_e|[[E_h(w)]]|^2ds+\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_\mathcal{D}}\frac{h_e}{\eta}\int_e|\{\{\nabla_h E_h(w)\}\}|^2ds$$
$$\lesssim\sum_{\kappa\in\mathcal{T}_h}(\|\nabla_h E_h(w)\|_{0,\kappa}^2+\eta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_\mathcal{D}}h_e^{-1}(\|[[E_h(w)]]\|_{0,e}^2+\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_\mathcal{D}}\frac{h_e}{\eta}(\|\{\{\nabla_h E_h(w)\}\}\|_{0,e}^2$$
$$:=I_1+I_2+I_3 \tag{2.25}$$

An estimation for $I_1$ can be obtained using (2.21)

$$\sum_{\kappa\in\mathcal{T}_h}\|\nabla_h E_h(w)\|_{0,\kappa}^2\lesssim\sum_{\kappa\in\mathcal{T}_h}\|E_h(w)\|_{1,\kappa}^2$$
$$\lesssim\sum_{\kappa\in\mathcal{T}_h}\left(h_\kappa^{\min(m+1,s_\kappa)-1}\|w\|_{s_\kappa,\kappa}\right)^2 \tag{2.26}$$

An estimation for $I_2$ can be obtained using (2.22)

$$\begin{aligned}\eta\sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}} h_e^{-1}\|[[E_h(w)]]\|_{0,e}^2 &\lesssim \sum_{\kappa\in\mathcal{T}_h}\left(\sum_{e\in\partial\kappa}\eta h_e^{-1}\|E_h(w)\|_{0,e}^2\right)\\ &\lesssim \sum_{\kappa\in\mathcal{T}_h}\eta h_\kappa^{-1}\left(h_\kappa^{\min(m+1,s_\kappa)-\frac{1}{2}}\|w\|_{s_\kappa,\kappa}\right)^2\\ &\lesssim \sum_{\kappa\in\mathcal{T}_h}\left(h_\kappa^{\min(m+1,s_\kappa)-1}\|w\|_{s_\kappa,\kappa}\right)^2\end{aligned} \tag{2.27}$$

Estimation for $I_3$ can be obtained, using (2.22), where $e=\kappa^+\cap\kappa^-$, $\kappa^+=\kappa^-$

$$\begin{aligned}\sum_{e\in\mathcal{E}_\mathcal{J}\cup\mathcal{E}_\mathcal{D}}\frac{h_e}{\eta}\|\{\{\nabla_h E_h(w)\}\}\|_{0,e}^2 &\lesssim \frac{1}{\eta}\left(h_\kappa\left(\|\nabla_h E_h(w)_{\kappa^+}\|_{0,\mathcal{E}_\mathcal{J}}^2+\|\nabla_h E_h(w)_{\kappa^-}\|_{0,\mathcal{E}_\mathcal{J}}^2\right)\right.\\ &\quad \left.+h_\kappa\|\nabla_h E_h(w)\|_{0,\mathcal{E}_\mathcal{D}}^2\right)\\ &\lesssim \sum_{\kappa\in\mathcal{T}_h} h_\kappa\frac{1}{\eta}\left(h_\kappa^{-1}\|\nabla_h E_h(w)\|_{0,\kappa}^2+h_\kappa^{2r-1}|\nabla_h E_h(w)|_{r,\kappa}^2\right)\\ &\lesssim \sum_{\kappa\in\mathcal{T}_h}\left(\left(h_\kappa^{\min(m+1,s_\kappa)-1}\|w\|_{s_\kappa,\kappa}\right)^2+h_\kappa^{2r}\left(h_\kappa^{\min(m+1,s_\kappa)-1-r}\|w\|_{s_\kappa,\kappa}\right)^2\right)\\ &\lesssim \sum_{\kappa\in\mathcal{T}_h}\left(h_\kappa^{\min(m+1,s_\kappa)-1}\|w\|_{s_\kappa,\kappa}\right)^2\end{aligned} \tag{2.28}$$

From (2.26), (2.27) and (2.28), we can obtain

$$\|w-\Pi_h w\|_h \lesssim \left(\sum_{\kappa\in\mathcal{T}_h}\left(h_\kappa^{\min(m+1,s_\kappa)-1}\|w\|_{s_\kappa,\kappa}\right)^2\right)^{\frac{1}{2}}$$

According to the error estimation formula and the interpolation error formula, we have:

$$\inf_{v_h\in V_h}\|w-v_h\|_h \lesssim \|w-\Pi_h w\|_h \tag{2.29}$$

From (2.23), (2.29) and the above equation, we can derive (2.24), the proof is completed.

**Theorem 2.2.** Let $w$ and $w_h$ be the solutions to (2.6) and (2.7), respectively. Assuming that $w$ satisfies $w|_\kappa\in H^s(\kappa)$, the following inequalities hold for all $\kappa\in\mathcal{T}_h$ and $s\geqslant 1+r$

$$\|w-w_h\|_{0,\Omega}\lesssim h^r\|w-w_h\|_h\,, \tag{2.30}$$

$$\|w-w_h\|_{0,\Omega}\lesssim h^{\min(m+1,s)+r-1}\|w\|_{s,\Omega}\,. \tag{2.31}$$

where $h=\max_{\kappa\in\mathcal{T}_h} h_\kappa$ .

**Proof**. Firstly, to prove equation (2.30), considering the primal problem of the dual problem (2.4) denoted as $a(v,w')=(v,g)$ , $\forall v\in H^1_{\Gamma_D}(\Omega)$ , for any fixed $g\in L^2(\Omega)$ ,where $w'\in H^{1+r}(\Omega)$ , a regularity estimate $\|w'\|_{1+r,\Omega}\lesssim\|g\|_{0,\Omega}$ holds. Let $w'_h=\Pi_h w'$,By utilizing (2.13), we can derive

$$\begin{aligned}(w-w_h,g)=a_h(w-w_h,w')&=a_h(w-w_h,w'-\Pi_h w')\\ &\lesssim\|w-w_h\|_h\|w'-\Pi_h w'\|_h.\end{aligned} \tag{2.32}$$

From (2.24) and the elliptic regularity estimate, we can obtain

$$\|w'-\Pi_h w'\|_h\lesssim h^r\|w'\|_{1+r,\Omega}\lesssim h^r\|g\|_{0,\Omega}. \tag{2.33}$$

From (2.32) and (2.33), we can obtain

$$\|w-w_h\|_{0,\Omega}=\sup_{g\in L^2(\Omega)}\frac{|(w-w_h,g)|}{\|g\|_{0,\Omega}}\lesssim h^r\|w-w_h\|_h$$

that is (2.30).

Next, we will prove (2.31). From (2.24) and (2.30) we have

$$\|w-w_h\|_{0,\Omega}\lesssim h^r\|w-w_h\|_h=h^{\min(m+1,s)+r-1}\|w\|_{s,\Omega}\,.$$

that is (2.31), the proof is completed.

From (2.24) and regularity estimate, we derive the stability estimation:

$$\|T_h f\|_h\lesssim\|f\|_{0,\Omega}$$

## III. A PRIORI ERROR ESTIMATES FOR THE EIGENVALUE PROBLEM

Assume λ is the jth eigenvalue of (2.4) with the algebraic multiplicity q and the ascent $\alpha=1$, $\lambda_j=\lambda_{j+1}=\cdots=\lambda_{j+q-1}$. When $\|T_h-T\|_{0,\Omega}\to 0$, q eigenvalues $\lambda_{j,h},\cdots,\lambda_{j+q-1,h}$of (2.6) will converge to $\lambda$. Let $M(\lambda)$be the space of generalized eigenvectors of (2.4) associated with $\lambda$, and $M_h(\lambda)$ be the direct sum of the generalized eigenspace of (2.8) associated with $\lambda_h$ that converge to $\lambda$.

Given two closed subspaces $V$ and $U$, denote

$$\delta(U,V)=\sup_{u\in V,\|u\|_{0,\Omega}=1}\inf_{v\in U}\|u-v\|_{0,\Omega},\hat{\delta}(U,V)=\max\{\delta(U,V),\delta(V,U)\}.$$

And denote the arithmetic mean $\hat{\lambda}_h=\frac{1}{q}\sum_{i=j}^{j+q-1}\lambda_{i,h}$.

**Theorem 3.1.** Assume $M(\lambda) \subset H^s(\Omega)(s \geq 1+r),\ t = min\{m+1, s\} - 1$, then

$$|\lambda_h - \lambda| \lesssim h^{2t}, \tag{3.1}$$

Suppose $u_h \in M_h(\lambda)$ is a direct sum of generalized eigenvector spaces for (2.5). Then there exists an eigenfunction $u$ for the eigenvalue problem (2.4) such that

$$\|u - u_h\|_{0,\Omega} \lesssim h^{t+r}, \tag{3.2}$$

$$\|u - u_h\|_h \lesssim h^t. \tag{3.3}$$

Note that $Tf = w$ and $T_h f = w_h$. With the operator form, regularity estimate, and (2.31), we can obtain

$$\| T - T_h \|_{0,\Omega} = \sup_{0 \neq f \in L^2(\Omega)} \frac{\| Tf - T_h f \|_{0,\Omega}}{\| f \|_{0,\Omega}} \lesssim \sup_{0 \neq f \in L^2(\Omega)} \frac{h^{t+r} \| f \|_{0,\Omega}}{\| f \|_{0,\Omega}} \lesssim h^{t+r} \to 0, \qquad (h \to 0).$$

From Theorem 7.1, Theorem 7.2, Theorem 7.3 and Theorem 7.4 in [9], we have

$$\hat{\delta}\big(M(\lambda), M_h(\lambda)\big) \lesssim \| (T - T_h)|_{M(\lambda)} \|_{0,\Omega}, \tag{3.4}$$

$$\left|\lambda - \hat{\lambda}_h\right| \lesssim \sum_{i,l=j}^{j+q-1} \left|\big((T - T_h)\varphi_i, \varphi_l\big)\right| + \| (T - T_h)|_{M(\lambda)} \|_{0,\Omega}^2, \tag{3.5}$$

$$|\lambda - \lambda_h| \lesssim \sum_{i,l=j}^{j+q-1} \left|\big((T - T_h)\varphi_i, \varphi_l\big)\right| + \|(T - T_h)|_{M(\lambda)}\|_{0,\Omega}^2, \tag{3.6}$$

$$\|u - u_h\|_{0,\Omega} \lesssim \| (T - T_h)|_{M(\lambda)} \|_{0,\Omega}. \tag{3.7}$$

where $\{\varphi_i\}_{i=j}^{j+q-1}$ is basis for $M(\lambda)$ .

From Theorem 2.1 and Theorem 2.2, we derive

$$\begin{aligned} \| (T - T_h)|_{M(\lambda)} \|_{0,\Omega} &= \sup_{f \in M(\lambda), \|f\|_{0,\Omega}=1} \| Tf - T_h f \|_{0,\Omega} \\ &\lesssim \sup_{f \in M(\lambda), \|f\|_{0,\Omega}=1} h^{t+r} \| Tf \|_{t+1,\Omega} \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we can obtain (3.2).

By utilizing the properties of the operator and regularity estimates, from (2.13), we can obtain that

$$\begin{aligned} ((T - T_h)\varphi_i, \varphi_l) &= a_h(T\varphi_i - T_h\varphi_i, T\varphi_l) \\ &= a_h(T\varphi_i - T_h\varphi_i, T\varphi_l - T_h\varphi_l) \\ &\lesssim \| T\varphi_i - T_h\varphi_i \|_h \| T\varphi_l - T_h\varphi_l \|_h \\ &\lesssim h^t \| T\varphi_i \|_{t+1}\, h^t \| T\varphi_l \|_{t+1} \\ &\lesssim h^{2t} \end{aligned} \tag{3.9}$$

Substituting (3.8), (3.9) into (3.6), we get (3.1).

From $u = \lambda T u$ and $u_h = \lambda_h T_h u_h$,using the triangle inequality, (2.31), (3.1) and (3.2), we derive

$$\left| \|u - u_h\|_h - \|u - \lambda T_h u\|_h \right| \lesssim \|u_h - \lambda T_h u\|_h = \|T_h(\lambda_h u_h - \lambda u)\|_h \lesssim \|\lambda_h u_h - \lambda u\|_{0,\Omega} \lesssim h^{t+r} \tag{3.10}$$

From (2.9) and (2.11), we get

$$\|u - \lambda T_h u\|_h = \|\lambda T u - \lambda T_h u\|_h \leqslant \lambda \|Tu - T_h u\|_h \lesssim \inf_{v_h \in V_h} \|Tu - v_h\|_h \lesssim \|u - u_h\|_h \lesssim h^t \tag{3.11}$$

From (3.10) and (3.11), we can obtain (3.3).

The proof is completed.

## IV. TWO-GRID DISCRETIZATION

In this section, we present a two-grid discretization scheme based on the shifted inverse iteration. We propose Scheme 4.1 and conduct a rigorous theoretical analysis. Denote $V_H \subset V_h$ , $h < H$ .

**Scheme 4.1**(Two-grid discretization based on shifted inverse iteration)

**Step 1:** Solve on the coarse grid $\pi_H$ (2.5): Find $\lambda_H, u_H) \in R \times V_H$, such that $\|u_H\|_H = 1$ and

$$a_H(u_H, v) = \lambda_H(u_H, v), \qquad \forall v \in V_H$$

**Step 2:** Solve a linear system on the $\pi_h$: Find $u \in V_h$, such that

$$a_h(u, v) - \lambda_H(u, v) = (u_H, v), \qquad \forall v \in V_h.$$

Set $u_j^h = \frac{u}{\|u\|_h}$ .

**Step 3:** Compute the Rayleigh quotient

$$\lambda_j^h = \frac{a_h(u_j^h, u_j^h)}{(u_j^h, u_j^h)}.$$

Next, we will perform an error analysis for scheme 4.1.

We first present the following lemma to prepare for the error analysis.

**Lemma 4.1.** Let $(\lambda, u)$ be an eigenpair of (2.4), then for any $v \in V_h$ and $\sqrt{(v,v)} \neq 0$, the Rayleigh quotient $R(v) = \frac{a_h(v,v)}{(v,v)}$ satisfies

$$R(v) - \lambda = \frac{a_h(v-u,v-u)}{(v,v)} - \lambda \frac{(v-u,v-u)}{(v,v)}. \tag{4.1}$$

**Proof**. From (2.13) we have

$$a_h(u,v) = (\lambda u, v) = (\lambda_h u, v), \quad \forall v \in V_h,$$

thus,

$$\begin{aligned} & a_h(v-u,v-u) - \lambda(v-u,v-u) \\ &= a_h(v,v) - 2a_h(u,v) + a_h(u,u) - \lambda(v,v) + 2\lambda(u,v) - \lambda(u,u) \\ &= a_h(v,v) - 2(\lambda u,v) + a(u,u) - \lambda(v,v) + 2\lambda(u,v) - \lambda(u,u) \\ &= a_h(v,v) - \lambda(v,v) \end{aligned}$$

dividing both sides by $(v,v)$ we get (4.1)

**Lemma 4.2.** For any non-zero elements $u$, $v$ in any normed linear space $(V, \|\cdot\|)$, it holds that:

$$\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\| \leq 2\frac{\|u-v\|}{\|u\|}, \qquad \left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\| \leq 2\frac{\|u-v\|}{\|v\|}.$$

Proof See Lemma 3.1 in [10]

**Lemma 4.3.** [6] Let $(\mu_0, u_0)$ be the jth approximate eigenpair of (2.4), where $\mu_0$ is not an eigenvalue of $T_h$, $u_0 \in V_h$, $\|u_0\|_h = 1$, such that

(C1) $dist(u_0, M_h(\mu_j)) \leq \frac{1}{2}$

(C2) $|\mu_0 - \mu_j| \leq \frac{\vartheta}{4}$, $|\mu_{k,h} - \mu_k| \leq \frac{\vartheta}{4}$, $k = j-1, j, j+q (k \neq 0)$, where $\vartheta = \min_{\mu_k \neq \mu_j} |\mu_k - \mu_j|$ is the separate constant of the eigenvalue $\mu_j$;

(C3) $u \in V_h$ and $u_j^h \in V_h$ satisfy

$$(\mu_0 - T_h)u = u_0, \quad u_j^h = \frac{u}{\|u\|_h}, \tag{4.2}$$

then

$$dist(u_j^h, M_h(\mu_j)) \leq \frac{4}{\vartheta} \max_{j \leq k \leq j+q-1} |\mu_0 - \mu_{k,h}| dist(u_0, M_h(\mu_j)).$$

Now we can use Theorem 3.1 and the above lemma to analyze the error of the two-grid discretization Scheme 4.1.

**Theorem 4.1.** Suppose that $M(\lambda_j) \subset H^s(\Omega)$ $(s > \frac{3}{2})$, and $t = min\{m+1, s\} - 1$. Let $(\lambda_j^h, u_j^h)$ be an approximate eigenpair obtained by Scheme 4.1 and $H$ is sufficiently small, then there exists $u_j \in M(\lambda_j)$ such that

$$\|u_j^h - u_j\|_h \leq C(H^{3t+r} + h^t) \tag{4.3}$$

$$\|u_j^h - u_j\|_{0,\Omega} \leq C(H^{3t+r} + h^{t+r}) \tag{4.4}$$

$$|\lambda_j^h - \lambda_j| \leq C(H^{3t+r} + h^t)^2 \tag{4.5}$$

**Proof.** We will use Lemma 4.3 to complete the proof. Take $\mu_0 = \frac{1}{\lambda_H}$, $u_0 = \frac{\lambda_H T_h u_H}{\|\lambda_H T_h u_H\|_h}$.From (3.3) we know that there exists $\bar{u} \in M(\lambda_j)$, such that $\lambda_H T_h u_H - \bar{u}$ satisfy (3.2) and (3.3). From (2.10), Schwarz's inequality and (3.2), we get

$$\begin{aligned} a_h(T_h(u_H - \bar{u}), T_h(u_H - \bar{u})) &= (u_H - \bar{u}, T_h(u_H - \bar{u})) \\ &\leq \|u_H - \bar{u}\|_{0,\Omega} \|T_h(u_H - \bar{u})\|_{0,\Omega} \\ &\leq C(H^{t+r})^2 \end{aligned}$$

then,

$$\|T_h(u_H - \bar{u})\|_h \leq CH^{t+r} \tag{4.6}$$

From (4.10), (3.1) and (3.3), we get

$$\begin{aligned} \|\lambda_H T_h u_H - \bar{u}\|_h &= \|\lambda_H(T_h u_H - T_h \bar{u}) + \lambda_H(T_h \bar{u} - T\bar{u}) + (\lambda_H - \lambda)T\bar{u}\|_h \\ &\leq C(\|T_h(u_H - \bar{u})\|_h + \|(T - T_h)|_{M(\lambda_j)}\|_h + |\lambda_H - \lambda|) \\ &\leq C(H^{t+r} + h^t + H^{2t}) \\ &\leq C(H^{t+r} + h^t) \end{aligned}$$

Denote $u' = \frac{\bar{u}}{\|\bar{u}\|_h}$, from Lemma 4.2 we can obtain

$$\|u_0 - u'\|_h = \left\| u_0 - \frac{\bar{u}}{\|\bar{u}\|_h} \right\|_h \leq C \left\| \lambda_H T_h u_H - \bar{u} \right\|_h \leq C(H^{t+r} + h^t) \tag{4.7}$$

From (3.3) we know that there exists $u_h \in M_h(\lambda_j)$ , such that

$$\|u_h - u'\|_h = \left\| u_h - \frac{\bar{u}}{\|\bar{u}\|_h} \right\|_h \leq Ch^t \tag{4.8}$$

From the triangle inequality, (4.11) and (4.12) we get

$$dist(u_0, M_h(\lambda_j)) \leq \|u_0 - u_h\|_h \leq \|u_0 - u'\|_h + \|u_h - u'\|_h \leq C(H^{t+r} + h^t) \tag{4.9}$$

where $H$ is small enough, the condition $(C1)$ in Lemma 4.3 is valid.

From (3.1) we get

$$|\mu_0 - \mu_j| = \frac{|\lambda_H - \lambda_j|}{|\lambda_H \lambda_j|} \leq CH^{2t} \leq \frac{\vartheta}{4};$$

$$|\mu_k - \mu_{k,h}| = \frac{|\lambda_{k,h} - \lambda_k|}{|\lambda_{k,h} \lambda_k|} \leq Ch^{2t} \leq \frac{\vartheta}{4}, \qquad k = j-1, j, \ldots, j+q, k \neq 0.$$

Then, the condition $(C2)$ in Lemma 4.3 is valid.

According to the definition of $T_h$, Step 3 of Scheme 4.1 is equivalent to

$$a_h(u, v) - \lambda_H a_h(T_h u, v) = a_h(T_h u_H, v), \qquad \forall v \in V_h\ ,$$

and $u_j^h = \frac{u}{\|u\|_h}$.

$$(\lambda_H^{-1} - T_h)u = \lambda_H^{-1} T_h u_H, \qquad u_j^h = \frac{u}{\|u\|_h}.$$

Note that $\lambda_H^{-1} T_h u_H$ and $u_0$ differ by only one constant, so Step 3 is equivalent to

$$(\lambda_H^{-1} - T_h)u = u_0, \qquad u_j^h = \frac{u}{\|u\|_h}.$$

Therefore, all the conditions in Lemma 4.3 are valid.

Since the dimension of $M_h(\lambda_j)$ is $q$, there exists $u^* \in M_h(\lambda_j)$ such that

$$\|u_j^h - u^*\|_h = dist(u_j^h, M_h(\lambda_j)).$$

where $k = j, j+1, \ldots, j+q-1$, according to (3.1), we have

$$\begin{aligned} |\mu_0 - \mu_{k,h}| &= \left| \frac{1}{\lambda_H} - \frac{1}{\lambda_{k,h}} \right| \leq \left| \frac{\lambda_H - \lambda_{k,h}}{\lambda_H \lambda_{k,h}} \right| \\ &\leq C(|\lambda_H - \lambda_j| + |\lambda_j - \lambda_{k,h}|) \\ &\leq C(H^{2t} + h^{2t}) \leq CH^{2t}. \end{aligned} \tag{4.10}$$

Therefore, from Lemma 4.3, (4.13) and (4.14) we get

$$\begin{aligned} \|u_j^h - u^*\|_h &= dist(u_j^h, M_h(\lambda_j)) \\ &\leq \frac{C}{\vartheta} \max_{j \leq k \leq j+q-1} |\mu_0 - \mu_{k,h}| dist(u_0, M_h(\lambda_j)) \\ &\leq CH^{2t}(H^{t+r} + h^t) = C(H^{3t+r} + h^t H^{2t}). \end{aligned} \tag{4.11}$$

From (3.3) we know exists $u_j \in M(\lambda_j)$,such that $\|u^* - u_j\|_h = dist(u^*, M(\lambda_j))$, and

$$\|u^* - u_j\|_h \leq Ch^t \tag{4.12}$$

Therefore, from (4.15) and (4.16) we get

$$\|u_j^h - u_j\|_h \leq \|u_j^h - u^*\|_h + \|u^* - u_j\|_h \leq C(H^{3t+r} + h^t)$$

This (4.3) is proven.

Next, we will prove (4.4), from (3.2) we have

$$\|u^* - u_j\|_{0,\Omega} \leq Ch^{t+r}$$

similarly，

$$\|u_j^h - u_j\|_{0,\Omega} \leq \|u_j^h - u^*\|_{0,\Omega} + \|u^* - u_j\|_{0,\Omega} \leq C(H^{3t+r} + h^{t+r})$$

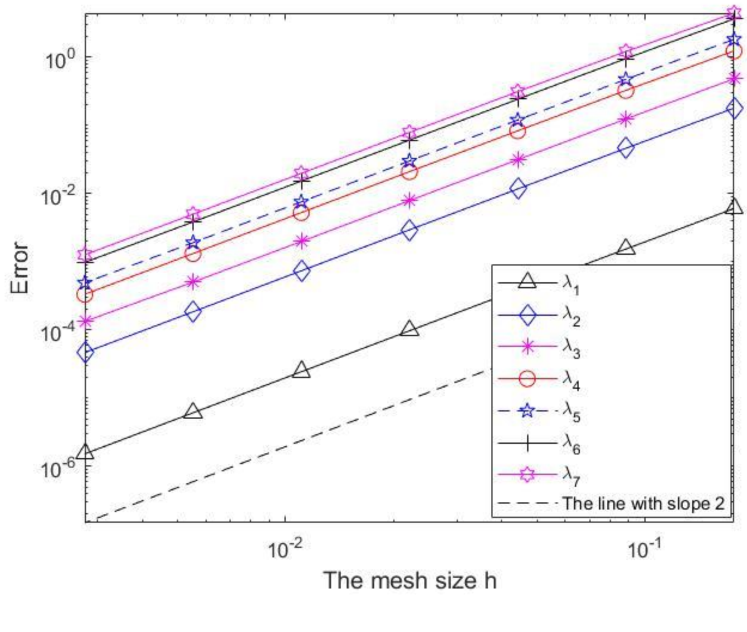
Finally, we use Lemma 4.1 to derive (4.5), from Step 4 of Scheme 4.1, Lemma 4.1, (4.7) and (4.8) we derive that

$$\begin{aligned}|\lambda_j^h-\lambda_j| &= \left|\frac{a_h(u_j^h-u_j,u_j^h-u_j)}{\|u_j^h\|_{0,\Omega}^2}-\lambda_j\frac{(u_j^h-u_j,u_j^h-u_j)}{\|u_j^h\|_{0,\Omega}^2}\right| \\ &\le C(\|u_j^h-u_j\|_h^2+|\lambda_j|\|u_j^h-u_j\|_{0,\Omega}^2) \\ &\le C(H^{3t+r}+h^t)^2\end{aligned}$$

The proof is completed.

## V. NUMERICAL EXPERIMENTS

In this section, we will report some numerical experiments to show the efficiency of our method. We consider problem (2.1), where the penalty parameter is set to 8. Our program was compiled under the iFEM[11] software package. We consider the following two test domains: a square region $\Omega_S=(0,1)^2$ and an L-shaped domain $(0,1)^2\setminus(\frac{1}{2},1)^2$. The initial mesh is a uniformly triangulated mesh with edge length 1/2, and the mesh is uniformly refined by dividing each triangle into four congruent triangles. We directly use linear elements to obtain an approximate solution for the eigenvalues. Since the exact eigenvalues are unknown, we use higher-dimensional computations to obtain their eigenvalues as reference values. For example, we take reference eigenvalues $\lambda_1\approx$ 2.467401100, $\lambda_2\approx$ 12.337005, $\lambda_3\approx$22.2066 in the square domain, and $\lambda_1\approx$ 1.26503, $\lambda_2\approx$ 10.4193, $\lambda_3\approx$ 24.1361 in the L-shaped domain. From the two figures below, the first figure shows the error curve obtained by solving the linear elements in a square region, while the second figure represents the error curve obtained by solving the linear elements in an L-shaped region. In the square region, the error curves of the eigenvalues are all parallel to a straight line with a slope of 2, whereas in the L-shaped region, the error curves of the first and second eigenvalues have slopes of 1.33 and 1.37 respectively, which are not parallel to the straight line with a slope of 2. Therefore, the first and second eigenvalues of the L-shaped region are singular, while the error curves of the remaining eigenvalues have slopes close to 2. Hence, except for the first and second eigenvalues of the L-shaped region, all other eigenvalues can achieve optimal convergence rates. In Tables 1 and 2, we list the solutions obtained directly using linear elements on fine meshes in the square domain and L-shaped domains, and the solutions obtained using the two-grid discretization method based on the shifted inverse iteration method in Section 4.1, along with the solution times required by these two methods. From the comparison of the data in the two tables, it is evident that our method in Section 4.1 is more efficient and the solutions obtained still maintain optimal accuracy.



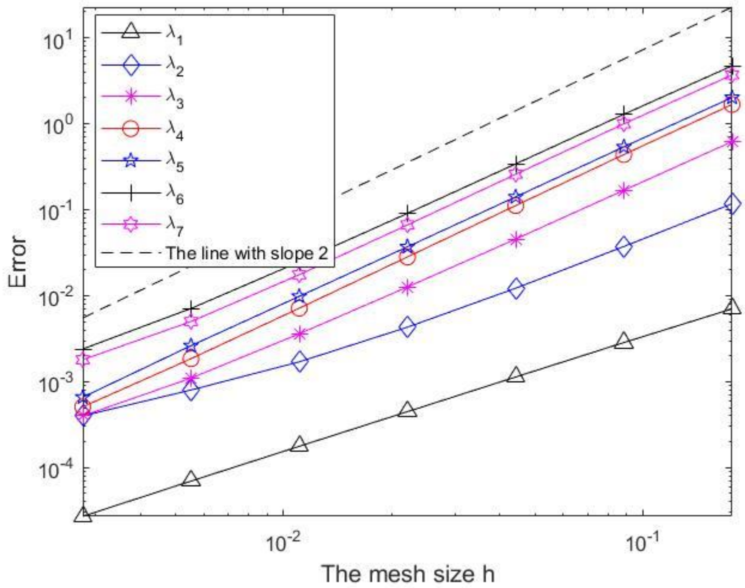



Figure: The above two graphs describe the error curves obtained by solving on linear elements. The first graph shows the error curve on the test domain $\Omega_S$, while the second graph shows the error curve on the test domain $\Omega_L$. The initial mesh has an edge length of 1/2.

Table 1: The first three eigenvalues of (2.1) solved using linear elements on domain $\Omega_S$ , based on scheme 4.1.

| $j$ | $H$ | $h$ | $\lambda_{j,H}$ | $\lambda_{j,h}$ | $CPU(s)$ | $\lambda_j^h$ | $CPU(s)$ |
|---|---|---|---|---|---|---|---|
| 1 | $\sqrt{2}/16$ | $\sqrt{2}/128$ | 2.4689387917 | 2.4674254560 | 1.35 | 2.4674254561 | 0.68 |
| 1 | $\sqrt{2}/16$ | $\sqrt{2}/256$ | 2.4689387917 | 2.4674071945 | 6.89 | 2.4674071949 | 3.61 |
| 1 | $\sqrt{2}/32$ | $\sqrt{2}/512$ | 2.4677886497 | 2.4674026245 | 49.57 | 2.4674026260 | 30.89 |
| 2 | $\sqrt{2}/16$ | $\sqrt{2}/128$ | 12.3832323 | 12.3377415 | 1.19 | 12.3377415 | 0.68 |
| 2 | $\sqrt{2}/16$ | $\sqrt{2}/256$ | 12.3832323 | 12.3371897 | 5.36 | 12.3371897 | 3.46 |
| 2 | $\sqrt{2}/32$ | $\sqrt{2}/512$ | 12.3486973 | 12.3370516 | 47.18 | 12.3370516 | 32.74 |
| 3 | $\sqrt{2}/16$ | $\sqrt{2}/128$ | 22.33084 | 22.20858 | 1.03 | 22.20858 | 0.64 |
| 3 | $\sqrt{2}/16$ | $\sqrt{2}/256$ | 22.33084 | 22.20710 | 5.26 | 22.20710 | 3.65 |
| 3 | $\sqrt{2}/32$ | $\sqrt{2}/512$ | 22.23798 | 22.20673 | 46.65 | 22.20673 | 30.78 |

Table 1: The first three eigenvalues of (2.1) solved using linear elements on domain $\Omega_L$ , based on scheme 4.1.

| $j$ | $H$ | $h$ | $\lambda_{j,H}$ | $\lambda_{j,h}$ | $CPU(s)$ | $\lambda_j^h$ | $CPU(s)$ |
|---|---|---|---|---|---|---|---|
| 1 | $\sqrt{2}/16$ | $\sqrt{2}/128$ | 1.267879655 | 1.265207079 | 0.74 | 1.265207079 | 0.49 |
| 1 | $\sqrt{2}/16$ | $\sqrt{2}/256$ | 1.267879655 | 1.265099724 | 3.45 | 1.265099725 | 2.23 |
| 1 | $\sqrt{2}/32$ | $\sqrt{2}/512$ | 1.266163395 | 1.265057232 | 21.81 | 1.265057233 | 13.23 |
| 2 | $\sqrt{2}/16$ | $\sqrt{2}/128$ | 10.4564 | 10.4208 | 0.71 | 10.4208 | 0.43 |
| 2 | $\sqrt{2}/16$ | $\sqrt{2}/256$ | 10.4564 | 10.4199 | 3.60 | 10.4199 | 2.23 |
| 2 | $\sqrt{2}/32$ | $\sqrt{2}/512$ | 10.4313 | 10.4195 | 21.61 | 10.4195 | 13.69 |
| 3 | $\sqrt{2}/16$ | $\sqrt{2}/128$ | 24.3042 | 24.1397 | 0.70 | 24.1397 | 0.48 |
| 3 | $\sqrt{2}/16$ | $\sqrt{2}/256$ | 24.3042 | 24.1372 | 3.51 | 24.1372 | 2.27 |
| 3 | $\sqrt{2}/32$ | $\sqrt{2}/512$ | 24.1816 | 24.1365 | 23.02 | 24.1365 | 14.14 |

## VI. CONCLUSIONS

This paper presents a study on the two-grid discretization of eigenvalue problems with homogeneous mixed boundary conditions using the discontinuous Galerkin method. Based on our approach, we solve the eigenvalue problem on the fine grid $\pi h$ using linear elements and also provide solutions using Scheme 4.1. Numerical experiments are conducted on $\Omega_S$ and $\Omega_L$. The numerical results show that compared to directly solving the eigenvalue problem on the fine grid, the two-grid discretization method based on shifted inverse iteration requires less CPU time. Furthermore, as the grid size decreases, the advantages of the two-grid discretization method with shifted inverse iteration become more apparent, indicating the efficiency of our approach. Therefore, this method has strong practical value for solving eigenvalue problems with homogeneous mixed boundary conditions.

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