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Discontinuous Finite Element Adaptive Methods for Biharmonic Eigenvalue Problems with Simply Supported Boundary Conditions

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***ABSTRACT**: Biharmonic eigenvalue equation is a typical fourth-order partial differential equation, which is an important partial differential equation model in elastic thin plate, biophysics and other fields, and its efficient numerical solution has been a hot spot and difficulty in related fields. The discontinuous finite element method has high plasticity and adaptability, and has become an important numerical method for solving various kinds of partial differential equations and practical problems. In this paper, we use the discontinuous finite element method to study the eigenvalue problem of biharmonic equations with simply supported boundary conditions, and introduce a posterior error index based on residual through discontinuous Galerkin discretization, and obtain the complete posterior error estimation results of this method. The performance of this index is verified in an adaptive mesh refiner.*

***KEYWORDS**: Biharmonic eigenvalue equation, Discontinuous Galerkin method, Posterior error, adaptive.*

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I. INTRODUCTION

The biharmonic equation originates from the elastic thin plate theory in the field of continuum mechanics. The fourth-order boundary value problem is a kind of special boundary value problem of partial differential equations, which often appears in thin plate theory of elasticity, phase field model and mathematical biology, which makes biharmonic equations widely used. Many scholars have also been committed to the numerical solution of biharmonic equations, and its solution methods are constantly optimized and innovative. The finite difference method was used to solve biharmonic equations[1]. Liu used the mixed finite element method to solve the biharmonic equation[2], that is, by introducing intermediate variables, the biharmonic equation was reduced to two second-order equations, and the mixed finite element space satisfying certain conditions was used to discretize corresponding mixed variational problem, so as to obtain the numerical solutions of the original variables and intermediate variables satisfying the original equation. Discontinuous Galerkin finite element method is a kind of finite element method using completely discontinuous basis function, which can solve more complex boundary problems, and is easy to realize the selection of local mesh and each element polynomial. Therefore, discontinuous Galerkin method is often used to solve various eigenvalue problems, such as Steklov eigenvalue problem, Laplacian eigenvalue problem, biharmonic eigenvalue problem, etc. Emmanuil derived the DG scheme of the biharmonic equation[3]. The internal penalty discontinuous finite element method is to penalty the jump of the approximating solution on the common edge or common surface of the element, which is more flexible than the finite element method. [4] constructed the hp internal penalty discontinuity Galerkin finite element method for biharmonic equations and analyzed the prior error of the method. In this paper, the biharmonic eigenvalue problem with simply supported boundary is studied by discontinuous finite element method in internal penalty discontinuous galerkin(IPDG) format, and a posterior error estimation is established to verify the reliability and validity of the posterior error estimation of the discontinuous finite element method. The results show that the adaptive algorithm can achieve the optimal convergence order.

II. BASIC THEORETICAL PREPARATION

$L^p(\omega)$ to represent a standard Lebesgue space, where $1 \leqslant p \leqslant \infty, \omega \subset \mathbb{R}^2$, The corresponding norm is expressed by $\|\cdot\|_{L^p(\omega)}$. In this paper, the norm of $L^2(\omega)$ is represented by $\|\cdot\|_\omega$. We also use $H^s(\omega)$ to express the standard Hilbert Sobolev space of real functions defined at $\omega \subset \mathbb{R}^2$ with index $s \geqslant 0$, and the corresponding norm and semi-norm are $\|\cdot\|_{s,\omega}$ and $|\cdot|_{s,\omega}$. Let Ω be the bounded open polygon region of R^2, and let $\partial\Omega$ represent its boundary. Consider the simply supported boundary condition eigenvalue problem: find $\lambda \in C$ and $u \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$\begin{cases} \Delta^2 u = \lambda u, & in\Omega, \\ u = \Delta u = 0, & on\, \partial\Omega. \end{cases} \tag{2.1}$$

Denote

$$(u, v) = \int_\Omega uvdx,$$

and define a continuous bilinear form

$$a(u, v) = (\Delta u, \Delta v), \forall u, v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.2}$$

Then, there exists two positive constants A and B independent of u and v, such that the bilinear form $a(\cdot,\cdot)$ is satisfied

$$\begin{aligned} |a(u, v)| &\leqslant A \parallel u \parallel \parallel v \parallel, \forall u, v \in H_0^1(\Omega) \cap H^2(\Omega), \\ | a(v, v) | &\geqslant B \parallel v \parallel^2, \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \end{aligned} \tag{2.3}$$

The weak form of (2.1) is to find $(\lambda, u) \in R \times H_0^1(\Omega) \cap H^2(\Omega), u \neq 0$, such that

$$a(u, v) = \lambda(u, v), \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.4}$$

Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint triangular or quadrilateral elements $\kappa \in \mathcal{T}$, on this assumption that the subdivision is shape regular and constructed by affine mapping $\mathcal{F}_\kappa$, where $\mathcal{F}_\kappa: \hat{\kappa} \to \kappa$, with nonsingular Jacobin, where $\hat{\kappa}$ is the reference triangle or quadrilateral. It is assumed that the mapping is constructed to ensure that $\overline{\Omega} = \quad _{\kappa \in \mathcal{T}} \bar{\kappa}$ and the elemental edges are straight line segments.

The broken Laplacian $\Delta_h u$ is defined by

$$(\Delta_h u) |_K = \Delta(u |_K), \ \ \forall \kappa \in \mathcal{T}$$

For a non-negative integer r, $\mathcal{P}_r(\hat{\kappa})$ is used to represent the set of all polynomials of degree at most r if $\hat{\kappa}$ is a reference triangle, and $\mathcal{P}_r(\hat{\kappa})$ is used to represent the set of polynomials of tensor product if $\hat{\kappa}$ is a reference quadrilateral. For $r = 2$, consider its finite element space

$$S^2 := \{v \in L^2(\Omega) : v |_K \circ F_K \in \mathcal{P}_2(\hat{\kappa}), \kappa \in \mathcal{T}\}.$$

We use Γ_h to represent the union (including the boundary) of all one-dimensional unit edges associated with the subdivision $\mathcal{T}$. In addition, we decompose Γ_h into two disjoint subsets, i.e. $\Gamma_h = \Gamma_\partial \cup \Gamma_{\text{int}}$, where $\Gamma_{\text{int}} := \Gamma_h \setminus \Gamma_\partial$.

Let $\kappa^+$and κ^- be two elements of the shared edge $e := \partial\kappa^+ \cap \partial\kappa^- \subset \Gamma_{\text{int}}$. Define the outward normal unit vectors on e corresponding to $\partial\kappa^+$ and $\partial\kappa^-$, respectively, as $\mathbf{n}^+$ and $\mathbf{n}^-$. For functions $v: \Omega \to \mathbb{R}$ and $\mathbf{q}: \Omega \to \mathbb{R}^2$, these functions may be discontinuous in Γ_h, the following is defined for $v^+ := v|_{e \subset \partial\kappa^+}, v^- := v|_{e \subset \partial\kappa^-}, \mathbf{q}^+ := \mathbf{q}|_{e \subset \partial\kappa^+}$,

$$\{v\} := \tfrac{1}{2}(v^+ + v^-), \ \ \{\mathbf{q}\} := \tfrac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), \ \ [v] := q^+\mathbf{n}^+ + q^-\mathbf{n}^-, \ \ [\mathbf{q}] := \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-.$$

If $e \in \partial\kappa \cap \Gamma_\partial$, then these definitions are changed as follows:

$$\{v\} := v^+, \quad \{\mathbf{q}\} := \mathbf{q}^+, \quad [v] := v^+\mathbf{n}, \quad [\mathbf{q}] := \mathbf{q}^+ \cdot \mathbf{n}.$$

With the above definition, it can be verified

$$\sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa} v\mathbf{q} \cdot \mathbf{n} \mathrm{d}s = \int_{\Gamma_h} [v] \cdot \{\mathbf{q}\} \mathrm{d}s + \int_{\Gamma_{\text{int}}} \{v\}[\mathbf{q}] \mathrm{d}s. \tag{2.5}$$

To define $h_\kappa := \operatorname{diam}(\kappa)$, and collect them into the elementwise constant function $\mathbf{h} : \Omega \to \mathbb{R}$, with $\mathbf{h} |_\kappa = h_\kappa, \kappa \in \mathcal{T}$, and $\mathbf{h} |_e = \{\mathbf{h}\}, e \subset \Gamma_h$. We always assume that the families of meshes considered are locally quasi-uniform, there are constants $c \geqslant 1$ independent of $\mathbf{h}$, for any pair of elements κ^+ and κ^- in $\mathcal{T}$, that share an edge, we have

$$c^{-1} \leqslant \frac{h_{\kappa^+}}{h_{\kappa^-}} \leqslant c.$$

We first introduce the lifting operator $\mathcal{L}: \mathcal{S} := S^2 + (H_0^1(\Omega) \cap H^2(\Omega)) \to S^2$ by

$$\int_\Omega \mathcal{L}(v) w \mathrm{d}x = \int_{\Gamma_h} [v] \cdot \{\nabla w\} \mathrm{d}s - \int_{\Gamma_{\text{int}}} \{w\}[\nabla v] \mathrm{d}s, \forall \mathrm{w} \in S^2. \tag{2.6}$$

And the lifting operator $\mathcal{L}$ has stability: for $w \in \mathcal{S}$, there is

$$\parallel \mathcal{L}(w) \parallel_\Omega^2 \leqslant C\big(\parallel \sqrt{\sigma}[w] \parallel_{\Gamma_h}^2 + \parallel \sqrt{\tau}[\nabla w] \parallel_{\Gamma_{\text{int}}}^2\big),$$

Where $\sigma = C_\sigma \mathbf{h}^3$, $\tau = C_\tau \mathbf{h}$.

Proof. See [5].

Define bilinear form as $a_h: \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ by

$$\begin{aligned} a_h(w_h, v_h) = &\int_\Omega (\Delta_h w_h \Delta_h v_h + \mathcal{L}(w_h)\Delta_h v_h + \Delta_h w_h \mathcal{L}(v_h)) \mathrm{d}x \\ &+ \int_{\Gamma_h} \sigma[w][v] \mathrm{d}s + \int_{\Gamma_{\text{int}}} \tau[\nabla w][\nabla v] \mathrm{d}s, \end{aligned} \tag{2.7}$$

here the internal penalty parameter $\sigma: \Gamma_h \to \mathbb{R}$, $\tau: \Gamma_{\text{int}} \to \mathbb{R}$ of the segmentation constant is defined as

$$\sigma|_{\Gamma_h} = \sigma_0(\mathbf{h}|_e)^{-3}, \tau|_{\Gamma_{\text{int}}} = \tau_0(\mathbf{h}|_e)^{-1}, \tag{2.8}$$

where $\sigma_0 > 0, \tau_0 > 0$, in order to guarantee the stability of the IPDG method defined in (2.7), σ, τ must be selectively large enough.
The finite element approximation of (2.4) is to find $(\lambda_h, u_h) \in R \times S^2$, such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h), \forall v_h \in S^2. \tag{2.9}$$

The source problem of (2.4) is to find $w \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$a(w, v) = (f, v), \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.10}$$

The DG approximation of (2.10) is to find $w_h \in S^2$, such that

$$a_h(w_h, v_h) = (f, v_h), \forall v_h \in S^2. \tag{2.11}$$

Define the linear bounded operator $T: L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$a(Tf, v) = (f, v), \forall f \in L^2(\Omega), v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.12}$$

The equivalent operator from of (2.4) is

$$Tu = \frac{1}{\lambda} u. \qquad (2,13)$$

By using (2.10), the corresponding discrete solution operator $T_h: L^2(\Omega) \to S^2$ can be defined:

$$a_h(T_h f, v) = (f, v), \forall f \in L^2(\Omega), \forall v \in S^2. \tag{2.14}$$

The equivalent operator from of (2.10) is

$$T_h u_h = \frac{1}{\lambda_h} u_h. \tag{2.15}$$

From the consistency of discontinuous finite element method, let w be the solution of (2.12), and $f \in L^2(\Omega)$, then

$$a_h(w, v_h) = (f, v_h), \forall v_h \in S^2. \tag{2.16}$$

From (2.11) and (2.16), we obtain

$$a_h(w - w_h, v_h) = 0, \forall v_h \in S^2. \tag{2.17}$$

For any function $w \in \mathcal{S}$, introduce sum space $\mathcal{S} = S^2 + (H_0^1(\Omega) \cap H^2(\Omega))$, that assigns a locally discontinuous finite element norm, where the energy norm is defined as follows:

$$\| w \|_G = (\| \Delta_h w \|_\Omega^2 + \| \sqrt{\sigma}[w] \|_{\Gamma_h}^2 + \| \sqrt{\tau}[\nabla w] \|_{\Gamma_{\text{int}}}^2)^{\frac{1}{2}}. \tag{2.18}$$

There is $a_h(\cdot,\cdot)$ is continuous and coercive：

$$|a_h(w, v)| \leqslant C_1 \| w \|_G \| v \|_G \; \forall w, v \in S^2, \tag{2.19}$$

$$a_h(w, w) \geqslant C_2 \| w \|_G^2 \; \forall w \in S^2. \tag{2.20}$$

where $\sigma: \Gamma_h \to \mathbb{R}, \tau: \Gamma_{\text{int}} \to \mathbb{R}$ is a piecewise continuous function, C_1 and C_2 are positive constants depending only on the mesh parameters.

Proof. For $w, v \in S^2$, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |a_h(w, v)| &\leqslant \| \Delta_h w \|_\Omega \| \Delta_h v \|_\Omega + \| \mathcal{L}(w) \|_\Omega \| \Delta_h v \|_\Omega + \| \mathcal{L}(v) \|_\Omega \| \Delta_h w \|_\Omega \\ &\quad + \| \sqrt{\sigma}[w] \|_{\Gamma_h} \| \sqrt{\sigma}[v] \|_{\Gamma_h} + \| \sqrt{\tau}[\nabla w] \|_{\Gamma_{\text{int}}} \| \sqrt{\tau}[\nabla v] \|_{\Gamma_{\text{int}}} \\ &\leqslant C \| w \|_G \| v \|_G. \end{aligned}$$

Continuity is valid.
Next, we prove (2.20), using the definition of norm and the Young's inequality, we obtain

$$\begin{aligned} a_h(w, w) &= \| w \|_G^2 + 2 \int_\Omega \mathcal{L}(w) \Delta_h w \mathrm{d}x \\ &\geqslant \| w \|_G^2 - 2 \| L(w) \|_\Omega^2 - \frac{1}{2} \| \Delta_h w \|_\Omega^2 \\ &\geqslant \frac{1}{2} \| \Delta_h w \|_\Omega^2 + \sqrt{1 - 2c} \left(\| \sqrt{\sigma}[w] \|_{\Gamma_h}^2 + \| \sqrt{\tau}[\nabla w] \|_{\Gamma_{\text{int}}}^2 \right). \end{aligned}$$

When $0 < c < \frac{1}{2}$, the proof is completed.

Let $w \in H^{2+r}(\Omega) (1 < r \leqslant 2)$ be the solution of (2.12), and $f \in L^2(\Omega)$, assuming the following regularity estimate holds:

$$\| w \|_{2+r} \lesssim \| f \|_{0,\Omega}. \tag{2.21}$$

Let w^I be the quadratic interpolation of w, then:

$$\| w - w^I \|_G \lesssim h \| w \|_{3,\Omega}, \tag{2.22}$$

also $[w - w^I] = 0$.

Lemma 2.1. (Proposition 4.9 in [6]) Let $\kappa \in \mathcal{T}$ and $v \in H^{s_\kappa}(\kappa), s_\kappa > 3$, then there exists the polynomial $\Pi v \in S^{h_\kappa}$, satisfying $(0 \leqslant m \leqslant s_\kappa)$

$$\| v - \Pi v \|_{m,\kappa} \lesssim h_\kappa^{s_\kappa - m} \| v \|_{s_\kappa,\kappa}, \tag{2.23}$$

$$\| v - \Pi v \|_{0,\partial\Omega} \lesssim h_\kappa^{s_\kappa - \frac{1}{2}} \| v \|_{s_\kappa,\kappa}. \tag{2.24}$$

Introduce the global interpolation operator $\Pi: (H_0^1(\Omega) \cap H^2(\Omega)) \to S^2$, such that $\Pi(u)|_\kappa = \Pi(u|_\kappa)$, for the vector-value function $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_d)$, define $\Pi(\mathbf{r})|_\kappa = (\Pi\mathbf{r}_1, \Pi\mathbf{r}_2, \cdots, \Pi\mathbf{r}_d)$.

Lemma 2.2. (lemma 2.1 in [7]) Let $\kappa \in \mathcal{T}$, $e \subset \partial\kappa$, and $0 < \xi < \frac{1}{2}$, for any $v \in H^{1+\xi}(\kappa)$ with $\Delta v \in L^2(\kappa)$, there exists a positive constant C independent of v such that

$$\| \nabla v \cdot \mathbf{n} \|_{\xi-\frac{1}{2},e} \leq C\left(\| \nabla v \|_{\xi,\kappa} + h_\kappa^{1-\xi} \| \Delta v \|_{0,\kappa}\right).$$

Theorem 2.1. Let w and w_h be the solution of (2.10) and (2.11), for all $\kappa \in \mathcal{T}$, and $s_\kappa > 3, 0 < \xi < \frac{1}{2}$, then there holds

$$\| w - w_h \|_G \lesssim \inf_{v_h \in S^2} \| w - v_h \|_G + h^{\xi+1} \| \nabla\Delta w \|_{\xi,\Omega} + h \| f \|_{0,\Omega}, \tag{2.25}$$

$$\| w - w_h \|_G \lesssim \sum_{\kappa\in\mathcal{T}} h_\kappa^{s_\kappa-2} \| w \|_{s_\kappa,\kappa}. \tag{2.26}$$

Proof. Firstly, we prove (2.25) by utilizing (2.17), (2.19) and (2.20), we obtain

$$\begin{aligned} \| v_h - w_h \|_G^2 &\lesssim \| a_h(v_h - w_h, v_h - w_h) \| \lesssim a_h(w - w_h, v_h - w_h) + a_h(v_h - w, v_h - w_h) \\ &\lesssim \| v_h - w \|_G \| v_h - w_h \|_G, \\ &+ \int_{\Gamma_h} (\{\nabla\Delta(v_h - w)\}[v_h - w_h] + \{\nabla\Delta(v_h - w_h)\}[v_h - w])\mathrm{d}s \\ &+ \int_{\Gamma_{\mathrm{int}}} (\{\Delta(v_h - w)\}[\nabla(v_h - w_h)] + \{\Delta(v_h - w_h)\}[\nabla(v_h - w)])\mathrm{d}s. \end{aligned}$$

From lemma 2.2, the inverse estimate and the definition of energy norm, we deduce

$$\begin{aligned} &\int_{\Gamma_h} \{\nabla\Delta(v_h - w)\}[v_h - w_h])\mathrm{d}s \\ &\lesssim \sum_{e\in\Gamma_h} \| \{\nabla\Delta(v_h - w)\} \cdot \mathbf{n} \|_{\xi-\frac{1}{2},e} \| [v_h - w_h] \|_{\frac{1}{2}-\xi,e} \\ &\lesssim \sum_{\kappa} \left(\| \nabla\Delta(v_h - w) \|_{\xi,\kappa} + h_\kappa^{1-\xi} \| \Delta^2(v_h - w) \|_{0,\kappa}\right) h^{\xi+1} \left(\| h^{-\frac{3}{2}}[v_h - w_h] \|_{0,e}^2\right)^{\frac{1}{2}} \\ &\lesssim \left(h^{\xi+1} \| \nabla\Delta w \|_{\xi,\Omega} + h^2 \| f \|_{0,\Omega}\right) \| v_h - w_h \|_G. \end{aligned} \tag{2.27}$$

Also

$$\int_{\Gamma_h} \{\nabla\Delta(v_h - w_h)\}[v_h - w])\mathrm{d}s = 0. \tag{2.28}$$

From the trace inequality, the definition of energy norm and (2.21), we deduce

$$\begin{aligned} &\int_{\Gamma_{\mathrm{int}}} \{\Delta(v_h - w)\}[\nabla(v_h - w_h)]\mathrm{d}s \\ &\lesssim \sum_{e\in\Gamma_{\mathrm{int}}} \| \Delta(v_h - w) \|_{0,e} \| [\nabla(v_h - w_h)] \|_{0,e} \\ &\lesssim \sum_{\kappa} \left(h^{-\frac{1}{2}} \| \Delta(v_h - w) \|_{0,k} + h_\kappa^{\frac{1}{2}} \| \nabla\Delta(v_h - w) \|_{0,k}\right) h^{\frac{1}{2}} (\| h^{-\frac{1}{2}}[\nabla(v_h - w_h)] \|_{0,e}^2)^{\frac{1}{2}} \\ &\lesssim (\| v_h - w \|_G + h \| \nabla\Delta w \|_{0,\Omega}) \| v_h - w_h \|_G \\ &\lesssim (\| v_h - w \|_G + h \| f \|_{0,\Omega}) \| v_h - w_h \|_G. \end{aligned} \tag{2.29}$$

Similarly,

$$\begin{aligned} &\int_{\Gamma_{\mathrm{int}}} \{\Delta(v_h - w_h)\}[\nabla(v_h - w)]\mathrm{d}s \\ &\lesssim \sum_{e\in\Gamma_{\mathrm{int}}} \| \Delta(v_h - w_h) \|_{0,e} \| [\nabla(v_h - w)] \|_{0,e} \\ &\lesssim \sum_{\kappa} \left(h^{-\frac{1}{2}} \| \Delta(v_h - w_h) \|_{0,k} + h_\kappa^{\frac{1}{2}} \| \nabla\Delta(v_h - w_h) \|_{0,k}\right) h^{\frac{1}{2}} (\| h^{-\frac{1}{2}}[\nabla(v_h - w)] \|_{0,e}^2)^{\frac{1}{2}} \\ &\lesssim \| v_h - w_h \|_G \| v_h - w \|_G. \end{aligned} \tag{2.30}$$

Then

$$\| v_h - w_h \|_G \lesssim \| v_h - w \|_G + h^{\xi+1} \| \nabla\Delta w \|_{\xi,\Omega} + h \| f \|_{0,\Omega}. \tag{2.31}$$

Using the triangle inequality, we get (2.25).
Next, we prove (2.26). By (2.18), let $E_h(w) = w - \Pi w$, having

$$\begin{aligned}\| E_h(w) \|_G^2 &\lesssim \sum_{\kappa\in\mathcal{T}} \| \Delta_h E_h(w) \|_{0,\kappa}^2 + \sum_{e\in\Gamma_h} \| h^{-\frac{3}{2}}[E_h(w)] \|_{0,\Gamma_h}^2 + \sum_{e\in\Gamma_{\rm int}} \| h^{-\frac{1}{2}}[\nabla E_h(w)] \|_{0,\Gamma_{\rm int}}^2 \\ &\lesssim \sum_{\kappa\in\mathcal{T}} \| \Delta_h E_h(w) \|_{0,\kappa}^2 + \sum_{e\in\Gamma_h} \| h^{-\frac{3}{2}}[E_h(w)] \|_{0,\Gamma_h}^2 + \sum_{e\in\Gamma_h} \| h^{-\frac{1}{2}}[\nabla E_h(w)] \|_{0,\Gamma_h}^2 \\ &:= I_1 + I_2 + I_3\end{aligned}$$

I_1 can be estimated from (2.23):

$$\| \Delta_h E_h(w) \|_{0,\kappa}^2 \lesssim (h^{s_\kappa-2} \| w \|_{s_\kappa,\kappa})^2. \tag{2.32}$$

I_2 can be estimated from (2.24), the trace inequality and the inverse estimate:

$$\begin{aligned}\| h^{-\frac{3}{2}}[E_h(w)] \|_{0,e}^2 &= \| h^{-\frac{3}{2}}((E_h(w))_{\kappa^+} - (E_h(w))_{\kappa^-}) \cdot \mathbf{n} \|_{0,e}^2 \\ &\lesssim h^{-3}(h_e^{-\frac{1}{2}} \| E_h(w) \|_{0,\kappa^+\cup\kappa^-} + h_e^{\frac{1}{2}} \| \nabla E_h(w) \|_{0,\kappa^+\cup\kappa^-})^2 \\ &\lesssim h^{-3}(h_e^{-\frac{1}{2}} \| E_h(w) \|_{0,\kappa^+\cup\kappa^-})^2 \\ &\lesssim (h^{s_\kappa-2} \| w \|_{s_\kappa,\kappa})^2.\end{aligned} \tag{2.33}$$

Similarly, we get I_3:

$$\begin{aligned}\| h^{-\frac{1}{2}}[\nabla E_h(w)] \|_{0,e}^2 &= \| h^{-\frac{1}{2}}((\nabla E_h(w))_{\kappa^+} - (\nabla E_h(w))_{\kappa^-}) \cdot \mathbf{n} \|_{0,e}^2 \\ &\lesssim h^{-1}(h_e^{-\frac{1}{2}} \| \nabla E_h(w) \|_{0,\kappa^+\cup\kappa^-})^2 \\ &\lesssim (h^{s_\kappa-2} \| w \|_{s_\kappa,\kappa})^2.\end{aligned} \tag{2.34}$$

Using (2.32), (2.33) and (2.34), we get

$$\| w - \Pi w \|_G \lesssim \sum_{\kappa\in\mathcal{T}} h^{s_\kappa-2} \| w \|_{s_\kappa,\kappa}. \tag{2.35}$$

By using the error estimate and the interpolation estimate, we obtained

$$\inf_{v_h\in S^2} \| w - v_h \|_G + h^{\xi+1} \| \nabla\Delta w \|_{\xi,\Omega} + h \| f \|_{0,\Omega} \lesssim \| w - \Pi w \|_G. \tag{2.36}$$

Then (2.26) directly from (2.25), (2.35) and (2.36), the proof is completed.

Theorem 2.2. Let w and w_h be the solution of (2.10) and (2.11), then there holds:

$$\| w - w_h \|_{0,\Omega} \lesssim h \| w - w_h \|_G + h^2 \| f \|_{0,\Omega}, \tag{2.37}$$

$$\| w - w_h \|_{0,\Omega} \lesssim h^2 \| w \|_{2+r,\Omega}. \tag{2.38}$$

Proof. w^I is the quadratic interpolation of w, form (2.17) and (2.22), we have

$$\begin{aligned}(w - w_h, f) &= a_h(w - w_h, w) = a_h(w - w_h, w - w^I) \\ &\lesssim \| w - w_h \|_G \| w - w^I \|_G \\ &\quad + \int_{\Gamma_h} (\{\nabla\Delta(w - w_h)\}[w - w^I] + \{\nabla\Delta(w - w^I)\}[w - w_h])\mathrm{d}s \\ &\quad + \int_{\Gamma_{\rm int}} (\{\Delta(w - w_h)\}[\nabla(w - w^I)] + \{\Delta(w - w^I)\}[\nabla(w - w_h)])\mathrm{d}s.\end{aligned} \tag{2.39}$$

From $[w - w^I] = 0$, we derive

$$\int_{\Gamma_h} \{\nabla\Delta(w - w_h)\}[w - w^I]\mathrm{d}s = 0. \tag{2.40}$$

From lemma 2.2, the inverse estimate, definition of energy norm, (2.21) and taking $\xi = r - 1$, we deduce

$$\begin{aligned}\int_{\Gamma_h} \{\nabla\Delta(w - w^I)\}[w - w_h]\mathrm{d}s &\lesssim \sum_{e\in\Gamma_h} \| \{\nabla\Delta w\} \cdot \mathbf{n} \|_{\xi-\frac{1}{2},e} \| [w - w_h] \|_{\frac{1}{2}-\xi,e} \\ &\lesssim \left(h^{\xi+1} \| \nabla\Delta w \|_{\xi,\Omega} + h^2 \| f \|_{0,\Omega}\right) \| w - w_h \|_G \\ &\lesssim h^r \| f \|_{0,\Omega} \| w - w_h \|_G.\end{aligned} \tag{2.41}$$

By the trace inequality with $\frac{1}{2} < \beta \leqslant 1$, the interpolation estimates and the definition of energy norm, we get

$$\int_{\Gamma_{\rm int}} \{\Delta(w - w^I)\}[\nabla(w - w_h)]\mathrm{d}s \lesssim \sum_{e\in\Gamma_{\rm int}} \| \{\Delta(w - w^I)\} \|_{0,e} \| [\nabla(w - w_h)] \|_{0,e}$$

$$\lesssim \sum_{\kappa} h^{\frac{1}{2}}\left(h^{-\frac{1}{2}} \| \Delta(w-w^I) \|_{0,\kappa}+ h_{\kappa}^{\beta-\frac{1}{2}}|\Delta(w-w^I)|_{\beta,\kappa}\right)\left(\| h^{-\frac{1}{2}}[\nabla(w-w_h)] \|_{0,e}^2\right)^{\frac{1}{2}}$$
$$\lesssim (h \| w \|_3+ h^r \| w \|_{2+r}) \| w-w_h \|_G$$
$$\lesssim h \| f \|_{0,\Omega}\| w-w_h \|_G . \quad (2.42)$$

From the trace inequality, (2.21), (2.22) and the definition of energy norm, we derive

$$\int_{\Gamma_{\text{int}}} \{\Delta(w-w_h)\}[\nabla(w-w^I)]\text{d}s \lesssim \sum_{e\in\Gamma_{\text{int}}} h^{\frac{1}{2}} \| \Delta(w-w_h) \|_{0,e} \left(\| h^{-\frac{1}{2}}[\nabla(w-w^I)] \|_{0,e}^2\right)^{\frac{1}{2}}$$
$$\lesssim \sum_{\kappa} h^{\frac{1}{2}} \| w-w^I \|_G \left(h^{-\frac{1}{2}} \| \Delta(w-w_h) \|_{0,\kappa}+ h_{\kappa}^{\frac{1}{2}} \| \nabla\Delta(w-w_h) \|_{0,\kappa}\right)$$
$$\lesssim h^{\frac{3}{2}} \| w \|_3 \left(h^{-\frac{1}{2}} \| (w-w_h) \|_G+ h_{\kappa}^{\frac{1}{2}} \| \nabla\Delta w \|_{0,\kappa}\right)$$
$$\lesssim h \| f \|_{0,\Omega}\| w-w_h \|_G+ h^2 \| f \|_{0,\Omega}^2 . \quad (2.43)$$

Then (2.37) directly from (2.39), (2.40), (2.41) and (2.43).
Next, we prove (2.38). From (2.26), (2.37) and (2.43), we get

$$\| w-w_h \|_{0,\Omega}\lesssim h^{r+3} \| w \|_{2+r,\Omega}+ h^2 \| \nabla\Delta w \|_{0,\Omega}$$
$$\lesssim h^{r+3} \| w \|_{2+r,\Omega}+ h^2 \| w \|_{2+r,\Omega}$$
$$\lesssim h^2 \| w \|_{2+r,\Omega} . \quad (2.44)$$

So, (2.38) is valid.

Taking $s = 2 + r(1 < r \leqslant 2)$ in (2.26), and the regularity estimate yields the following stable estimate:

$$\| T_hf \|_G\lesssim\| T_hf-Tf \|_G +\| Tf \|_G\lesssim\| T_hf-Tf \|_G +\| Tf \|_2\lesssim h^r \| Tf \|_{2+r} +\| Tf \|_2\lesssim\| f \|_{0,\Omega}. \quad (2.45)$$

Let λ be the jth eigenvalue of (2.4), with algebraic multiplicities q and the ascent $\alpha = 1$,where $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+q-1}$. When $\| T_h - T \|_{0,\Omega}\rightarrow 0$, q eigenvalue $\lambda_{j,h}, \cdots \lambda_{j+q-1,h}$ of (2.9) will converge to λ. Let $M(\lambda)$ be the generalized eigenvector space of (2.4) related to λ, $M_h(\lambda)$ be the direct sum of the generalized eigenvector space of (2.9) related to λ_h, and λ_h converge to λ.

The following theorem can be proven using a similar method as proof Theorem 3.1 in reference [8].

Theorem 2.3. The following inequality holds

$$|\lambda_h - \lambda| \lesssim h^2. \quad (2.46)$$

Let $u_h \in M_h(\lambda)$ be the direct sum of the generalized eigenvector space of (2.9), with $0 < \xi < \frac{1}{2}$,then there exists eigenvalue function u of (2.4) such that

$$\| u-u_h \|_{0,\Omega}\lesssim h^2, \quad (2.47)$$
$$\| u-u_h \|_G\lesssim h \| u \|_{2+r,\Omega}+ h^{\xi+1} \| \nabla\Delta u \|_{\xi,\Omega}+ h^4. \quad (2.48)$$

III. POSTERIOR ERROR ESTIMATION

i. Estimators of eigenfunctions and their reliability

Let (λ_h, u_h) be the eigenpair of (2.9), and define element residuals and surface residuals on each element $\kappa \in \mathcal{T}$ and $e \in \Gamma_h$, respectively, as follows,

$$R_\kappa = \lambda_h u_h - \Delta_h^2 u_h,$$
$$J_{F,1} = [u_h], \forall e \in \Gamma_h, J_{F,2} = [\nabla u_h], \forall e \in \Gamma_{int},$$
$$J_{F,3} = [\nabla\Delta u_h], \forall e \in \Gamma_{int}, J_{F,4} = [\Delta u_h], \forall e \in \Gamma_{int}.$$

Define local error indicators on the $\kappa \in \mathcal{T}$ of each unit

$$\eta_\kappa^2 = \sum_{\kappa} C(h^4 \| (\lambda_h u_h - \Delta_h^2 u_h) \|_{0,\kappa}^2+ C_p(\sum_{e\in\Gamma_h} h^{-3} \| J_{F,1} \|_{0,e}^2+ \sum_{e\in\Gamma_{int}} h^{-1} \| J_{F,2} \|_{0,e})$$
$$+ \sum_{e\in\Gamma_{int}} h^3 \| J_{F,3} \| |_{0,e}^2 + \sum_{e\in\Gamma_{int}} h \| J_{F,4} \|_{0,e}^2). \quad (3.1)$$

where $C_p := max\{1, \sigma_0, \tau_0, \sigma_0^2, \tau_0^2\}$.
The global error indicator is

$$\eta(u_h) = (\sum_{\kappa\in\mathcal{T}} \eta_\kappa^2)^{\frac{1}{2}}. \quad (3.2)$$

Lemma 3.1. We assume that the mesh $\mathcal{T}$ is constructed as above. Then there exists an operator $E: S^2 \to \tilde{S}^4 \cap (H_0^1(\Omega) \cap H^2(\Omega))$ that satisfies

$$\sum_{\kappa\in\mathcal{T}} |u_h - E(u_h)|_{a,\kappa}^2 \leqslant C\left(\| \mathbf{h}^{\frac{1}{2}-\alpha}[u_h] \|_{\Gamma_h}^2 + \| \mathbf{h}^{\frac{3}{2}-\alpha}[\nabla u_h] \|_{\Gamma_{int}}^2\right),$$

with $\alpha = 0,1,2$ and $C > 0$ being a constant that is independent of $\mathbf{h}$ and u_h.
Note that the recovery operator E maps elements of S^2 onto a C^1-conforming space consisting of macro-elements of degree 4.
Proof. See [3].

Theorem 3.1. Let (λ, u) and (λ_h, u_h) be the eigenpairs of (2.4) and (2.9), for any $v \in H_0^1(\Omega) \cap H^2(\Omega)$, the following formula holds

$$\| u - u_h \|_G^2 \leqslant \eta(u_h)^2 + \| \lambda u - \lambda_h u_h \|_\Omega^2. \tag{3.3}$$

Proof. Let $v_h \in S^2, v \in H_0^1(\Omega) \cap H^2(\Omega), \eta = v - v_h$, with $E(u_h) \in \tilde{S}^4 \cap (H_0^1(\Omega) \cap H^2(\Omega))$ in lemma 3.1, then the error can be decomposed into

$$b = u - u_h = \big(u - E(u_h)\big) + (E(u_h) - u_h) \equiv b_c + b_d.$$

Since u is the solution to the weak-form problem, we have $a_h(u,v) = \lambda(u,v)$, where $\mathcal{L}(u) = \mathcal{L}(v) = 0$. We have

$$\begin{aligned} a_h(b,v) &= a_h(u,v) - a_h(u_h,v) \\ &= \lambda(u,v) - a_h(u_h, v - v_h) - a_h(u_h, v_h) \\ &= \lambda(u,\eta) - a_h(u_h,\eta). \end{aligned} \tag{3.4}$$

Then

$$a_h(b_c, v) = \lambda(u,\eta) - a_h(u_h,\eta) - a_h(b_d, v). \tag{3.5}$$

By $u \in H_0^1(\Omega) \cap H^2(\Omega), E(u_h) \in \tilde{S}^4 \cap (H_0^1(\Omega) \cap H^2(\Omega))$, there is $b_c = u - E(u_h) \in H_0^1(\Omega) \cap H^2(\Omega)$, then $\mathcal{L}(b_c) = 0$, and by $v = b_c$ in (3.5), there is

$$\begin{aligned} \| \Delta_h b_c \|^2 &= a_h(b_c, b_c) \\ &= (\lambda(u,\eta) - (\lambda_h u_h, \eta)) + ((\lambda_h u_h, \eta) - a_h(u_h,\eta)) - a_h(b_d . b_c) \\ &= B_1 + B_2 + B_3. \end{aligned} \tag{3.6}$$

We have $v = b_c$, v_h is a linear approximation to b_c, then $C > 0$ is a constant independent of $\mathcal{T}$, $0 \leqslant j \leqslant m \leqslant 2, \kappa \in \mathcal{T}$, from [9] we get

$$b_c - v_h|_{j,\kappa} \leqslant C h_\kappa^{m-j}\ b_c|_{m,\kappa}. \tag{3.7}$$

By (3.7), then

$$| B_1 | \leqslant \| \lambda u - \lambda_h u_h \| \| \eta \| \leqslant \| h^2(\lambda u - \lambda_h u_h) \|_\Omega | b_c |_{2,\Omega}. \tag{3.8}$$

By (2.5), (2.7), Green's formula and the definition of the lifting operator, there is

$$\begin{aligned} B_2 = &\int_\Omega (\lambda_h u_h - \Delta_h^2 u_h)\eta \mathrm{d}x - \int_\Omega \mathcal{L}(u_h)\Delta_h \eta \mathrm{d}x - \int_{\Gamma_{\text{int}}} \{\nabla_h \eta\}[\Delta_h u_h]\mathrm{d}s \\ &+ \int_{\Gamma_{\text{int}}} \{\eta\}[\nabla \Delta_h u_h]\mathrm{d}s - \int_{\Gamma_h} \sigma[u_h][\eta]\mathrm{d}s - \int_{\Gamma_{\text{int}}} \tau[\nabla u_h][\nabla \eta]\mathrm{d}s, \end{aligned}$$

From the inverse estimate, the stability of the lifting operator, the trace inequality, (3.7) and Poincaré-Friedrichs inequalities, we get

$$\begin{aligned} |B_2| \lesssim &\left(\| h^2(\lambda_h u_h - \Delta_h^2 u_h) \|^2 + \| \sqrt{\sigma}[u_h] \|_{\Gamma_h}^2 + \| \sqrt{\tau}[\nabla u_h] \|_{\Gamma_{\text{int}}}^2\right)^{\frac{1}{2}} |b_c|_{2,\Omega} \\ &+ \| h^{\frac{1}{2}}[\Delta_h u_h]] \|_{\Gamma_{\text{int}}} |b_c|_{2,\Omega} + \| h^{\frac{3}{2}}[\nabla \Delta_h u_h]] \|_{\Gamma_{\text{int}}} |b_c|_{2,\Omega} \\ &+ (\| h^{-\frac{3}{2}}[u_h] \|_{\Gamma_h}^2 + \| h^{-\frac{1}{2}}[\nabla u_h] \|_{\Gamma_{\text{int}}}^2)^{\frac{1}{2}} |b_c|_{2,\Omega}. \end{aligned} \tag{3.9}$$

Using $\mathcal{L}(b_c) = [b_c]|_{\Gamma_h} = [\nabla b_c]|_{\Gamma_{\text{int}}} = 0$, the triangle inequality and the stability of the lifting operator

$$\begin{aligned} |B_3| = &\left| \int_\Omega (\Delta_h b_d \Delta_h b_c + \mathcal{L}(b_c)\Delta_h b_d + \mathcal{L}(b_d)\Delta_h b_c)\mathrm{d}x \right. \\ &\left. + \int_{\Gamma_h} \sigma[b_d][b_c]\mathrm{d}s - \int_{\Gamma_{\text{int}}} \tau[\nabla b_d][\nabla b_c]\mathrm{d}s \right| \\ \lesssim &(\| \Delta_h b_d \|_\Omega^2 + \| \sqrt{\sigma}[u_h] \|_{\Gamma_h}^2 + \| \sqrt{\tau}[\nabla u_h] \|_{\Gamma_{\text{int}}}^2)^{\frac{1}{2}} \| \Delta_h b_c \|_\Omega. \end{aligned} \tag{3.10}$$

Substituting (3.8), (3.9) and (3.10) into (3.6), and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \| \Delta_h b_c \|^2 \lesssim &\|h^2(\lambda u - \lambda_h u_h)\|_\Omega \\ &+ \| h^2(\lambda_h u_h - \Delta_h^2 u_h) \|_\Omega^2 + \| h^{-\frac{3}{2}}[u_h] \|_{\Gamma_h}^2 + \| h^{-\frac{1}{2}}[\nabla u_h] \|_{\Gamma_{\text{int}}}^2 \\ &+ \| h^{\frac{3}{2}}[\nabla \Delta_h u_h] \|_{\Gamma_{\text{int}}}^2 + \| h^{\frac{1}{2}}[\Delta_h u_h] \|_{\Gamma_{\text{int}}}^2 + \| \Delta_h b_d \|_\Omega^2. \end{aligned} \tag{3.11}$$

Then

$$\| \Delta b_d \|_\Omega^2 \leqslant \sum_{\kappa \in \mathcal{T}} |u_h - E(u_h)|_{2,\kappa}^2 \leqslant C(\| \mathbf{h}^{\frac{1}{2}-\alpha}[u_h] \|_{\Gamma_h}^2 + \| \mathbf{h}^{\frac{3}{2}-\alpha}[\nabla u_h] \|_{\Gamma_{\text{int}}}^2). \qquad (3.12)$$

Theorem 3.1 can be proved by Lemma 3.1, (3.11), (3.12) and the triangle inequality.

For the residual term $\| h^2(\lambda_h u_h - \Delta_h^2 u_h) \|_\Omega$, reference [3] shows that it does not affect the upper bound, and it can be seen from theorem 2.3 that when ascent $\alpha = 1$, $\| \lambda u - \lambda_h u_h \|_{0,\Omega}$ and $\| u - u_h \|_{0,\Omega}$ are both small quantities of higher order $\| u - u_h \|_G$. Therefore, it can be seen from (3.3) that the indicator of error estimation $\eta(u_h)$ is one of the upper bounds of the discontinuous finite element energy norm, so the error estimation is reliable.

ii. Effectiveness of the eigenfunction estimator

Theorem 3.2. Under theorem 3.1, there is

(i) for any $\kappa \in \mathcal{T}$,

$$\| \mathbf{h}^2(\lambda_h u_h - \Delta^2 u_h) \|_\kappa^2 \lesssim \| \mathbf{h}^2(\lambda u - \lambda_h u_h) \|_\kappa^2 + \| \Delta b \|_\kappa^2,$$

(ii) for any $e \in \Gamma_h$,

$$\mathbf{h}^{-3} \| J_{F,1} \|_{0,e}^2 = \mathbf{h}^{-3} \| [u_h] \|_{0,e}^2 = \mathbf{h}^{-3} \| [b] \|_{0,e}^2,$$

(iii) for any $e \in \Gamma_{\text{int}}$,

$$\mathbf{h}^{-1} \| J_{F,2} \|_{0,e}^2 = \mathbf{h}^{-1} \| [\nabla u_h] \|_{0,e}^2 = \mathbf{h}^{-1} \| [\nabla b] \|_{0,e}^2,$$

(iv) for any $e \in \Gamma_{\text{int}}$,

$$\| \mathbf{h}^{\frac{3}{2}} J_{F,3} \|_{\Gamma_{\text{int}}}^2 \lesssim \| \mathbf{h}^2(\lambda u - \lambda_h u_h) \|_{\kappa_1 \cup \kappa_2}^2 + \| \Delta b \|_{\kappa_1 \cup \kappa_2}^2,$$

(v) for any $e \in \Gamma_{\text{int}}$,

$$\| \mathbf{h}^{\frac{1}{2}} J_{F,4} \|_{\Gamma_{\text{int}}}^2 \lesssim \| \mathbf{h}^2(\lambda u - \lambda_h u_h) \|_{\kappa_1 \cup \kappa_2}^2 + \| \Delta b \|_{\kappa_1 \cup \kappa_2}^2.$$

Proof. First prove (i). Given that $H_0^2(\Omega)$ is a subspace of $H_0^1(\Omega) \cap H^2(\Omega)$. Fix $\kappa \in \mathcal{T}$, and let $v \in H_0^2(\Omega) \cap H_0^2(\kappa)$, with $v|_{\Omega \setminus \kappa} = 0$, be a polynomial function on κ. Setting $v_h = 0$ and taking v as above in (3.4) yields

$$\int_\kappa \Delta b \Delta v \mathrm{d}x = \int_\kappa (\lambda u - \lambda_h u_h) v \mathrm{d}x + \int_\kappa (\lambda_h u_h - \Delta^2 u_h) v \mathrm{d}s = \int_\kappa (\lambda u - \Delta^2 u_h) v \mathrm{d}s, \qquad (3.13)$$

noting that $\mathcal{L}(u) = \mathcal{L}(v) = 0$ on Ω and that $[v]|_{\Gamma_h} = [\nabla v]|_{\Gamma_h} = \{v\}|_{\Gamma_h} = \{\nabla v\}|_{\Gamma_h} = 0$. We have

$$\left| \int_\kappa (\lambda_h u_h - \Delta^2 u_h) v \mathrm{d}s \right| \lesssim (\| \Delta b \|_\kappa + \| \mathbf{h}^2(\lambda u - \lambda_h u_h) \|_\kappa) \| \mathbf{h}^{-2} v \|_\kappa. \qquad (3.14)$$

Letting $v|_\kappa = (\lambda_h u_h - \Delta^2 u_h) b_\kappa^2$, where $b_K: \kappa \to \mathbb{R}$ is the standard internal bubble function (which is defined by $b_\kappa := b_{\hat{\kappa}} \circ F_\kappa$, where $\lambda_1, \lambda_2, \lambda_3$ are the barycentric coordinates of the reference triangle $\hat{\kappa}$, then $b_{\hat{\kappa}} := 27\lambda_1\lambda_2\lambda_3$, and if $\hat{\kappa}$ is the reference rectangle, then $b_{\hat{\kappa}} := (1 - \lambda_1^2)(1 - \lambda_2^2)$. We have

$$\| (\lambda_h u_h - \Delta^2 u_h) \|_\kappa^2 \leqslant C \int_k (\lambda_h u_h - \Delta^2 u_h) b_\kappa^2 \mathrm{d}x = C \int_k (\lambda_h u_h - \Delta^2 u_h) v \mathrm{d}x. \qquad (3.15)$$

Then applying (3.14), (3.15) and the Cauchy-Schwarz inequality yields

$$\begin{aligned} \| (\lambda_h u_h - \Delta^2 u_h) \|_\kappa^2 &\lesssim (\| \Delta b \|_k + \| \mathbf{h}^2(\lambda u - \lambda_h u_h) \|_k^2) \| \mathbf{h}^{-2} v \|_k \\ &\lesssim \mathbf{h}^{-4}(\| \Delta b \|_k^2 + \| \mathbf{h}^2(\lambda u - \lambda_h u_h) \|_k^2), \end{aligned} \qquad (3.16)$$

(i) is valid.

For any $e \in \Gamma_h$, we have $[u]|_{\Gamma_h} = 0$, which gives (ii).For any $e \in \Gamma_{\text{int}}$, we have $[\nabla u]|_{\Gamma_{int}} = 0$, then we get (iii).

Next prove (iv). For each inner edge e, we define the largest diamond in $\kappa_1 \cup \kappa_2$ as $\tilde{\kappa}$, where e is the diagonal of the diamond $\tilde{\kappa}$. And we define the bubble function $b_{\tilde{\kappa}}: \tilde{\kappa} \to \mathbb{R}$ on the diamond $\tilde{\kappa}$. And there is an affine $b_l: \tilde{\kappa} \to \mathbb{R}$ which has a value of 0 along edge e, i.e. $(\nabla b_l \cdot \mathbf{n})|_e = \mathbf{h}^{-2}|_e$. Thus b_l is fully defined as a symbol, which is irrelevant to the discussion. The above definition gives the function $b_e: \Omega \to \mathbb{R}$, where $b_e|_{\tilde{\kappa}} := b_l b_{\tilde{\kappa}}^3$, and on $\Omega \setminus \tilde{\kappa}$, where $b_e := 0$, then we have the following properties:

$$\begin{aligned} &b_e \in C^2(\Omega) \cap H_0^2(\Omega), \qquad [b_e]|_{\Gamma_h} = [\nabla b_e]|_{\Gamma_h} = \{b_e\}|_{\Gamma_h} = 0, \\ &(\{\nabla b_e\} \cdot \mathbf{n})|_e = (\mathbf{h}^{-1} b_{\tilde{\kappa}}^3)|_e, \qquad \{\nabla b_e\}|_{\Gamma_h \setminus e} = 0, \end{aligned} \qquad (3.17)$$

and along edge e we have $\nabla b_{\tilde{\kappa}} \cdot \mathbf{n} = 0$.

We set $v = \psi b_e$, where ψ is a constant function in the direction of e normal, i.e., $(\nabla \psi \cdot \mathbf{n})|_e = 0$, and substitute v and $v_h = 0$ into (3.4), we deduce

$$\int_{\Gamma_{int}} [\Delta u_h] \cdot \{\nabla \eta\} \mathrm{d}s = \int_{k_1 \cup k_2} (\lambda u - \lambda_h u_h) v \mathrm{d}x + \int_{k_1 \cup k_2} (\lambda_h u_h - \Delta^2 u_h) v \mathrm{d}x - \int_{k_1 \cup k_2} \Delta b \Delta v \mathrm{d}x. \qquad (3.18)$$

Letting $\psi|_e = (\mathbf{h}^{-1}[\Delta u_h] \cdot \mathbf{n})|_e$ in (3.18), we derive

$$\int_{\Gamma_{int}} [\Delta u_h] \cdot \{\nabla \eta\} ds = \| b_{\tilde{\kappa}}^{\frac{3}{2}} \mathbf{h}^{-1} [\Delta u_h] \|_{\Gamma_{int}}^2 \geqslant C \| \mathbf{h}^{-1} [\Delta u_h] \|_{\Gamma_{int}}^2. \quad (3.19)$$

From scaling argument and norm equivalence, let $m: e \to \mathbb{R}$, where $m(p)$ represents the length of a line perpendicular to e in $\tilde{\kappa}$ intersecting at point $p \in e$, so there is

$$\| v \|_{\kappa_1 \cup \kappa_2} \leqslant C \| \psi \|_{\kappa_1 \cup \kappa_2} = C(\int_e \psi^2(p) m(p) ds)^{\frac{1}{2}} \leqslant C \| \mathbf{h}^{\frac{1}{2}} \psi \|_e = C \| \mathbf{h}^{-\frac{1}{2}} [\Delta u_h] \|_e. \quad (3.20)$$

From (3.18) and (3.19), we have

$$\begin{aligned} &\| \mathbf{h}^{-1} [\Delta u_h] \|_{\Gamma_{int}}^2 \\ &\lesssim \| (\lambda u - \lambda_h u_h) \|_{k_1 \cup k_2} \| v \|_{k_1 \cup k_2} + \| \lambda_h u_h - \Delta^2 u_h \|_{k_1 \cup k_2} \| v \|_{k_1 \cup k_2} + \| \Delta b \|_{k_1 \cup k_2} \| \Delta v \|_{k_1 \cup k_2} \\ &\lesssim \left(\| \mathbf{h}^{\frac{1}{2}} (\lambda u - \lambda_h u_h) \|_{k_1 \cup k_2} + \| \mathbf{h}^{\frac{1}{2}} (\lambda_h u_h - \Delta^2 u_h) \|_{k_1 \cup k_2} + \| \mathbf{h}^{-\frac{3}{2}} \Delta b \|_{k_1 \cup k_2} \right) \| \mathbf{h}^{-\frac{1}{2}} \Delta v \|_{k_1 \cup k_2}. \end{aligned} \quad (3.21)$$

Substitute (3.20) and (i) into (3.21), by the Cauchy-Schwarz inequality, and multiply (3.21) by $\mathbf{h}^3$, so (iv) is proved.

Similarly, (v) the same as the above, have

$$b_{\tilde{\kappa}}^3 \in C^2(\Omega) \cap H_0^2(\Omega), \qquad [b_{\tilde{\kappa}}^3]|_{\Gamma_h} = [\nabla b_{\tilde{\kappa}}^3]|_{\Gamma_h} = (\{\nabla b_{\tilde{\kappa}}^3\} \cdot \mathbf{n})|_{\Gamma_h} = 0, \quad \{b_{\tilde{\kappa}}^3\}|_{\Gamma_h \backslash e} = 0, \quad (3.22)$$

Letting $v = \zeta b_{\tilde{\kappa}}^3$, ζ is defined the same as ψ, and substitute v and $v_h = 0$ into (3.4), we have

$$\int_{\Gamma_{int}} [\nabla \Delta u_h] \cdot \{v\} ds = \int_{k_1 \cup k_2} (\lambda u - \lambda_h u_h) v dx + \int_{k_1 \cup k_2} (\lambda_h u_h - \Delta^2 u_h) v dx - \int_{k_1 \cup k_2} \Delta b \Delta v dx. \quad (3.23)$$

Let $\zeta|_e = ([\nabla \Delta u_h]|_e$ into (3.23), there is

$$\int_{\Gamma_{int}} [\nabla \Delta u_h] \cdot \{\eta\} ds = \| b_{\tilde{\kappa}}^{\frac{3}{2}} [\nabla \Delta u_h] \|_{\Gamma_{int}}^2 \geqslant C \| [\nabla \Delta u_h] \|_{\Gamma_{int}}^2. \quad (3.24)$$

From the above, there are the following

$$\| v \|_{\kappa_1 \cup \kappa_2} \leqslant C \| \zeta \|_{\kappa_1 \cup \kappa_2} \leqslant C \| \mathbf{h}^{\frac{1}{2}} \xi \|_e \leqslant C \| \mathbf{h}^{\frac{1}{2}} [\nabla \Delta u_h] \|_e. \quad (3.25)$$

The following can be obtained by (3.23) and (3.24)

$$\begin{aligned} \| [\nabla \Delta u_h] \|_{\Gamma_{int}}^2 \lesssim & (\| \mathbf{h}^{\frac{1}{2}} (\lambda u - \lambda_h u_h) \|_{k_1 \cup k_2}^2) \| \mathbf{h}^{-\frac{1}{2}} v \|_{k_1 \cup k_2} \\ &+ (\| \mathbf{h}^{\frac{1}{2}} (\lambda_h u_h - \Delta^2 u_h) \|_{k_1 \cup k_2}^2 + \| \mathbf{h}^{-\frac{3}{2}} \Delta b \|_{k_1 \cup k_2}^2) \| \mathbf{h}^{-\frac{1}{2}} v \|_{k_1 \cup k_2}. \end{aligned} \quad (3.26)$$

By substituting (i) and (3.25) into (3.26) and multiplying both sides of (3.26) by $\mathbf{h}^3$, (v) is proved.

Theorem 3.3. Under Theorem 3.1 and theorem 3.2, we have

$$\begin{aligned} \eta_\kappa^2 \lesssim & \sum_{\kappa \in \mathcal{T}} (\| \mathbf{h}^2 (\lambda u - \lambda_h u_h) \|_\kappa^2 + \| \Delta b \|_\kappa^2) + \sum_{e \in \Gamma_h} \| \mathbf{h}^{-\frac{3}{2}} [u - u_h] \|_e^2 \\ &+ \sum_{e \in \Gamma_{int}} \| \mathbf{h}^{-\frac{1}{2}} [\nabla (u - u_h)] \|_e^2, \end{aligned} \quad (3.27)$$

$$\eta(u_h)^2 \lesssim \| u - u_h \|_G^2 + \| \mathbf{h}^2 (\lambda u - \lambda_h u_h) \|_\Omega^2. \quad (3.28)$$

Proof. According to the definition of η_κ and theorem 3.2, (3.27) can be obtained, and using the definition of energy norm, (3.28) can be obtained.

Theorem 3.3 shows that the error estimation indicator is valid.

iii. The reliability of the estimators for the eigenvalues

Lemma 3.2. Let (λ, u) and (λ_h, u_h) be the eigenpairs of (2.4) and (2.9), respectively, then

$$\lambda - \lambda_h = \frac{\lambda(u - u_h, u - u_h)}{(u_h, u_h)} - \frac{a_h(u - u_h, u - u_h)}{(u_h, u_h)}. \quad (3.29)$$

Theorem 3.4. Under the condition of lemma 3.2, let $M(\lambda) \subset H^{2+r}(\Omega)$, $0 < \xi < \frac{1}{2}$, then

$$\| \lambda - \lambda_h \| \lesssim \eta(u_h)^2 + \| \lambda u - \lambda_h u_h \|_\Omega^2 + h^2 \| \nabla \Delta u \|_{\xi,\Omega}^2 + h^4 \| \Delta^2 u \|_{0,\Omega}^2. \quad (3.30)$$

Proof. Theorem 2.3 shows that $\| u - u_h \|_{0,\Omega}$ is a term higher than $\| u - u_h \|_G$, so from lemma 3.1 and (3.3), we have

$$| \lambda - \lambda_h | \lesssim \| u - u_h \|_G^2 + \sum_{e \in \Gamma_h} \int_e \{\nabla \Delta (u - u_h)\} [u - u_h] ds + \sum_{e \in \Gamma_{int}} \int_e \{\Delta (u - u_h)\} [\nabla (u - u_h)] ds. \quad (3.31)$$

From lemma 2.2, the inverse estimate and the definition of energy norm, we deduce

$$\sum_{e\in\Gamma_h}\int_e \{\nabla\Delta(u-u_h)\}[u-u_h]\mathrm{d}s$$
$$\lesssim \sum_{e\in\Gamma_h} \| \{\nabla\Delta(u-u_h)\}\cdot\mathbf{n}\|_{\xi-\frac{1}{2},e}\| [u-u_h]\|_{\frac{1}{2}-\xi,e}$$
$$\lesssim \sum_{\kappa} h^{\xi+1}\left(\| \nabla\Delta(u-u_h)\|_{\xi,\kappa}+h_\kappa^{1-\xi}\|\Delta^2(u-u_h)\|_{0,\kappa}\right)(\| h^{-\frac{3}{2}}[u-u_h]\|_{0,e}^2)^{\frac{1}{2}}$$
$$\lesssim \left(h^{\xi+1}\|\nabla\Delta u\|_{\xi,\Omega}+h^2\|\Delta^2 u\|_{0,\Omega}\right)\| u-u_h\|_G. \quad (3.32)$$

From the trace inequality and the definition of energy norm, we derive

$$\sum_{e\in\Gamma_{int}}\int_e \{\Delta(u-u_h)\}[\nabla(u-u_h)]\mathrm{d}s$$
$$\lesssim \sum_{e\in\Gamma_{int}} h^{\frac{1}{2}}\|\{\Delta(u-u_h)\}\|_{0,e}\left(\| h^{-\frac{1}{2}}[\nabla(u-u_h)]\|_{0,e}^2\right)^{\frac{1}{2}}$$
$$\lesssim \sum_{\kappa} h^{\frac{1}{2}}\left(h^{-\frac{1}{2}}\|\Delta(u-u_h)\|_{0,\kappa}+h^{\frac{1}{2}}\|\nabla\Delta(u-u_h)\|_{0,\kappa}\right)\| u-u_h\|_G$$
$$\lesssim \left(\| u-u_h\|_G+h\|\nabla\Delta(u-u_h)\|_{0,\Omega}\right)\| u-u_h\|_G. \quad (3.33)$$

Substituting (3.32) and (3.33) into (3.31), and then from (3.3) and the Cauchy-Schwarz inequality, we get (3.30), that is, the proof is complete.

From theorem 3.1 and theorem 3.3, we know that the estimator $\eta(u_h)^2$ of the eigenfunction error $\| u-u_h\|_G^2$ is reliable and efficient. Therefore, an adaptive algorithm based on this estimator indicator can generate a good gradient grid such that the approximate eigenfunction reaches the optimal convergence rate $O(dof^{-1})$ in $\|\cdot\|_G^2$. Thus, we expect:

$$h^2\|\nabla\Delta u\|_{\xi,\Omega}^2+h^4\|\Delta^2 u\|_{0,\Omega}^2\leq O(dof^{-1}).$$

Therefore, from (3.30), we get $|\lambda-\lambda_h|<O(dof^{-1})$. Thus, $\eta(u_h)^2$ can be regarded as the error estimation indicator of λ_h. The following numerical experiments show that $\eta(u_h)^2$as the error estimation indicator of $\lambda_h$is reliable and efficient.

IV. NUMBERICAL EXPERIMENTS

In this section, we report some numerical experiments to demonstrate the effectiveness of our approach. Considering the problem (2.1), our program is compiled under the iFEM package and we use the DG method where the penalty coefficient is $\sigma=70,\tau=70$ to do the calculation. Consider the following two test domain: square domain Ω_S with vertex of $(0,0),(1,0),(1,1),(0,1)$, hexagonal domains Ω_H with vertex of $(1,7,2),(2,7,3),(3,7,4),(7,5,4),(7,6,5),(1,6,7)$. Since the exact eigenvalue is unknown, we take the reference eigenvalue $\lambda_1=389.6365$ in the square domain and the first two reference eigenvalues $\lambda_1=51.198878119786$, $\lambda_2=328.757742218653$ in the hexagon domain.

Table 1: Results of numerical solutions of quadratic eigenvalues for region Ω_S,with an initial grid of 1/8

Domin	*l*	*dof*	λ_1	*Error*
Ω_S	1	768	1.0e+02*4.804360813618129	90.7995813618128
	2	1056	1.0e+02*4.0162473161865	11.98823162
	4	1728	1.0e+02*3.92381659951859	2.74515995185873
	6	3888	1.0e+02*3.90616393229752	2.62161409558774
	8	8760	1.0e+02*3.92258114095587	0.979893229753657
	10	18564	1.0e+02*3.9012227109147	0.485771091471861
	12	42378	1.0e+02*3.89847285964222	0.210785965330899
	14	87588	1.0e+02*3.89736006335979	0.099506810507876
	16	206172	1.0e+02*3.89680850296493	0.044593973099722

Table 2: Results of numerical solutions of quadratic eigenvalues for region Ω_H,with an initial grid of 1/8

Domin	*l*	*dof*	λ_1	*Error*
	1	2304	56.681054076591394	5.482175956805392
	2	2616	54.063730945910883	2.864852826124881
	4	4512	52.744740755403164	1.545862635617162

Ω_H	6	8334	52.153313504680995	0.954435384894992
	8	15468	51.745061666227073	0.546183546441071
	10	28248	51.509294035452548	0.310415915666546
	12	53400	51.376378012289926	0.177499892503924
	14	99072	51.290365005390711	0.091486885604709
	15	136656	51.266349425181744	0.067471305395742

Table 3: Results of numerical solutions of quadratic eigenvalues for region Ω_H,with an initial grid of 1/8

$Domin$	l	dof	λ_2	$Error$
Ω_H	1	2304	1.0e+02 *3.614939016628592	32.736159444206237
	2	2838	1.0e+02*3.425149701035349	13.757227884881900
	4	5502	1.0e+02* 3.364259155266177	7.668173307964651
	6	11460	1.0e+02* 3.333056194237199	4.547877205066868
	8	22872	1.0e+02*3.313142058416707	2.556463623017692
	10	45060	1.0e+02* 3.300261319939643	1.268389775311334
	12	89652	1.0e+02*3.294558000135493	0.698057794896329
	14	174900	1.0e+02*3.291024325952683	0.344690376615290
	16	341610	1.0e+02*3.289234693276596	0.165727109006639

Figure 1: On the test domain Ω_S, the initial grid is 1/8 quadratic adaptive mesh and error curve

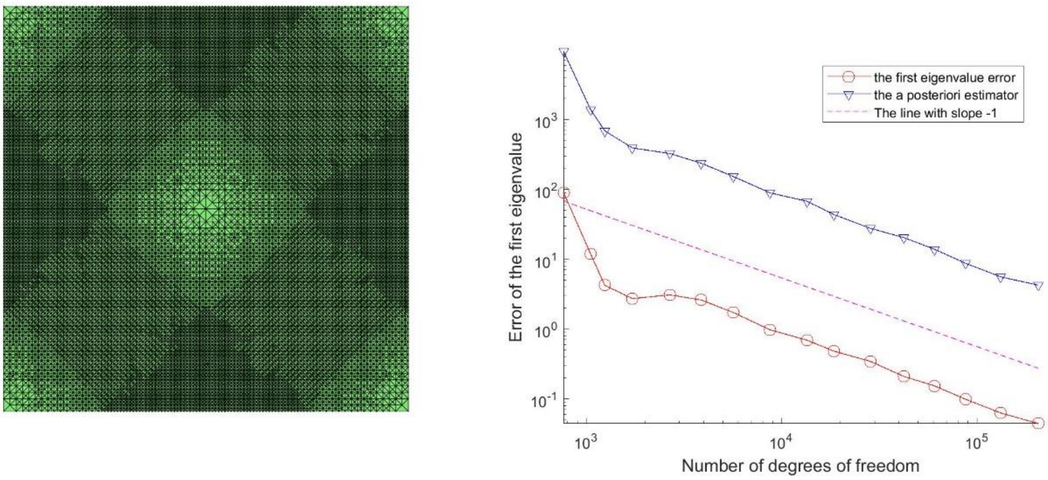


Figure 2: On the test domain Ω_H, the reference eigenvalue is λ_1 with an initial grid of 1/8 quadratic adaptive mesh and error curve

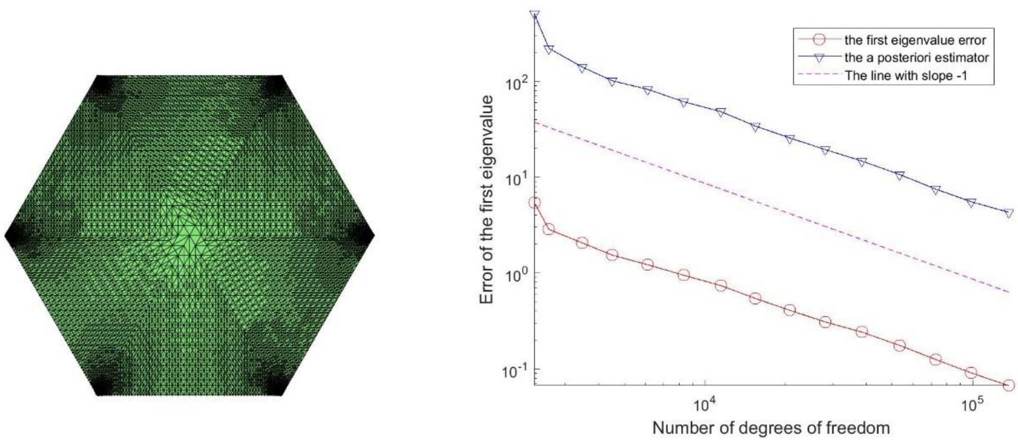
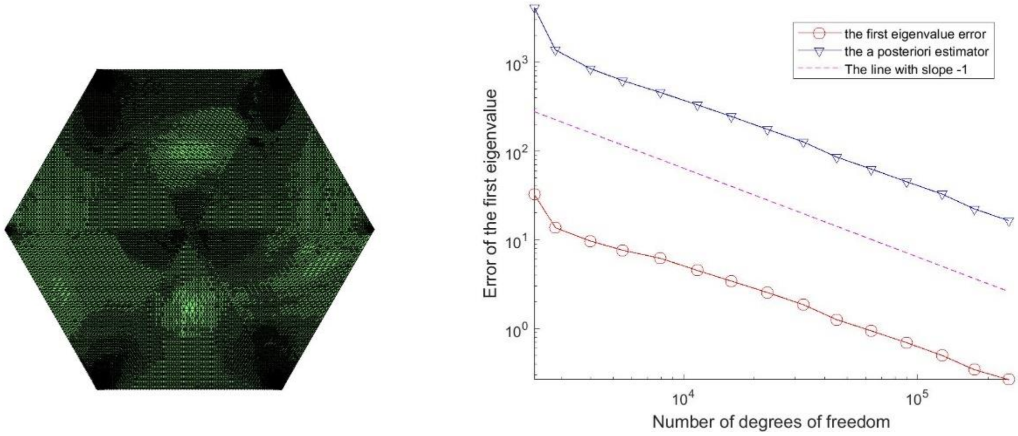


Figure 3: On the test domain Ω_H, the reference eigenvalue is λ_2 with an initial grid of 1/8 quadratic adaptive mesh and error curve



The numerical solution results of eigenvalues obtained through adaptive calculation are listed in table 1 to Table 3, and the figure illustrates the adaptive mesh and error curve. From Figure1 to Figure3, we can see that the error curve of the numerical solution for eigenvalues is approximately parallel to the error index curve to a certain extent, the error curve of the quadratic discontinuity element exhibits a nearly parallel relationship with a line having a slope of -1. It shows that all the posterior error indexes of numerical eigenvalues are reliable and effective. The results show that the adaptive algorithm can achieve the optimal convergence order, you can also see from the error curve that for the same degree of freedom(dof^{-1}), the approximation obtained by the adaptive algorithm is more accurate than that obtained by the uniform grid calculation.

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