International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 12 Issue 3 || May.-June. 2024 || PP-06-20

Novel Results on Generalized Hilfer Fractional for Nonlinear Volterra-Fredholm Systems

Shakir M. Atshan[1], Ahmed A. Hamoud[2]
[1]*Thi-Qar Directorates of Education, Department of Mathematics, Ministry of Education, Iraq.*
[2]*Department of Mathematics, Taiz University, Taiz P.O. Box 6803, Yemen.*
E-mail: s11h32@yahoo.com; ahmed.hamoud@taiz.edu.ye

Date of Submission: 01-05-2024 Date of acceptance: 12-05-2024

The existence and uniqueness of solutions for a novel modelling of fractional boundary value problem (FBVP) of generalized Hilfer fractional Volterra-Fredholm integro-differential equations (HFV-FIDEs) are examined. This novel approach is based on the Krasnosel'skii, and Laray-Schauder alternative algorithms of fixed points, and Banach's contraction mapping principle which are applied to get the intended outcomes in finite dimensional space. In order to bolster the validity of the theoretical claims, we elaborate on the cases that are shown at the conclusion of the text. Finally, an example demonstrate the validity of the obtained theoretical results. These studies are important in the context of the development of the theory of fractional nonlinear Volterra-Fredholm systems.

Keywords: Generalized Hilfer fractional derivative, Volterra-Fredholm system, fixed point techniques, existence and uniqueness.

I. Introduction

The modelling of many physical processes that occur in the practical and social sciences benefits greatly from the use of fractional-order differential operators, as demonstrated by the recent development of fractional calculus. We direct the reader to the books [2, 3, 15, 17, 19] for information on the theory and applications of fractional differential equations, and to the literature [4] for a current description of non-linear and non-local FBVP. There are several different fractional integral operators, which are used to define fractional derivative operators. See the text [2] for specific examples, such as Riemann-Liouville, Erdelyi-Kober, Caputo, Hilfer, Hadamard, fractional derivatives (FD), etc.

The mathematical modelling of systems and processes in the domains of electro-chemistry, signal processing, physics, control theory, chemistry, electromagnetic, aerodynamics, viscoelasticity, porous media, and so on gives rise to fractional differential equations (FDEs) in many scientific and engineering disciplines (see [1, 5, 7, 8, 9, 10, 11] and the references therein). In recent years, FDEs have seen a substantial theoretical development (see [6, 14, 15, 20] and the references therein).

*s11h32@yahoo.com; ahmed.hamoud@taiz.edu.ye

The notion of generalised and Caputo type generalised fractional integrals and derivatives of a function with regard to another function was covered by the writers of a recent paper [8]. In [9], an initial value problem for nonlinear implicit FDEs of the generalized Ψ-Hilfer type was studied. The attractivity of solutions for a problem using the Hilfer FD was covered by the writers in [10].

It is essential to note that the Hilfer FD is most broadly represented by the $(\Im,\Upsilon)$-Hilfer fractional derivative, which is specialised to $(\Im,\Upsilon)$-Caputo, $(\Im,\Upsilon)$-Riemann-Liouville, $\Im$-Hadamard-Hilfer and $\Im$-Katugampola-Hilfer FD operators [12, 13] for further information). While the Hilfer fractional diffusion advection equation with the power-law starting condition is addressed in [19], [18] provides an example of a physical system model using the Hilfer FD.

Now, let's focus on the literature that addresses starting and boundary value issues of the $(\Im,\Upsilon)$-Hilfer FD kind. [6, 11] introduced the $(\Im,\Upsilon)$-Riemann-Liouville fractional integral and derivative operators, respectively. In [12], a $(\Im,\Upsilon)$-Hilfer FD operator initial value problem was examined. The authors of [13] looked at if the following $(\Im,\Upsilon)$-Hilfer multi-point nonlocal fractional has any solutions.

$$\begin{cases} {}^{\Im,H}D^{\alpha,\beta;\Upsilon}\Phi(t)=\Xi_1(t,\Phi(t)), \quad t\in(\Lambda,\theta], \\ \Phi(\Lambda)=0, \quad \Phi(\theta)=\sum_{i=1}^{n}\mu_i\int_{\Lambda}^{\eta_i}\Upsilon'(\hbar)\Phi(\hbar)d\hbar+\sum_{j=1}^{m}\zeta_j{}^{\Im}I^{\phi_j;}(z_j), \end{cases}$$

where $\Lambda,\theta\in\mathbb{R},\Lambda<\theta, {}^{\Im,H}D^{\alpha,\beta;\Upsilon}$ denotes the $(\Im,\Upsilon)$-Hilfer FD operator of order $\alpha, 1<\alpha<2, 0\leqslant\beta\leqslant1, k>0, \Xi_1\in C([\Lambda,\theta]\times\mathbb{R},\mathbb{R}), {}^{\Im}I^{\phi_j;\Upsilon}$ is the $(\Im,\Upsilon)$-Riemann-Liouville fractional integral operator of order $\phi_j>0, \mu_i,\zeta_j\in\mathbb{R}$, and $\Lambda<\eta_i, z_j<\theta, i=1,2,\ldots,n, j=1,2,\ldots,m$

In [17], a coupled system of FDEs with multi-point non-local boundary conditions and $(\Im,\Upsilon)$-Hilfer FD operators was covered. We direct the reader to the paper [12] for some features of $(\Im,\Upsilon)$-Hilfer FDEs. Nonetheless, it has been noted that there is a dearth of research on systems of $(\Im,\Upsilon)$-Hilfer FDEs and that more needs to be done in this area.

In this research, we explore a $(\Im,\Upsilon)$-HFV-FIDE equipped with nonlocal ordinary and FBVP provided by inspired by the work presented in [17]:

$$\begin{cases} {}^{\Im,H}D^{\alpha,\beta;\Upsilon}\ \Phi(t)=\Xi_1\left(t,\Phi(t),\int_{\Lambda}^{t}\Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_{\Lambda}^{\theta}\Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right), \quad t\in(\Lambda,\theta], \\ {}^{\Im,H}D^{p,q;\Upsilon}\vartheta(t)=\Xi_2\left(t,\vartheta(t),\int_{\Lambda}^{t}\Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_{\Lambda}^{\theta}\Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right), \quad t\in(\Lambda,\theta], \\ \Phi(\Lambda)=0, \quad \Phi(\theta)=\sum_{i=1}^{n}\mu_i\int_{\Lambda}^{\eta_i}\Upsilon'(\hbar)\vartheta(\hbar)d\hbar+\sum_{j=1}^{m}\zeta_j{}^{\Im}I^{\phi_j,\Upsilon}(z_j), \\ \vartheta(\Lambda)=0, \quad \vartheta(\theta)=\sum_{l=1}^{v}r_l\int_{\Lambda}^{\gamma_l}\Upsilon'(\hbar)\Phi(\hbar)d\hbar+\sum_{u=1}^{\lambda}\delta_u{}^{\Im}I^{\epsilon_u,\Upsilon}\Phi(\xi_u), \end{cases} \tag{1}$$

where ${}^{\Im,H}D^{\alpha,\beta;\Upsilon}, {}^{\Im,H}D^{p,q;\Upsilon}$ denote the $(\Im,\Upsilon)-$ Hilfer FD operator of orders $\alpha,p, 1<\alpha,p<2$ and parameters $\beta,q, 0\leqslant\beta,q\leqslant1$, respectively, $\Xi_1,\Xi_2:[\Lambda,\theta]\times\mathbb{R}^3\to\mathbb{R}$ are continuous functions, ${}^{\Im}I^{\phi_j,\Upsilon}, {}^{\Im}I^{\epsilon_u,\Upsilon}$ are the $(\Im,\Upsilon)$-Riemann-Liouville fractional integrals of order $\phi_j,\epsilon_u>0$, respectively, $\mu_i,\zeta_j,r_l,\xi_u\in\mathbb{R}$, and $\Lambda<\eta_i,z_j,\gamma_l,\xi_u<\theta$, $i=1,2,\ldots,n, j=1,2,\ldots,m, l=1,2,\ldots,v, u=1,2,\ldots,\lambda$.

The Boyd-Wong fixed point theorem for nonlinear contractions and Banach's contraction mapping principle are utilised to demonstrate the uniqueness results for system (1) under various conditions, while the Krasnosel'skii fxed-point theorem and an alternative of Leray-Schauder are used to establish the existence results for the given problem. The key findings are illustrated by the examples that have been created. It is important to note that, as specific circumstances, our results provide numerous additional conclusions that are novel in the current environment.

II. Preliminaries

Let's start this part with a few definitions and lemmas that are relevant to our research.

Definition 2..1 *[11] Assume that $\Upsilon : [\Lambda, \theta] \to \mathbb{R}$ is a function with $\Upsilon'(t) \neq 0$ for all $t \in [\Lambda, \theta]$ and increasing on $[\Lambda, \theta]$. Then, the $(\Im, \Upsilon)$-Riemann-Liouville fractional integral of order $\alpha > 0 (\alpha \in \mathbb{R})$ of a function $w \in L^1([\Lambda, \theta], \mathbb{R})$ is defined by*

$$^{\Im}I_{a+}^{\alpha;\Upsilon} w(t) = \frac{1}{\Im \Gamma_{\Im}(\alpha)} \int_a^t \Upsilon'(u)(\Upsilon(t) - \Upsilon(u))^{\frac{\alpha}{\Im}-1} w(u)du, \Im > 0,$$

where $\Gamma_{\Im}(t) = \int_0^{\infty} \hbar^{t-1} e^{-\frac{\hbar^{\Im}}{\Im}} d\hbar$.

Definition 2..2 *[12] Assume that $\Upsilon \in C^n([\Lambda, \theta], \mathbb{R}), \Upsilon'(t) \neq 0, t \in [\Lambda, \theta], w \in C^n([\Lambda, \theta], \mathbb{R})$, and $\alpha, \Im \in \mathbb{R}^+ = (0, \infty), \beta \in [0, 1]$. Then, the $(\Im, \Upsilon)$-Hilfer FD of the function w of order α and type β is given as*

$$^{\Im,H}D^{\alpha,\beta;\Upsilon} w(t) = {}^{\Im}I_{a+}^{\beta(n\Im-\alpha);\Upsilon} \left(\frac{\Im}{\Upsilon'(t)} \frac{d}{dt} \right)^n {}^{\Im}I_{a+}^{(1-\beta)(n\Im-\alpha);\Upsilon} w(t), \quad n = \left\lceil \frac{\alpha}{\Im} \right\rceil.$$

Lemma 2..1 *Let $\Lambda < \theta, \Im > 0, 1 < \alpha, p \leqslant 2, \beta, q \in [0, 1], t_{\Im} = \alpha + \beta(2\Im - \alpha)$, $w_{\Im} = p + q(2\Im - p)$ and $\Phi, \vartheta \in AC^2([\Lambda, \theta], \mathbb{R})$ and $h_1, h_2 \in AC([\Lambda, \theta], \mathbb{R})$. If*

$$\mathbb{A} := A_1 A_4 - A_2 A_3 \neq 0, \tag{2}$$

then the unique solution of $(\Im, \Upsilon)$-Hilfer system:

$$\begin{cases} ^{\Im,H}D^{\alpha,\beta;\Upsilon}\Phi(t) = h_1(t), \quad t \in (\Lambda, \theta], \\ ^{\Im,H}D^{p,q;\Upsilon}\vartheta(t) = h_2(t), \quad t \in (\Lambda, \theta], \\ \Phi(\Lambda) = 0, \quad \Phi(\theta) = \sum_{i=1}^n \mu_i \int_{\Lambda}^{\eta_i} \Upsilon'(\hbar)\vartheta(\hbar)d\hbar + \sum_{j=1}^m \zeta_j {}^{\Im}I^{\phi_j;\Upsilon}\vartheta(z_j), \\ \vartheta(\Lambda) = 0, \quad \vartheta(\theta) = \sum_{l=1}^v r_l \int_a^{\gamma_l} \Upsilon'(\hbar)\Phi(\hbar)d\hbar + \sum_{u=1}^{\lambda} \delta_u {}^{\Im}I^{\epsilon_u,\Upsilon}\Phi(\xi_u), \end{cases} \tag{3}$$

is given by

$$\begin{aligned} \Phi(t) = {}^{\Im}I^{\alpha;\Upsilon}h_1(t) + \frac{(\Upsilon(t) - \Upsilon(\Lambda))^{\frac{t_{\Im}}{\Im}} - 1}{\mathbb{A}\Gamma_{\Im}(t_{\Im})} \Bigg[A_4 \Bigg(\sum_{i=1}^n \mu_i \int_{\Lambda}^{\eta_i} \Upsilon'(\hbar) {}^{\Im}I^{p;\Upsilon}h_2(\hbar)d\hbar \\ + \sum_{j=1}^m \zeta_j {}^{\Im}I^{\phi_j+p;\Upsilon}h_2(z_j) - {}^{\Im}I^{\alpha;\Upsilon}h_1(\theta) \Bigg) + A_2 \Bigg(\sum_{l=1}^v r_l \int_{\Lambda}^{\gamma_l} \Upsilon'(\hbar) {}^{\Im}I^{\alpha;\Upsilon}h_1(\hbar)d\hbar \\ + \sum_{u=1}^{\lambda} \delta_u {}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}h_1(\xi_u) - {}^{\Im}I^{p;\Upsilon}h_2(\theta) \Bigg) \Bigg], \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathcal{W}(t) = {}^{\Im}I^{p;\Upsilon}h_1(t) + \frac{(\Upsilon(t) - \Upsilon(\Lambda))^{\frac{w_{\Im}}{\Im}} - 1}{\mathbb{A}\Gamma_{\Im}(w_{\Im})} \Bigg[A_1 \Bigg(\sum_{l=1}^v r_l \int_{\Lambda}^{\gamma_l} \Upsilon'(\hbar) {}^{\Im}I^{\alpha;\Upsilon}h_1(\hbar)d\hbar \\ + \sum_{u=1}^{\lambda} \delta_u {}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}h_1(\xi_u) - {}^{\Im}I^{p;\Upsilon}h_2(\theta) \Bigg) + A_3 \Bigg(\sum_{i=1}^n \mu_i \int_{\Lambda}^{\eta_i} \Upsilon'(\hbar) {}^{\Im}I^{p;\Upsilon}h_2(\hbar)d\hbar \\ + \sum_{j=1}^m \zeta_j {}^{\Im}I^{\phi_j+p;\Upsilon}h_2(z_j) - {}^{\Im}I^{\alpha;\Upsilon}h_1(\theta) \Bigg) \Bigg], \end{aligned} \tag{5}$$

where

$$\begin{aligned} A_1 &= \frac{(\Upsilon(\theta) - \Upsilon(\Lambda))^{\frac{t_{\Im}}{\Im}-1}}{\Gamma_{\Im}(t_{\Im})}, \\ A_2 &= \Im \sum_{i=1}^n \mu_i \frac{(\Upsilon(\eta_i) - \Upsilon(\Lambda))^{\frac{w_{\Im}}{\Im}}}{\Gamma_{\Im}(w_{\Im} + \Im)} + \sum_{j=1}^m \zeta_j \frac{(\Upsilon(z_j) - \Upsilon(\Lambda))^{\frac{w_{\Im}+\phi_j}{\Im}} - 1}{\Gamma(w_{\Im} + \phi_j)}, \\ A_3 &= \Im \sum_{l=1}^v r_l \frac{(\Upsilon(\gamma_l) - \Upsilon(\Lambda))^{\frac{t_{\Im}}{\Im}}}{\Gamma_{\Im}(t_{\Im} + \Im)} + \sum_{u=1}^{\lambda} \delta_u \frac{(\Upsilon(\xi_u) - \Upsilon(\Lambda))^{\frac{t_{\Im}+\epsilon_u}{\Im}} - 1}{\Gamma(t_{\Im} + \epsilon_u)} \\ A_4 &= \frac{(\Upsilon(\theta) - \Upsilon(\Lambda))^{\frac{w_{\Im}}{\Im}} - 1}{\Gamma_{\Im}(w_{\Im})}. \end{aligned} \tag{6}$$

Proof. Let (Φ, ϑ) be a solution of the system (3). Applying the operators ${}^{\Im}I^{\alpha;\Upsilon}$ and ${}^{\Im}I^{p;\Upsilon}$, as well, on both sides of the first and second Eqs. in (3), we have

$$\Phi(t) = {}^{\Im}I^{\alpha;\Upsilon}h_1(t) + c_0\frac{(\Upsilon(t)-\Upsilon(\Lambda))^{\frac{t_\Im}{\Im}-1}}{\Gamma_\Im(t_\Im)} + c_1\frac{(\Upsilon(t)-\Upsilon(\Lambda))^{\frac{t_\Im}{\Im}-2}}{\Gamma_\Im(t_\Im-\Im)} \tag{7}$$

and

$$\vartheta(t) = {}^{\Im}I^{p;}\Upsilon h_2(t) + d_0\frac{(\Upsilon(t)-\Upsilon(\Lambda))^{\frac{w_\Im}{\Im}}-1}{\Gamma_\Im(w_\Im)} + d_1\frac{(\Upsilon(t)-\Upsilon(\Lambda))^{\frac{w_\Im}{\Im}}-2}{\Gamma_\Im(w_\Im-\Im)}, \tag{8}$$

where c_0, c_1, d_0 and d_1 are unknown arbitrary constants.

By the conditions $\Phi(\Lambda) = 0$ and $\vartheta(\Lambda) = 0$ in (7) and (8), we get $c_1 = 0$ and $d_1 = 0$, since $\frac{t_\Im}{\Im} - 2 < 0, \frac{w_\Im}{\Im} - 2 < 0$. Taking (7) and (8) with $c_1 = 0$ and $d_1 = 0$ in the boundary conditions:

$$\Phi(\theta) = \sum_{i=1}^{n}\mu_i\int_\Lambda^{\eta_i}\Upsilon'(\hbar)\vartheta(\hbar)d\hbar + \sum_{j=1}^{m}\zeta_j^{\Im}I^{\phi_j;\Upsilon}\vartheta(z_j)$$

and

$$\vartheta(\theta) = \sum_{l=1}^{v}r_l\int_a^{\gamma_l}\Upsilon'(h)\Phi(h)dh + \sum_{u=1}^{\lambda}\delta_u{}^{\Im}I^{\epsilon_u;\Upsilon}\Phi(\xi_u)$$

Combined with (6) notation, we get

$$\begin{aligned} A_1c_0 - A_2d_0 &= \sum_{i=1}^{n}\mu_i\int_\Lambda^{\eta_i}\Upsilon'(\hbar){}^{\Im}I^{p;\Upsilon}h_2(\hbar)d\hbar + \sum_{j=1}^{m}\zeta_j{}^{\Im}I^{\phi_j+p;\Upsilon}h_2(z_j) - {}^{\Im}I^{\alpha;\Upsilon}h_1(\theta), \\ -A_3c_0 + A_4d_0 &= \sum_{l=1}^{v}r_l\int_\Lambda^{\gamma_l}\Upsilon'(\hbar){}^{\Im}I^{\alpha;\Upsilon}h_1(\hbar)d\hbar + \sum_{u=1}^{\lambda}\delta_u{}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}h_1(\xi_u) - {}^{\Im}I^{p;};h_2(\theta). \end{aligned} \tag{9}$$

Solving system (9) for c_0 and d_0, we get

$$\begin{aligned} c_0 =& \frac{1}{\mathbb{A}}\Bigg[A_4\left(\sum_{i=1}^{n}\mu_i\int_\Lambda^{\eta_i}\Upsilon'(\hbar){}^{\Im}I^{p;\Upsilon}h_2(\hbar)d\hbar + \sum_{j=1}^{m}\zeta_j{}^{\Im}I^{\phi_j+p;\Upsilon}h_2(z_j) - {}^{\Im}I^{\alpha;\Upsilon}h_1(\theta)\right) \\ &+A_2\left(\sum_{l=1}^{v}r_l\int_\Lambda^{\gamma_l}\Upsilon'(h){}^{\Im}I^{\alpha;\Upsilon}h_1(h)dh + \sum_{u=1}^{\lambda}\delta_u{}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}h_1(\xi_u) - {}^{\Im}I^{p;\Upsilon}h_2(\theta)\right)\Bigg], \\ d_0 =& \frac{1}{\mathbb{A}}\Bigg[A_1\left(\sum_{l=1}^{v}r_l\int_\Lambda^{\gamma_l}\Upsilon'(\hbar){}^{\Im}I^{\alpha;\Upsilon}h_1(\hbar)d\hbar + \sum_{u=1}^{\lambda}\delta_u{}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}h_1(\xi_u) - {}^{\Im}I^{p;\Upsilon}h_2(\theta)\right) \\ &+A_3\left(\sum_{i=1}^{n}\mu_i\int_\Lambda^{\eta_i}\Upsilon'(\hbar){}^{\Im}I^{p;\Upsilon}h_2(\hbar)d\hbar + \sum_{j=1}^{m}\zeta_j{}^{\Im}I^{\phi_j+p;\Upsilon}h_2(z_j) - {}^{\Im}I^{\alpha;\Upsilon}h_1(\theta)\right)\Bigg]. \end{aligned}$$

Substituting the values of c_0, c_1 and d_0, d_1 in (7) and (8), respectively, we get the solution (4) and (5). The contrary is easily shown by direct computation.

Definition 2..3 *Assume that W be a Banach space. Then, the operator $G : W \to W$ is called a contraction nonlinear if there exists a non-decreasing and continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Omega(0) = 0$ and $\Omega(t) < t, \forall\ t > 0$ with the property:*

$$\|Gx - Gy\| \leqslant \Omega(\|x-y\|), \quad \forall x, y \in W.$$

III. Existence and Uniqueness Results

Let be the space in Banach algebra of all continuous functions Φ, ϑ from $[\Lambda, \theta]$ to $\mathbb{R}$ endowed with the norm $\|\Phi\| = \max\{|\Phi(t)|, t \in [\Lambda, \theta]\}$ and $\|\vartheta\| = \max\{|\vartheta(t)|, t \in [\Lambda, \theta]\}$ be define by $\mathbb{X} = C([\Lambda, \theta], \mathbb{R})$. Thus, the product space $(\mathbb{X} \times \mathbb{X}, \|(\Phi, \vartheta)\|)$ is a Banach space with norm $\|(\Phi, \vartheta)\| = \|\Phi\| + \|\vartheta\|$.

Lemma 2..1 notwithstanding, we create a map $\mathbb{W} : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ associated with model (1) by

$$\mathbb{W}(\Phi, \vartheta)(t) = \begin{pmatrix} \mathbb{W}_1(\Phi, \vartheta)(t) \\ \mathbb{W}_2(\Phi, \vartheta)(t) \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}
\mathbb{W}_1(\Phi, \vartheta)(t) =& {}^{\Im}I^{\alpha;\Upsilon}\Xi_1\left(t, \Phi(t), \int_\Lambda^t \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right) + \frac{(\Upsilon(t) - \Upsilon(\Lambda))^{\frac{t_\Im}{\Im}-1}}{\mathbb{A}\Gamma_\Im(t_\Im)} \\
&\times \left[A_4\left(\sum_{i=1}^n \mu_i \int_\Lambda^{\eta_i} \Upsilon'(\hbar){}^{\Im}I^{p;\Upsilon}\Xi_2\left(\hbar, \vartheta(\hbar), \int_\Lambda^\hbar \Upsilon'(x)\Phi(x)dx, \int_\Lambda^\theta \Upsilon_1'(x)\Phi(x)dx\right)d\hbar\right.\right. \\
&+ \sum_{j=1}^m \zeta_j {}^{\Im}I^{\phi_j+p;\Upsilon}\Xi_2\left(z_j, \vartheta(z_j), \int_\Lambda^{z_j} \Upsilon'(\hbar)\Phi(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right) \\
&\left. - {}^{\Im}I^{\alpha;\Upsilon}\Xi_1\left(\theta, \Phi(\theta), \int_\Lambda^\theta \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right) \\
&+ A_2\left(\sum_{l=1}^v r_l \int_\Lambda^{\gamma_l} \Upsilon'(\hbar){}^{\Im}I^{\alpha;\Upsilon}\Xi_1\left(\hbar, \Phi(\hbar), \int_\Lambda^\hbar \Upsilon'(x)\vartheta(x)dx, \int_\Lambda^\theta \Upsilon_1'(x)\vartheta(x)dx\right)d\hbar\right. \\
&+ \sum_{u=1}^\lambda \delta_u {}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}\Xi_1\left(\xi_u, \Phi(\xi_u), \int_\Lambda^{\xi_u} \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right) \\
&\left.\left. - {}^{\Im}I^{p;\Upsilon}\Xi_2\left(\theta, \vartheta(\theta), \int_\Lambda^\theta \Upsilon'(\hbar)\Phi(\hbar)d\hbar \int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right)\right],
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\mathbb{W}_2(\Phi, \vartheta)(t) =& {}^{\Im}I^{p;\Upsilon}\Xi_2\left(t, \vartheta(t), \int_\Lambda^t \Upsilon'(\hbar)\Phi(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right) + \frac{(\Upsilon(t) - \Upsilon(\Lambda))^{\frac{w_\Im}{\Im}} - 1}{\mathbb{A}\Gamma_\Im(w_\Im)} \\
&\times \left[A_1\left(\sum_{l=1}^v r_l \int_\Lambda^{\gamma_l} \Upsilon'(\hbar){}^{\Im}I^{\alpha;\Upsilon}\Xi_1\left(\hbar, \Phi(\hbar), \int_\Lambda^\hbar \Upsilon'(x)\vartheta(x)dx, \int_\Lambda^\theta \Upsilon_1'(x)\vartheta(x)dx\right)d\hbar\right.\right. \\
&+ \sum_{u=1}^\lambda \delta_u {}^{\Im}I^{\epsilon_u+\alpha;\Upsilon}\Xi_1\left(\xi_u, \Phi(\xi_u), \int_\Lambda^{\xi_u} \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right) \\
&\left. - {}^{\Im}I^{p;\Upsilon}\Xi_2\left(\theta, \vartheta(\theta), \int_\Lambda^\theta \Upsilon'(\hbar)\Phi(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right) \\
&+ A_3\left(\sum_{i=1}^n \mu_i \int_\Lambda^{\eta_i} \Upsilon'(\hbar){}^{\Im}I^{p;\Upsilon}\Xi_2\left(\hbar, \vartheta(\hbar), \int_\Lambda^\hbar \Upsilon'(x)\Phi(x)dxd\hbar, \int_\Lambda^\theta \Upsilon_1'(x)\Phi(x)dx\right)d\hbar\right. \\
&+ \sum_{j=1}^m \zeta_j {}^{\Im}I^{\phi_j+p;\Upsilon}\Xi_2\left(z_j, \vartheta(z_j), \int_\Lambda^{z_j} \Upsilon'(\hbar)\Phi(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)
\end{aligned} \tag{12}$$

$$\left.\left. - {}^{\Im}I^{\alpha;\Upsilon}\Xi_1\left(\theta, \Phi(\theta), \int_\Lambda^\theta \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right)\right]. \tag{13}$$

To facilitate calculation, we employ the following notation:

$$
\begin{aligned}
Q_0 &= \Upsilon(\theta) - \Upsilon(\Lambda) \\
Q_1 &= \frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)} + \frac{Q_0^{\frac{t_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\left[|A_4|\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right. \\
&\quad \left. + |A_2|\left(\Im\sum_{l=1}^{v}|\gamma_l|\frac{(\Upsilon(\gamma_l)-\Upsilon(\Lambda))^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)} + \sum_{u=1}^{\lambda}|\delta_u|\frac{(\Upsilon(\xi_u)-\Upsilon(\Lambda))^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im(\epsilon_u+\alpha+\Im)}\right)\right], \\
Q_2 &= \frac{Q_0^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\left[|A_2|\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}\right. \\
&\quad \left. + |A_4|\left(\Im\sum_{i=1}^{n}|\mu_i|\frac{(\Upsilon(\eta_i)-\Upsilon(\Lambda))^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)} + \sum_{j=1}^{m}|\zeta_j|\frac{(\Upsilon(z_j)-\Upsilon(\Lambda))^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im(\phi_j+p+\Im)}\right)\right], \\
Q_3 &= \frac{Q_0^{\frac{w_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(w_\Im)}\left[|A_3|\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right. \\
&\quad \left. + |A_1|\left(\Im\sum_{l=1}^{v}|r_l|\frac{(\Upsilon(\gamma_l)-\Upsilon(\Lambda))^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)} + \sum_{u=1}^{\lambda}|\delta_u|\frac{(\Upsilon(\xi_u)-\Upsilon(\Lambda))^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im(\epsilon_u+\alpha+\Im)}\right)\right], \\
Q_4 &= \frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)} + \frac{Q_0^{\frac{w_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im(w_\Im)}\left[|A_1|\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}\right. \\
&\quad \left. + |A_3|\left(\Im\sum_{i=1}^{n}|\mu_i|\frac{(\Upsilon(\eta_i)-\Upsilon(\Lambda))^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)} + \sum_{j=1}^{m}|\zeta_j|\frac{(\Upsilon(z_j)-\Upsilon(\Lambda))^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im(\phi_j+p+\Im)}\right)\right], \\
Q_1^* &= Q_1 - \frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}, Q_4^* = Q_4 - \frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}.
\end{aligned} \tag{14}
$$

1. Uniqueness Results

First, we use Banach's contraction mapping principle [20] to demonstrate the uniqueness of solutions for the nonlocal $(\Im, \Upsilon)$-HFV-FIDE (1).

Theorem 3..1 *Suppose that $\Xi_1, \Xi_2 : [\Lambda, \theta] \times \mathbb{R}^3 \to \mathbb{R}$ satisfy the following assumption:*
(H_1) there exist constants $m_i, n_i, i = 1, 2, 3$, such that, $\forall\ t \in [\Lambda, \theta]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2, 3$,

$$|\Xi_1(t, x_1, x_2, x_3) - \Xi_1(t, y_1, y_2, y_3)| \leqslant m_1|x_1 - y_1| + m_2|x_2 - y_2| + m_3|x_3 - y_3|,$$

and

$$|\Xi_2(t, x_1, x_2, x_3) - \Xi_2(t, y_1, y_2, y_3)| \leqslant n_1|x_1 - y_1| + n_2|x_2 - y_2| + n_3|x_3 - y_3|.$$

Then, the $(\Im, \Upsilon)$-HFV-FIDE (1) has a unique solution on $[\Lambda, \theta]$, satisfy

$$Q_0[(Q_1 + Q_3)(m_1 + m_2 + m_3) + (Q_2 + Q_4)(n_1 + n_2 + n_3)] < 1, \tag{15}$$

where $Q_i, i = 0, 1, 2, 3, 4$, are given in (14).

Proof. First, let us demonstrate that $\mathbb{W}B_r \subset B_r$, where the map $\mathbb{W}$ is given in (10) and $B_r = \{(\Phi,\vartheta) \in \mathbb{X}\times\mathbb{X} : \|(\Phi,\vartheta)\| \leqslant r\}$, with

$$r \geqslant \frac{(Q_1+Q_3)\,N+(Q_2+Q_4)\,N_1}{1-[(Q_1+Q_3)\,(m_1+Q_0(m_2+m_3))+(Q_2+Q_4)\,(n_1+Q_0(n_2+n_3))]}$$

$\sup_{t\in[\Lambda,\theta]}\Xi_1(t,0,0,0)=N<\infty, \sup_{t\in[\Lambda,\theta]}\Xi_2(t,0,0,0)=N_1<\infty$. Thus, we get

$$\begin{aligned}
&\left|\Xi_1\left(t,\Phi(t),\int_\Lambda^t \Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right|\\
&\leqslant \left|\Xi_1\left(t,\Phi(t),\int_\Lambda^t \Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)-\Xi_1(t,0,0,0)\right|+|\Xi_1(t,0,0,0)|\\
&\leqslant m_1\|\Phi\|+m_2\left|\int_\Lambda^t \Upsilon'(\hbar)\vartheta(\hbar)d\hbar\right|+m_3\left|\int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right|+N\\
&\leqslant m_1\|\Phi\|+(m_2+m_3)Q_0\|\boldsymbol{\vartheta}\|+N
\end{aligned}$$

and similarly

$$\left|\Xi_2\left(t,\vartheta(t),\int_\Lambda^t \Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right|\leqslant n_1\|\boldsymbol{\vartheta}\|+(n_2+n_3)Q_0\|\Phi\|+N_1.$$

Next, for $(\Phi,\vartheta)\in B_r$, we have

$$\begin{aligned}
&|\mathbb{W}_1(\Phi,\infty)(t)|\\
&\leqslant {}^{\Im}I^{\alpha;\Upsilon}\left|\Xi_1\left(t,\Phi(t),\int_\Lambda^t \Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right|+\frac{(\Upsilon(t)-\Upsilon(\Lambda))^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\\
&\times\left[|A_4|\left(\sum_{i=1}^n|\mu_i|\int_\Lambda^{\eta_i}\Upsilon'(\hbar)^{\Im}I^{p;\Upsilon}\left|\Xi_2\left(\hbar,\vartheta(\hbar),\int_\Lambda^\hbar \Upsilon'(x)\Phi(x)dx,\int_\Lambda^\theta \Upsilon_1'(x)\Phi(x)dx\right)\right|d\hbar\right.\right.\\
&+\sum_{j=1}^m|\zeta_j|\,{}^{\Im}I^{\phi_j+p;\Upsilon}\left|\Xi_2\left(z_j,\vartheta(z_j),\int_\Lambda^{z_j}\Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right|\\
&\left.+{}^{\Im}I^{\alpha;\Upsilon}\left|\Xi_1\left(\theta,\Phi(\theta),\int_\Lambda^\theta \Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right|\right)\\
&+|A_2|\left(\sum_{l=1}^v|r_l|\int_\Lambda^{\gamma_l}\Upsilon'(\hbar)^{\Im}I^{\alpha;\Upsilon}\left|\Xi_1\left(\hbar,\Phi(\hbar),\int_\Lambda^\hbar \Upsilon'(x)\vartheta(x)dx,\int_\Lambda^\theta \Upsilon_1'(x)\vartheta(x)dx\right)\right|d\hbar\right.\\
&+\sum_{u=1}^\lambda|\delta_u|^{\Im}I^{\epsilon_u+\alpha;\Upsilon}\left|\Xi_1\left(\xi_u,\Phi(\xi_u),\int_\Lambda^\theta \Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^{\xi_u}\Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right|\\
&\left.\left.+{}^{\Im}I^{p;\Upsilon}\left|\Xi_2\left(\theta,\boldsymbol{\vartheta}(\theta),\int_\Lambda^\theta \Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_\Lambda^\theta \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right|\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\left[m_1\|\Phi\|+(m_2+m_3)Q_0\|\vartheta\|+N\right] \\
&+\frac{Q_0^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im\left(t_\Im\right)}\left[|A_4|\left(\Im\left[n_1\|\boldsymbol{\vartheta}\|+(n_2+n_3)Q_0\|\Phi\|+N_1\right]\sum_{i=1}^{n}|\mu_i|\frac{\left(\Upsilon\left(\eta_i\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}\right.\right. \\
&+\left[n_1\|\boldsymbol{\vartheta}\|+(n_2+n_3)Q_0\|\Phi\|+N_1\right]\sum_{j=1}^{m}|\zeta_j|\frac{\left(\Upsilon\left(z_j\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im\left(\phi_j+p+\Im\right)} \\
&\left.+\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\left[m_1\|\Phi\|+(m_2+m_3)Q_0\|\vartheta\|+N\right]\right) \\
&+|A_2|\left(\Im\left[m_1\|\Phi\|+(m_2+m_3)Q_0\|\vartheta\|+N\right]\sum_{l=1}^{v}|r_l|\frac{\left(\Upsilon\left(\gamma_l\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}\right. \\
&+\left[m_1\|\Phi\|+(m_2+m_3)Q_0\|\infty\|+N\right]\sum_{u=1}^{\lambda}|\delta_u|\frac{\left(\Upsilon\left(\xi_u\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im\left(\epsilon_u+\alpha+\Im\right)} \\
&\left.\left.+\left[n_1\|\vartheta\|+(n_2+n_3)Q_0\|\Phi\|+N_1\right]\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}\right)\right] \\
&\leqslant\left\{\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}+\frac{Q_0^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im\left(t_\Im\right)}\left[|A_4|\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right.\right. \\
&\left.\left.+|A_2|\left(\Im\sum_{l=1}^{\nu}|\gamma_l|\frac{\left(\Upsilon\left(\gamma_l\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{\left(\Upsilon\left(\xi_u\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im\left(\epsilon_u+\alpha+\Im\right)}\right)\right]\right\} \\
&\quad\times\left[m_1\|\Phi\|+m_2Q_0\|\vartheta\|+N\right] \\
&\quad+\left\{\frac{Q_0^{\frac{t_\Im}{\Im}-1}}{|A|\Gamma_\Im(t_\Im)}\left[|A_2|\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}+|A_4|\left(\Im\sum_{i=1}^{n}|\mu_i|\frac{\left(\Upsilon\left(\eta_i\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}\right.\right.\right. \\
&\quad\left.\left.\left.+\sum_{j=1}^{m}|\zeta_j|\frac{\left(\Upsilon\left(z_j\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im\left(\phi_j+p+\Im\right)}\right)\right]\right\}\left[n_1\|\boldsymbol{\vartheta}\|+(n_2+n_3)Q_0\|\Phi\|+N_1\right] \\
&=Q_1\left[m_1\|\Phi\|+Q_0(m_2+m_3)\|\boldsymbol{\vartheta}\|+N\right]+Q_2\left[n_1\|\boldsymbol{\vartheta}\|+Q_0(n_2+n_3)\|\Phi\|+N_1\right] \\
&=\left(Q_1m_1+Q_0Q_2(n_2+n_3)\right)\|\Phi\|+\left(Q_2n_1+Q_0Q_1(m_2+m_3)\right)\|\boldsymbol{\vartheta}\|+Q_1N+Q_2N_1 \\
&\leqslant\left(Q_1m_1+Q_0Q_2(n_2+n_3)+Q_2n_1+Q_0Q_1(m_2+m_3)\right)r+Q_1N+Q_2N_1.
\end{aligned}
$$

In a similar vein, one finds that

$$
|\mathbb{W}_2(\Phi,\vartheta)(t)|\leqslant\left(Q_3m_1+Q_0Q_4(n_2+n_3)+Q_4n_1+Q_0Q_3(m_2+m_3)\right)r+Q_3N+Q_4N_1.
$$

Considering the aforementioned discrepancies, we arrive at

$$
\begin{aligned}
\|\mathbb{W}(\Phi,\vartheta)\|=&\|\mathbb{W}_1(\Phi,\boldsymbol{\vartheta})\|+\|\mathbb{W}_2(\Phi,\vartheta)\| \\
\leqslant&\left[\left(Q_1+Q_3\right)\left(m_1+Q_0(m_2+m_3)\right)+\left(Q_2+Q_4\right)\left(n_1+Q_0(n_2+n_3)\right)\right]r \\
&+\left(Q_1+Q_3\right)N+\left(Q_2+Q_4\right)N_1\leqslant r,
\end{aligned}
$$

it suggests that $\mathbb{W}B_r \subset B_r$. We now demonstrate that the operator $\mathbb{W}$ given in (10) is a contraction. For (Φ_2, ϑ_2), $(\Phi_1, \vartheta_1) \in \mathbb{X} \times \mathbb{X}$, and $t \in [\Lambda, \theta]$, we have

$$
\begin{aligned}
&\left|\mathbb{W}_1\left(\Phi_2, \vartheta_2\right)(t)-\mathbb{W}_1\left(\Phi_1, \vartheta_1\right)(t)\right| \\
\leqslant & {}^{\Im}I^{\alpha;\Upsilon}\Big|\Xi_1\left(t, \Phi_2(t), \int_\Lambda^t \Upsilon'(\hbar)\vartheta_2(\hbar)d\hbar, \int_\Lambda^{b_1} \Upsilon_1'(\hbar)\vartheta_2(\hbar)d\hbar\right) \\
& -\Xi_1\left(t, \Phi_1(t), \int_\Lambda^t \Upsilon'(\hbar)\vartheta_1(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\vartheta_1(\hbar)d\hbar\right)\Big| \\
& +\frac{\left(\Upsilon(t)-\Upsilon(\Lambda)\right)^{t_\Im}-1}{|A|\Gamma_\Im\left(t_\Im\right)} \\
& \times\Bigg[|A_4|\left(\sum_{i=1}^n |\mu_i| \int_\Lambda^{\eta_i} \Upsilon'(\hbar)^{\Im}I^{p;\Upsilon} \mid \Xi_2\left(\hbar, \vartheta_2(\hbar), \int_\Lambda^{\hbar} \Upsilon'(x)\Phi_2(x)dx, \int_\Lambda^{\theta} \Upsilon_1'(x)\Phi_2(x)dx\right)\right. \\
& -\Xi_2\left(\hbar, \vartheta_1(\hbar), \int_\Lambda^{\hbar} \Upsilon'(x)\Phi_1(x)dx, \int_\Lambda^{\theta} \Upsilon_1'(x)\Phi_1(x)dx\right) \mid d\hbar \\
& +{}^{\Im}I^{\alpha;\Upsilon}\Big|\Xi_1\left(\theta, \Phi_2(\theta), \int_\Lambda^{\theta} \Upsilon'(\hbar)\vartheta_2(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\vartheta_2(\hbar)d\hbar\right) \\
& \left.-\Xi_1\left(\theta, \Phi_1(\theta), \int_\Lambda^{\theta} \Upsilon'(\hbar)\vartheta_1(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\vartheta_1(\hbar)d\hbar\right)\Big|\right) \\
& +|A_2|\left(\sum_{l=1}^{v} |r_l| \int_\Lambda^{\gamma_l} \Upsilon'(\hbar)^{\Im}I^{\alpha;\Upsilon} \mid \Xi_1\left(\hbar, \Phi_2(\hbar), \int_\Lambda^{\hbar} \Upsilon'(x)\vartheta_2(x)dx, \int_\Lambda^{\theta} \Upsilon_1'(x)\vartheta_2(x)dx\right)\right. \\
& -\Xi_1\left(\hbar, \Phi_1(\hbar), \int_\Lambda^{\hbar} \Upsilon'(x)\vartheta_1(x)dx, \int_\Lambda^{\theta} \Upsilon_1'(x)\vartheta_1(x)dx\right) \mid d\hbar \\
& +\sum_{u=1}^{\lambda} |\delta_u| {}^{\Im}I^{\epsilon_u+\alpha;\Upsilon} \mid \Xi_1\left(\xi_u, \Phi_2(\xi_u), \int_\Lambda^{\xi_u} \Upsilon'(\hbar)\vartheta_2(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\vartheta_2(\hbar)d\hbar\right) \\
& -\Xi_1\left(\xi_u, \Phi_1(\xi_u), \int_\Lambda^{\xi_u} \Upsilon'(\hbar)\vartheta_1(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\vartheta_1(\hbar)d\hbar\right) \mid \\
& +{}^{\Im}I^{p;\Upsilon}\Big|\Xi_2\left(\theta, \vartheta_2(\theta), \int_\Lambda^{\theta} \Upsilon'(\hbar)\Phi_2(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\Phi_2(\hbar)d\hbar\right) \\
& \left.-\Xi_2\left(\theta, \vartheta_1(\theta), \int_\Lambda^{\theta} \Upsilon'(\hbar)\Phi_1(\hbar)d\hbar, \int_\Lambda^{\theta} \Upsilon_1'(\hbar)\Phi_1(\hbar)d\hbar\right)\Big|\right)\Bigg] \\
\leqslant & \Bigg\{\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}+\frac{Q_0^{\frac{t_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im\left(t_\Im\right)}\Bigg[|A_4| \frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)} \\
& +|A_2|\left(\Im \sum_{l=1}^{v} |\gamma_l| \frac{\left(\Upsilon(\gamma_l)-\Upsilon(\Lambda)\right)^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}+\sum_{u=1}^{\lambda} |\delta_u| \frac{\left(\Upsilon(\xi_u)-\Upsilon(\Lambda)\right)^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im\left(\epsilon_u+\alpha+\Im\right)}\right)\Bigg]\Bigg\} \\
& \times Q_0\left(m_1\left\|\Phi_2-\Phi_1\right\|+\left(m_2+m_3\right)\left\|\vartheta_2-\vartheta_1\right\|\right) \\
& +\Bigg\{\frac{Q_0^{\frac{t_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\Bigg[|A_2| \frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}+|A_4|\left(\Im \sum_{i=1}^{n} |\mu_i| \frac{\left(\Upsilon(\eta_i)-\Upsilon(\Lambda)\right)^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}\right. \\
& \left.+\sum_{j=1}^{m} |\zeta_j| \frac{\left(\Upsilon(z_j)-\Upsilon(\Lambda)\right)^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im\left(\phi_j+p+\Im\right)}\right)\Bigg]\Bigg\} Q_0\left(n_1\left\|\vartheta_2-\vartheta_1\right\|+\left(n_2+n_3\right)\left\|\Phi_2-\Phi_1\right\|\right) \\
= & Q_1\left(m_1\left\|\Phi_2-\Phi_1\right\|+Q_0(m_2+m_3)\left\|\vartheta_2-\vartheta_1\right\|\right)+Q_2\left(n_1\left\|\vartheta_2-\vartheta_1\right\|+Q_0(n_2+n_3)\left\|\Phi_2-\Phi_1\right\|\right) \\
= & \left(Q_1 m_1+Q_0 Q_2(n_2+n_3)\right)\left\|\Phi_2-\Phi_1\right\|+\left(Q_2 n_1+Q_0 Q_1 m_1\right)\left\|\vartheta_2-\vartheta_1\right\|,
\end{aligned}
$$

which yields

$$\begin{aligned}&\|\mathbb{W}_1(\Phi_2,\vartheta_2)-\mathbb{W}_1(\Phi_1,\vartheta_1)\|\\&\leqslant(Q_1m_1+Q_0Q_2(n_2+n_3)+Q_2n_1+Q_0Q_1(m_2+m_3))\left[\|\Phi_2-\Phi_1\|+\|\vartheta_2-\vartheta_1\|\right].\quad(16)\end{aligned}$$

In a similar vein, one finds that

$$\begin{aligned}&\|\mathbb{W}_2(\Phi_2,\vartheta_2)-\mathbb{W}_2(\Phi_1,\vartheta_1)\|\\&\leqslant(Q_3m_1+Q_0Q_4(n_2+n_3)+Q_4n_1+Q_0Q_3(m_2+m_3))\left[\|\Phi_2-\Phi_1\|+\|\vartheta_2-\vartheta_1\|\right].\quad(17)\end{aligned}$$

From (16) and (17), we get

$$\begin{aligned}&\|\mathbb{W}(\Phi_2,\vartheta_2)-\mathbb{W}(\Phi_1,\vartheta_1)\|\\\leqslant&Q_0\left[(Q_1+Q_3)(m_1+m_2+m_3)+(Q_2+Q_4)(n_1+n_2+n_3)\right](\|\Phi_2-\Phi_1\|+\|\vartheta_2-\vartheta_1\|)\end{aligned}$$

It demonstrates that the operator $\mathbb{W}$ is a contraction in light of the requirement (15). According to Banach's contraction mapping principle, there is only one fixed point for the operator $\mathbb{W}$. Consequently, the $(\Im,\Upsilon)-$HFV-FIDE (1) has a unique solution.

2. Existence Results

The conditions guaranteeing the existence of solutions for the $(\Im,\Upsilon)$-HFV-FIDE (1) are given in this subsection. The Leray-Schauder alternative is the foundation of our first result [21].

Theorem 3..2 *Let:*

$(H_2)\,\Xi_1,\Xi_2:[\Lambda,\theta]\times\mathbb{R}^3\to\mathbb{R}$ *be continuous functions and* $\exists\ \Im_i,v_i\geqslant0, i=1,2,3,$ *and* $\Im_0,v_0>0$*, such that,* $\forall w_i\in\mathbb{R},(i=1,2,3)$,

$$\begin{aligned}|\Xi_1(t,w_1,w_2,w_3)|&\leqslant\Im_0+\Im_1|w_1|+\Im_2|w_2|+\Im_3|w_3|\\|\Xi_2(t,w_1,w_2,w_3)|&\leqslant v_0+v_1|w_1|+v_2|w_2|+v_3|w_3|\end{aligned}$$

$(H_3)\,(Q_1+Q_3)\Im_1+Q_0(Q_2+Q_4)(v_2+v_3)<1$ *and* $Q_0(Q_1+Q_3)(\Im_2+\Im_3)+(Q_2+Q_4)v_1<1$, $Q_i, i=0,...,4,$ *are defined in* (14).

Then, the $(\Im,\Upsilon)$*-HFV-FIDE* (1) *has at least one solution on* $[\Lambda,\theta]$.

Proof. Apparently, $\mathbb{W}$ is continuous by the continuity of Ξ_1 and Ξ_2. Now, we show that the map $\mathbb{W}$ is completely continuous. For that, let $\mathbb{O}_r=\{(\Phi,\boldsymbol{\vartheta})\in\mathbb{X}\times\mathbb{X}:\|(\Phi,\boldsymbol{\vartheta})\|\leqslant r\}\subset\mathbb{X}\times\mathbb{X}$ be a bounded set. Then, by (H_2), we get

$$\begin{aligned}|\Xi_1(t,w_1,w_2,w_3)|&\leqslant\Im_0+\Im_1|w_1|+\Im_2|w_2|+\Im_3|w_3|\\&\leqslant\Im_0+\Im_1(\|\Phi\|+\|\vartheta\|)+Q_0(\Im_2+\Im_3)(\|\Phi\|+\|\vartheta\|)\\&\leqslant\Im_0+(\Im_1+(\Im_2+\Im_3)Q_0)\,r=L_1.\end{aligned}$$

Likewise, we have that $|\Xi_2(t,w_1,w_2,w_3)|\leqslant v_0+(v_1+(v_2+v_3)Q_0)\,r=L_2$. Thus, for any $(\Phi,\infty)\in\mathbb{O}$, we have

$$\begin{aligned}&|\mathbb{W}_1(\Phi,\vartheta)(t)|\\\leqslant&\left\{\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}+\frac{Q_0^{\frac{t_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\left[|A_4|\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right.\right.\\&\left.\left.+|A_2|\left(\Im\sum_{l=1}^{v}|\gamma_l|\frac{(\Upsilon(\gamma_l)-\Upsilon(\Lambda))^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\Upsilon(\xi_u)-\Upsilon(\Lambda))^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im(\epsilon_u+\alpha+\Im)}\right)\right]\right\}\mathbb{L}_1\\&+\left\{\frac{Q_0^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\left[|A_2|\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}\right.\right.\\&\left.\left.+|A_4|\left(\Im\sum_{i=1}^{n}|\mu_i|\frac{(\Upsilon(\eta_i)-\Upsilon(\Lambda))^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}+\sum_{j=1}^{m}|\zeta_j|\frac{(\Upsilon(z_j)-\Upsilon(\Lambda))^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im(\phi_j+p+\Im)}\right)\right]\right\}\mathbb{L}_2\\=&Q_1\mathbb{L}_1+Q_2\mathbb{L}_2,\end{aligned}$$

and hence

$$\|\mathbb{W}_1(\Phi,\boldsymbol{\vartheta})\| \leqslant Q_1\mathbb{L}_1 + Q_2\mathbb{L}_2$$

$$\|\mathbb{W}_2(\Phi,\vartheta)\| \leqslant Q_3\mathbb{L}_1 + Q_4\mathbb{L}_2.$$

Then, we get

$$\|\mathbb{W}(\Phi,\boldsymbol{\vartheta})\| = \|\mathbb{W}_1(\Phi,\boldsymbol{\vartheta})\| + \|\mathbb{W}_2(\Phi,\boldsymbol{\vartheta})\| \leqslant (Q_1+Q_3)\,\mathbb{L}_1 + (Q_2+Q_4)\,\mathbb{L}_2.$$

Then $\mathbb{W}$ is uniformly bounded. Now, let $t_1,t_2\in[\Lambda,\theta]$ with $t_1<t_2$. Thus, we get

$$\begin{aligned}
&\left|\mathbb{W}_1\left(\Phi\left(t_2\right),\vartheta\left(t_2\right)\right)-\mathbb{W}_1\left(\Phi\left(t_1\right),\vartheta\left(t_1\right)\right)\right| \\
&\leqslant \frac{1}{\Gamma_{\Im}(\alpha)}\left|\int_a^{t_1}\Upsilon'(\hbar)\left[\left(\Upsilon\left(t_2\right)-\Upsilon(\hbar)\right)^{\frac{\alpha}{\Im}-1}-\left(\Upsilon\left(t_1\right)-\Upsilon(\hbar)\right)^{\frac{\alpha}{\Im}-1}\right]\right. \\
&\times \Xi_1\left(\hbar,\Phi(\hbar),\int_\Lambda^\hbar\Upsilon'(r)\vartheta(r)dr,\int_\Lambda^\theta\Upsilon_1'(r)\vartheta(r)dr\right)d\hbar \\
&+\int_{t_1}^{t_2}\Upsilon'(\hbar)\left(\Upsilon\left(t_2\right)-\Upsilon(\hbar)\right)^{\frac{\alpha}{\Im}-1}\Xi_1\left(\hbar,\Phi(\hbar),\int_\Lambda^\hbar\Upsilon'(r)\vartheta(r)dr,\int_\Lambda^\theta\Upsilon_1'(r)\vartheta(r)dr\right)d\hbar \\
&+\frac{\left(\Upsilon\left(t_2\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{t_\Im}{\Im}-1}-\left(\Upsilon\left(t_1\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im\left(t_\Im\right)} \\
&\times\left[|A_4|\left(\sum_{i=1}^n|\mu_i|\int_\Lambda^{\eta_i}\Upsilon'(h)^{\Im}I^{p;\Upsilon}\left|\Xi_2\left(\hbar,\vartheta(\hbar),\int_\Lambda^\hbar\Upsilon'(r)\Phi(r)dr,\int_\Lambda^\theta\Upsilon_1'(r)\Phi(r)dr\right)\right|d\hbar\right.\right. \\
&+\sum_{j=1}^m|\zeta_j|\,{}^{\Im}I^{\phi_j+p;\Upsilon}\left|\Xi_2\left(z_j,\vartheta\left(z_j\right),\int_\Lambda^{z_j}\Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_\Lambda^\theta\Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right| \\
&\left.+{}^{\Im}I^{\alpha;\Upsilon}\left|\Xi_1\left(\theta,\Phi\left(\theta\right),\int_\Lambda^\theta\Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^\theta\Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right|\right) \\
&+|A_2|\left(\sum_{l=1}^v|r_l|\int_\Lambda^{\gamma_l}\Upsilon'(\hbar)^{\Im}I^{\alpha;\Upsilon}\left|\Xi_1\left(\hbar,\Phi(\hbar),\int_\Lambda^\hbar\Upsilon'(r)\vartheta(r)dr,\int_\Lambda^\theta\Upsilon_1'(r)\vartheta(r)dr\right)\right|d\hbar\right. \\
&+\sum_{u=1}^\lambda|\delta_u|^{\Im}\,I^{\epsilon_u+\alpha;\Upsilon}\left|\Xi_1\left(\xi_u,\Phi\left(\xi_u\right),\int_\Lambda^{\xi_u}\Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_\Lambda^\theta\Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)\right| \\
&\left.\left.+{}^{\Im}I^{p;\Upsilon}\left|\Xi_2\left(\theta,\vartheta\left(\theta\right),\int_\Lambda^\theta\Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_\Lambda^\theta\Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)\right|\right)\right] \\
&\leqslant\frac{\mathbb{L}_1}{\Gamma_\Im(\alpha+\Im)}\left[2\left(\Upsilon\left(t_2\right)-\Upsilon\left(t_1\right)\right)^{\frac{\alpha}{\Im}}+\left|\left(\Upsilon\left(t_2\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\alpha}{\Im}}-\left(\Upsilon\left(t_1\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\alpha}{\Im}}\right|\right] \\
&+\frac{\left(\Upsilon\left(t_2\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{t_\Im}{\Im}-1}-\left(\Upsilon\left(t_1\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{t_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im\left(t_\Im\right)}\left[|A_4|\left(\Im\mathbb{L}_2\sum_{i=1}^n|\mu_i|\frac{\left(\Upsilon\left(\eta_i\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}\right.\right. \\
&\left.+\mathbb{L}_2\sum_{j=1}^m|\zeta_j|\frac{\left(\Upsilon\left(z_j\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\phi_j}{\Im}+p}}{\Gamma_\Im\left(\phi_j+p+\Im\right)}+\mathbb{L}_1\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right) \\
&+|A_2|\left(\Im\mathbb{L}_1\sum_{l=1}^v|r_l|\frac{\left(\Upsilon\left(\gamma_l\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}+\mathbb{L}_1\sum_{u=1}^\lambda|\delta_u|\frac{\left(\Upsilon\left(\xi_u\right)-\Upsilon\left(\Lambda\right)\right)^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im\left(\epsilon_u+\alpha+\Im\right)}Q_0^{\frac{p}{\Im}}\Gamma_\Im(p+\Im)\right) \\
&\longrightarrow 0 \quad \text{as} \quad t_2-t_1\to 0.
\end{aligned}$$

Then, $\mathbb{W}_1(\Phi,\vartheta)$ is equicontinuous. The equicontinuity of the map $\mathbb{W}_2(\Phi,\vartheta)$ can be established in an analogous manner. As a result, we conclude that the map $\mathbb{W}(\Phi,\vartheta)$ is completely continuous.

In order to apply the Laray-Schauder alternative's conclusion, we must demonstrate that the set $\mathbb{E}=\{(\Phi,\vartheta)\in\mathbb{X}\times\mathbb{X}:(\Phi,\vartheta)=\lambda\mathbb{W}(\Phi,\vartheta),0\leqslant\lambda\leqslant1\}$ is bounded. Assume that $(\Phi,\vartheta)\in\mathbb{E}$, thus $(\Phi,\vartheta)=\lambda\mathbb{W}(\Phi,\vartheta)$. For any $t\in[\Lambda,\theta]$, we get

$$\Phi(t)=\lambda\mathbb{W}_1(\Phi,\vartheta)(t),\quad \vartheta(t)=\lambda\mathbb{W}_2(\Phi,\vartheta)(t).$$

Thus, we get

$$\begin{aligned}|\Phi(t)|\leqslant&\left\{\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}+\frac{Q_0^{\frac{t_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\left[|A_4|\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right.\right.\\&\left.\left.+|A_2|\left(\Im\sum_{l=1}^{v}|\gamma_l|\frac{(\Upsilon(\gamma_l)-\Upsilon(\Lambda))^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\Upsilon(\xi_u)-\Upsilon(\Lambda))^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im(\epsilon_u+\alpha+\Im)}\right)\right]\right\}\\&\times(\Im_0+\Im_1\|\Phi\|+Q_0\Im_2\|\vartheta\|)\\&+\left\{\frac{Q_0^{\frac{t_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(t_\Im)}\left[|A_2|\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}+|A_4|\left(\Im\sum_{i=1}^{n}|\mu_i|\frac{(\Upsilon(\eta_i)-\Upsilon(\Lambda))^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}\right.\right.\right.\\&\left.\left.\left.+\sum_{j=1}^{m}|\zeta_j|\frac{(\Upsilon(z_j)-\Upsilon(\Lambda))^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im(\phi_j+p+\Im)}\right)\right]\right\}(v_0+v_1\|\vartheta\|+Q_0v_2\|\Phi\|),\end{aligned}$$

and

$$\begin{aligned}|\vartheta(t)|\leqslant&\left\{\frac{Q_0^{\frac{w_\Im}{\Im}-1}}{|\mathbb{A}|\Gamma_\Im(w_\Im)}\left[A_3\frac{Q_0^{\frac{\alpha}{\Im}}}{\Gamma_\Im(\alpha+\Im)}\right.\right.\\&\left.\left.+|A_1|\left(\Im\sum_{l=1}^{v}|r_l|\frac{(\Upsilon(\gamma_l)-\Upsilon(\Lambda))^{\frac{\alpha}{\Im}+1}}{\Gamma_\Im(\alpha+2\Im)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\Upsilon(\xi_u)-\Upsilon(\Lambda))^{\frac{\epsilon_u+\alpha}{\Im}}}{\Gamma_\Im(\epsilon_u+\alpha+\Im)}\right)\right]\right\}\\&\times(\Im_0+\Im_1\|\Phi\|+Q_0(\Im_2+\Im_3)\|\vartheta\|)\\&+\left\{\frac{Q_0^{p}}{\Gamma_\Im(p+\Im)}+\frac{Q_0^{\frac{w_\Im}{\Im}}-1}{|\mathbb{A}|\Gamma_\Im(w_\Im)}\left[|A_1|\frac{Q_0^{\frac{p}{\Im}}}{\Gamma_\Im(p+\Im)}\right.\right.\\&\left.\left.+|A_3|\left(\Im\sum_{i=1}^{n}|\mu_i|\frac{(\Upsilon(\eta_i)-\Upsilon(\Lambda))^{\frac{p}{\Im}+1}}{\Gamma_\Im(p+2\Im)}+\sum_{j=1}^{m}|\zeta_j|\frac{(\Upsilon(z_j)-\Upsilon(\Lambda))^{\frac{\phi_j+p}{\Im}}}{\Gamma_\Im(\phi_j+p+\Im)}\right)\right]\right\}\\&\times(v_0+v_1\|\vartheta\|+Q_0(v_2+v_3)\|\Phi\|).\end{aligned}$$

In consequence, we get

$$\|\Phi\|\leqslant Q_1(\Im_0+\Im_1\|\Phi\|+Q_0(\Im_2+\Im_3)\|\vartheta\|)+Q_2(v_0+v_1\|\vartheta\|+Q_0(v_2+v_3)\|\Phi\|),$$

and

$$\|\vartheta\|\leqslant Q_3(\Im_0+\Im_1\|\Phi\|+Q_0(\Im_2+\Im_3)\|\vartheta\|)+Q_4(v_0+v_1\|\vartheta\|+Q_0(v_2+v_3)\|\Phi\|),$$

which imply that

$$\begin{aligned}\|\Phi\|+\|\vartheta\|\leqslant&(Q_1+Q_3)\Im_0+(Q_2+Q_4)v_0+[(Q_1+Q_3)\Im_1+Q_0(Q_2+Q_4)(v_2+v_3)]\|\Phi\|\\&+[Q_0(Q_1+Q_3)(\Im_2+\Im_3)+(Q_2+Q_4)v_1]\|\vartheta\|.\end{aligned}$$

Thus, for any $t\in[\Lambda,\theta]$, we have

$$\|(\Phi,\vartheta)\|\leqslant\frac{(Q_1+Q_3)\Im_0+(Q_2+Q_4)v_0}{M_0},$$

where M_0 is defined by

$$M_0=\min\{1-[(Q_1+Q_3)\Im_1+Q_0(Q_2+Q_4)(v_2+v_3)],1-[Q_0(Q_1+Q_3)(\Im_2+\Im_3)+(Q_2+Q_4)v_1]\}$$

Then, $\mathbb{E}$ is bounded. Then, the $(\Im,\Upsilon)$-HFV-FIDE (1) has at least one solution on $[\Lambda,\theta]$.

4. Illustrative Example

Example 1. Consider the following $(\Im, \Upsilon)$-HFV-FIDE:

$$\begin{aligned}
&{}^{\frac{5}{4},H}D^{\frac{4}{3},\frac{2}{3};t^2e^{-t/2}}\Phi(t) = \Xi_1\left(t, \Phi(t), \int_{\frac{2}{9}}^{t} \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_{\frac{2}{9}}^{\frac{14}{9}} \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right), \quad t \in \left(\frac{2}{9}, \frac{14}{9}\right], \\
&{}^{\frac{5}{4},H}D^{\frac{5}{3},\frac{1}{3};t^2e^{-t/2}}\vartheta(t) = \Xi_2\left(t, \vartheta(t), \int_{\frac{2}{9}}^{t} \Upsilon'(\hbar)\Phi(\hbar)d\hbar, \int_{\frac{2}{9}}^{\frac{14}{9}} \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right), \quad t \in \left(\frac{2}{9}, \frac{14}{9}\right], \\
&x\left(\frac{2}{9}\right) = 0, \quad x\left(\frac{14}{9}\right) = \frac{1}{25}\int_{2/9}^{5/9}\left[-\frac{\left(\hbar^2e^{-\hbar/2}\right)}{2} + 2\hbar e^{-\hbar/2}\right]\vartheta(\hbar)d\hbar \\
&+ \frac{2}{35}\int_{\frac{2}{9}}^{8/9}\left[-\frac{\left(\hbar^2e^{-\hbar/2}\right)}{2} + 2\hbar e^{-\hbar/2}\right]\vartheta(\hbar)d\hbar + \frac{3}{45}\int_{2/9}^{11/9}\left[-\frac{\left(\hbar^2e^{-\hbar/2}\right)}{2} + 2\hbar e^{-\hbar/2}\right]\vartheta(\hbar)d\hbar \\
&+ \frac{2}{54}{}^{\Im}I^{3/4;t^2e^{-t/2}}\vartheta(1/3) + \frac{3}{65}{}^{\Im}I^{4/5;t^2e^{-t/2}}\vartheta(13/9), \\
&y\left(\frac{2}{9}\right) = 0, \quad y\left(\frac{14}{7}\right) = \frac{3}{37}\int_{2/9}^{4/9}\left[-\frac{\left(\hbar^2e^{-\hbar/2}\right)}{2} + 2\hbar e^{-\hbar/2}\right]\Phi(\hbar)d\hbar \\
&+ \frac{4}{49}\int_{2/9}^{7/9}\left[-\frac{\left(\hbar^2e^{-\hbar/2}\right)}{2} + 2\hbar e^{-\hbar/2}\right]\vartheta(\hbar)d\hbar + \frac{5}{61}\int_{2/9}^{13/9}\left[-\frac{\left(\hbar^2e^{-\hbar/2}\right)}{2} + 2\hbar e^{-\hbar/2}\right]\vartheta(\hbar)d\hbar \\
&+ \frac{4}{51}\Im I^{5/6;t^2e^{-t/2}}\vartheta(2/3) + \frac{5}{62}{}^{\Im}I^{6/7;t^2e^{-t/2}}\vartheta(1).
\end{aligned} \tag{18}$$

Here, $\Im = 5/4, \alpha = 4/3, \beta = 2/3, p = 5/3, q = 1/3, \Upsilon(t) = t^2e^{-t/2}, a = 2/9, b = 14/9, n = 3, m = 2, v = 3, \lambda = 2, \mu_1 = 1/25, \mu_2 = 2/35, \mu_3 = 3/45, \eta_1 = 5/9, \eta_2 = 8/9, \eta_3 = 11/9, \zeta_1 = 2/54, \zeta_2 = 3/65, z_1 = 1/3, z_2 = 13/9, \phi_1 = 3/4, \phi_2 = 4/5, r_1 = 3/37, r_2 = 4/49, r_3 = 5/61, \gamma_1 = 4/9, \gamma_2 = 7/9, \gamma_3 = 13/9, \delta_1 = 4/51, \delta_2 = 5/62, \xi_1 = 2/3, \xi_2 = 1, \epsilon_1 = 5/6, \epsilon_2 = 6/7$.

Using the given values, we get $t_{\frac{5}{4}} = 19/9$, $w_{\frac{5}{4}} = 29/18, A_1 \approx 0.9894137797, A_2 \approx 0.0737375203, A_3 \approx 0.0833890039, A_4 \approx 1.062508916, \mathbb{A}^4 \approx 1.045112065, Q_1 \approx 1.067507459, Q_1 \approx 1.649053094, Q_2 \approx 0.0681355524, Q_3 \approx 0.1370135866, Q_4 \approx 1.425499425, Q_1^* \approx 0.8286319543, Q_4^* \approx 0.7156209862$.

(*) From Theorem 3..1, we have $\Xi_1, \Xi_2 : [\frac{2}{9}, \frac{14}{9}] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ as

$$\begin{aligned}
\Xi_1\left(t, \Phi(t), \int_{\frac{2}{9}}^{t} \Upsilon'(\hbar)\vartheta(\hbar)d\hbar, \int_{\frac{2}{9}}^{\frac{14}{9}} \Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right) =& \frac{e^{-(9t-2)^3}}{2(9t+1)^3}\left(\frac{8\Phi^2 + 9|\Phi|}{1+|\Phi|}\right) + \frac{1}{3}t + \frac{3}{5} \\
&+ \frac{1}{7}\sin^2\left|\int_{2/9}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2} + 2\hbar e^{-\hbar/2}\right)\vartheta(\hbar)d\hbar\right| \\
&+ \frac{1}{7}\cos^2\left|\int_{\frac{2}{9}}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2} + 2\hbar e^{-\hbar/2}\right)\vartheta(\hbar)d\hbar\right|,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\Xi_2\left(t, \vartheta(t), \int_{\frac{2}{9}}^{t} \Upsilon'(\hbar)\Phi(\hbar)d\hbar, \int_{\frac{2}{9}}^{\frac{14}{9}} \Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right) \\
&= \frac{\sin\left|\int_{\frac{2}{9}}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2} + 2\hbar e^{-\hbar/2}\right)\Phi(\hbar)d\hbar\right| + \cos\left|\int_{\frac{2}{9}}^{\frac{14}{9}}\left(\frac{-\hbar^2e^{-\hbar/2}}{2} + 2\hbar e^{-\hbar/2}\right)\Phi(\hbar)d\hbar\right|}{+\frac{\cos^2(t+\Phi/2)}{8}\left(\frac{\vartheta^2+|\vartheta|}{1+|\vartheta|}\right) + \frac{1}{4}t + \frac{1}{3}}.
\end{aligned} \tag{20}$$

Notice that $m_1 = 1/6, m_2 = m_3 = 1/7, n_1 = 1/9$ and $n_2 = n_3 = 1/8$ as

$$|\Xi_1(t, \Phi_1, \vartheta_1, V_1) - \Xi_1(t, \Phi_2, \vartheta_2, V_2)| \leqslant \frac{1}{6}|\Phi_1 - \Phi_2| + \frac{1}{7}|\vartheta_1 - \vartheta_2| + \frac{1}{7}|V_1 - V_2|,$$

and

$$|\Xi_2(t,\vartheta_1,\Phi_1,V_1)-\Xi_2(t,\vartheta_2,\Phi_2,V_2)|\leqslant\frac{1}{9}|\Phi_1-\Phi_2|+\frac{1}{8}|\vartheta_1-\vartheta_2|+\frac{1}{8}|V_1-V_2|,$$

for all $\Phi_1,\Phi_2,\vartheta_1,\vartheta_2\in\mathbb{R}$. Then, we get

$$Q_0\left[(Q_1+Q_3)(m_1+m_2)+(Q_2+Q_4)(n_1+n_2)\right]\approx 0.867<1.$$

Then, from Theorem 3..1, the $(\Im,\Upsilon)$-HFV-FIDE (18) with Ξ_1 and Ξ_2 given by (19) and (20) respectively, has a unique solution on $[\frac{2}{9},\frac{14}{9}]$.

(*) The functions $\Xi_1,\Xi_2:[\frac{2}{9},\frac{14}{9}]\times\mathbb{R}^3\longrightarrow\mathbb{R}$ are given by

$$\begin{aligned}\Xi_1\left(t,\Phi(t),\int_{\frac{2}{9}}^{t}\Upsilon'(\hbar)\vartheta(\hbar)d\hbar,\int_{\frac{2}{9}}^{\frac{14}{9}}\Upsilon_1'(\hbar)\vartheta(\hbar)d\hbar\right)&=\frac{1}{9t+1}+\frac{1}{3}|\Phi|+(\frac{2}{5}\cos^2\Phi+\frac{2}{5}\sin^2\Phi)\\&\times\frac{\left|\int_{\frac{2}{9}}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2}+2\hbar e^{-\hbar/2}\right)\vartheta(\hbar)d\hbar\right|^{150}}{1+\left|\int_{\frac{2}{9}}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2}+2\hbar e^{-\hbar/2}\right)\vartheta(\hbar)d\hbar\right|^{149}}\\\Xi_2\left(t,\vartheta(t),\int_{\frac{2}{9}}^{t}\Upsilon'(\hbar)\Phi(\hbar)d\hbar,\int_{\frac{2}{9}}^{\frac{14}{9}}\Upsilon_1'(\hbar)\Phi(\hbar)d\hbar\right)&=\frac{1}{18t+1}+\frac{1}{7}\vartheta+(\frac{1}{4}\tan^{-1}|\vartheta|+\frac{1}{4}\cot^{-1}|\vartheta|)\\&\times\left(\frac{\left|\int_{\frac{2}{9}}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2}+2\hbar e^{-\hbar/2}\right)\Phi(\hbar)d\hbar\right|^{17}}{1+\left|\int_{\frac{2}{9}}^{t}\left(\frac{-\hbar^2e^{-\hbar/2}}{2}+2\hbar e^{-\hbar/2}\right)\Phi(\hbar)d\hbar\right|^{16}}\right),\end{aligned}\tag{21}$$

and

$$|\Xi_1(t,\Phi,\vartheta,V)|\leqslant\frac{1}{3}+\frac{1}{3}|\Phi|+\frac{2}{5}|\vartheta|+\frac{2}{5}|V|\text{ and }|\Xi_2(t,\vartheta,\Phi,V)|\leqslant\frac{1}{5}+\frac{1}{4}|\Phi|+\frac{1}{7}|\vartheta|+\frac{1}{7}|V|.$$

Fixing $\Im_0=1/3,\Im_1=1/3,\Im_2=\Im_3=2/5,v_0=1/5,v_1=1/4,v_2=v_3=1/7$, we have $(Q_1+Q_3)\Im_1$
$+Q_0(Q_2+Q_4)v_1\approx 0.837<1$ and $Q_0(Q_1+Q_3)(\Im_2+\Im_3)+(Q_2+Q_4)(v_2+v_3)\approx 0.826<1$. Hence, from Theorem 3..2, the $(\Im,\Upsilon)-$HFV-FIDE (18) with Ξ_1 and Ξ_2 given in (21) respectively, has at least one solution on $[\frac{2}{9},\frac{14}{9}]$.

References

[1] Alesemi, M.; Iqbal, N.; Hamoud, A.A. The analysis of fractional-order proportional delay physical models via a novel transform, Complexity, 2022 (2022), 1-12.

[2] Ali, I.; Malik, N.A. Hilfer fractional advection-diffusion equations with power-law initial condition; a numerical study using variational iteration method. Comput. Math. Appl. 2014, 68, 1161-1179.

[3] Ahmad, B.; Ntouyas, S.K. Nonlocal Nonlinear Fractional-Order Boundary Value Problems; World Scientifc: Singapore, 2021.

[4] Deimling, K. Nonlinear Functional Analysis; Springer: New York, NY, USA, 1985.

[5] Diethelm, K. The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type; Springer Science, Business Media: Berlin/Heidelberg, Germany, 2010.

[6] Dorrego, G.A. An alternative defnition for the k-Riemann-Liouville fractional derivative. Appl. Math. Sci. 2015, 9, 481-491.

[7] Granas, A.; Dugundji, J. Fixed Point Theory; Springer: New York, NY, USA, 2005.

[8] Hilfer, R. Experimental evidence for fractional time evolution in glass forming materials. Chem. Phys. 2002, 284, 399-408.

[9] Hamoud, A.A.; Ghadle, K.P. The approximate solutions of fractional Volterra-Fredholm integro-differential equations by using analytical techniques, Probl. Anal. Issues Anal. 7, no. 25 (2018), 41-58.

[10] Hamoud, A.A.; Ghadle, K.P. Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations, Journal of Mathematical Modeling 6, no. 1 (2018), 91-104.

[11] Hamoud, A.A.; A. Azeez; Ghadle, K. A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations, Indonesian Journal of Electrical Engineering and Computer Science 11, no. 3 (2018): 1228-1235.

[12] Hamoud, A.A.; Ghadle, K. Homotopy analysis method for the first order fuzzy Volterra-Fredholm integro-differential equations, Indonesian Journal of Electrical Engineering and Computer Science 11, no. 3 (2018), 857-867.

[13] Bani Issa, M.; Hamoud, A.; Ghadle, K. Numerical solutions of fuzzy integro-differential equations of the second kind, Journal of Mathematics and Computer Science 23, no. 1 (2021), 67-74.

[14] Osman, M.; Yonghui, X.; Omer, A. O; Hamoud, A. On the fuzzy solution of linear-nonlinear partial differential equations, Mathematics 10, no. 13 (2022), 2295.

[15] Hamoud, A.; Bani Issa, M.SH.; Ghadle, K. Existence and uniqueness results for nonlinear Volterra-Fredholm integro-differential equations, Nonlinear Funct. Anal. Appl., 23(4) (2018), 797-805.

[16] Hamoud, A.; Khandagale, A.; Shah, R.; Ghadle, K. Some new results on Hadamard neutral fractional nonlinear Volterra-Fredholm integro-differential equations, Discontinuity, Nonlinearity, and Complexity, 12(4) (2023) 893-903.

[17] Ivaz, K.; Alasadi, I.; Hamoud, A. On the Hilfer fractional Volterra-Fredholm integro differential equations, IAENG International Journal of Applied Mathematics, 52(2) (2022), 426-431.

[18] Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations; North-Holland Mathematics Studies: Amsterdam, The Netherlands, 2006.

[19] Kwun, Y.C.; Farid, G.; Nazeer, W.; Ullah, S.; Kang, S.M. Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities. IEEE Access 2018, 6, 64946-64953.

[20] Kucche, K.D.; Mali, A.D. On the nonlinear $(\Im, \Upsilon)$-Hilfer fractional differential equations. Chaos Solitons Fractals 2021, 152, 111335.

[21] Mubeen, S.; Habibullah, G.M. k-fractional integrals and applications. Int. J. Contemp. Math. Sci. 2012, 7, 89-94.

[22] Magin, R.L. Fractional Calculus in Bioengineering; Begell House Publishers: Danbury, CT, USA, 2006.

[23] Nieto, J.J.; Alghanmi, M.; Ahmad, B.; Alsaedi, A.; Alharbi, B. On fractional integrals and derivatives of a function with respect to another function. Fractals 2023, 31, 2340066.

[24] Ntouyas, S.K.; Ahmad, B.; Tariboon, J. $(\Im, \Upsilon)$-Hilfer nonlocal integro-multi-point boundary value problems for fractional differential equations and inclusions. Mathematics 2022, 10, 2615.

[25] Pourhadi, E.; Saadati, R.; Nieto, J.J. On the attractivity of the solutions of a problem involving Hilfer fractional derivative via the measure of noncompactness. Fixed Point Theory 2023, 24, 343-366.

[26] Sousa, J.V.d.C.; de Oliveira, E.C. On the Υ-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. 2018, 60, 72-91.

[27] Salim, A.; Lazreg, J.E.; Ahmad, B.; Benchohra, M.; Nieto, J.J. A Study on k-Generalized Υ-Hilfer derivative operator. Vietnam J. Math. 2022, 52, 25-43.

[28] Samadi, A.; Ntouyas, S.K.; Tariboon, J. On a nonlocal coupled system of Hilfer generalized proportional fractional differential equations. Symmetry 2022, 14, 738.

[29] Samadi, A.; Ntouyas, S.K.; Tariboon, J. Nonlocal coupled system for $(\Im, \Upsilon)$-Hilfer fractional differential equations. Fractal Fract. 2022, 6, 234.

[30] Zaslavsky, G.M. Hamiltonian Chaos and Fractional Dynamics; Oxford University Press: Oxford, UK, 2005.