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Fuzzy Events of Theoretic Characteristics by Employing Pseudo Probabilistic Information Measure of Type Alpha

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ABSTRACT: *This study first examines a number of widely used fuzziness metrics for discrete fuzzy collections. We present a novel informative measure for discriminating between two fuzzy sets after examining a few of the current fuzzy set measures. We demonstrate that every class fulfills five well-known fuzziness measure axioms and show how a number of current measurements are related to these classes. The non-negative, monotony rising concave functions form the foundation of the multiplicative class. The only functions needed for the additive class are non-negative concave functions. There are also some connections between the many chances that now exist.*

KEYWORDS: *fuzzy entropy, Shannon measure, fuzzy set, information measure*

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I. INTRODUCTION

Zadeh's[31] research launched one of the most innovative and successful representational capacities of logic for quantifying fuzzy uncertainty. "Fuzzy Sets." Beginning with the concept of gradual membership, it served as the foundation for the logic of gradualness in properties as well as a brand-new, incredibly straightforward and useful uncertainty calculus known as "Possibility Theory," which handles the concepts of possibility and certainty as gradual modalities. Zadeh specifically focused on fuzzy sets' potential contributions to pattern categorization, information processing and communication, abstraction, and summarization when he proposed them. When the assertions that fuzzy sets were pertinent in these fields were initially made, in the early 1960s, they didn't seem to be supported. Although the assertions that fuzzy sets were pertinent in these fields didn't seem to be supported when they were initially made, namely in surpassing all predictions, the subsequent advancements in engineering and information sciences demonstrated that these intuitions were correct in the early 1960s. With the use of examples, Kapur has provided a clear explanation of fuzzy uncertainty. In general, fuzzy entropy is the quantitative representation of fuzziness in fuzzy sets, while Shannon's entropy quantifies the average uncertainty in bits related to the prediction of results in a random experiment. Some criteria that reflect our perception regarding the degree of fuzziness were given by De Luca and Termini .One of the key digital characteristics of fuzzy sets is fuzzy entropy, which is also crucial for system modeling and design. For instance, efficient structural parameters are rapidly obtained when generalized fuzzy entropy is employed as the learning criterion for neural networks. Stated differently, the guidance function of generalized fuzzy entropy in neural network system architecture is superior. Following Zadeh's introduction of the theory of fuzzy sets [31], other scholars began focusing on this area after it was well accepted by various quarters. De Luca and Termini [6] thus established a fuzzy entropy measure that corresponds to Shannon's [29] measure, keeping in mind the concept of fuzzy sets

The concept of fuzzy sets (FS) was first introduced by Zadeh (1965). Its primary purpose is to model non-statistical vague phenomena, and as a result, the theory of FS has drawn interest from a wide range of scientific fields, including but not limited to engineering, image processing, data mining, medical science, clustering, information technology, and statistical information theory. Fuzziness as a feature of uncertainty can be explained as the result of a particular decision regarding whether or not an event should be considered a member of a set; in these situations, the event is regarded as a fuzzy rather than a sharply defined collection of points (Zadeh, 1968). It is used as a measure of such fuzziness (Bhat and Baig, 2017). The information theory has been used extensively in fuzzy set theory due to the principle of entropy's ability to deal with a lack of information models. A number of FE measures have been discussed in the literature. De-Luca and Termini (1972) proposed the first entropy extension of Shannon's (1948) entropy; by proposing a nonprobability FE, they also defined the fundamental characteristics of the proposed FE as sharpness, maximality, resolutions, and symmetry. This served as a roadmap for the development of any new FE measures. Later, a number of authors introduced modified FE measures (Ohlan, 2015; Naidu et al, 2017; Zhang et al, 2012; Al-Sharhan et al. 2001; Bhatri and Pal, 1993; Kapur, 1997; Parkash and Sharma, 2002).

some application of entropy in goodness of fit tests for non-fuzzy datasetsare possible to be generalized to the fuzzy entropy. (Zamanzadeand Mahdizadeh,(2016, 2017); Zamanzade andArghami, 2011;Zamanzade, 2014) as other applications were generalized to the fuzzy sets and fuzzy entropyin different fields, such as sampling (see; Greenfield, 2012;Cetintav, 2016), goodness of fit (see; Grzegorzewski and Szymanowski, 2014;Eliason and Stryker, 2009), testing (see; Xie, 2010), and many other fields.

II. NEW RESULT

A fuzzy entropy measure that correlates to Shannon's measure was presented by De Luca and Termini [6] and is provided by

$$H(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\ln\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\ln\left(1-\mu_A(x_i)\right)\right]$$
(1)

Following this development, numerous fuzzy entropy measures were examined, described, and expanded upon by different writers. The fuzzy entropy metric that Kapur introduced is as follows:

$$K_{\alpha,\beta}(A) = \frac{1}{\beta-\alpha}\log\frac{\sum_{i=1}^{n}H_i^{\alpha}(A)}{\sum_{i=1}^{n}H_i^{\beta}(A)} \quad ,\alpha\neq\beta\,,\alpha>0,\beta>0$$

$$= \frac{1}{\beta-\alpha}\log\frac{\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\ln\mu_A^{\alpha}(x_i)+\left(1-\mu_A^{\alpha}(x_i)\right)\ln\left(1-\mu_A^{\alpha}(x_i)\right)\right]}{\sum_{i=1}^{n}\left[\mu_A^{\beta}(x_i)\ln\mu_A^{\beta}(x_i)+\left(1-\mu_A^{\beta}(x_i)\right)\ln\left(1-\mu_A^{\beta}(x_i)\right)\right]}$$
(2)

Sharma and Taneja Suggested the following measure of entropy involving two real parameters

$$\mathrm{H_{S-T}}(\mathrm{A}) = \frac{1}{\beta-\alpha}\left[\sum_{i=1}^{n}H_i^{\alpha}(A) - \sum_{i=1}^{n}H_i^{\beta}(A)\right],\alpha\neq\beta$$

$$= \frac{1}{\beta-\alpha}\left\{\left[\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\ln\mu_A^{\alpha}(x_i) + \left(1-\mu_A^{\alpha}(x_i)\right)\ln\left(1-\mu_A^{\alpha}(x_i)\right)\right] - \sum_{i=1}^{n}\left[\mu_A^{\beta}(x_i)\ln\mu_A^{\beta}(x_i) + \left(1-\mu_A^{\beta}(x_i)\right)\ln\left(1-\mu_A^{\beta}(x_i)\right)\right]\right]\right\}$$

$$= \frac{1}{\beta-\alpha}\left\{\left[\sum_{i=1}^{n}\mu_A^{\alpha}(x_i)\ln\mu_A^{\alpha}(x_i) - \sum_{i=1}^{n}\mu_A^{\beta}(x_i)\ln\mu_A^{\beta}(x_i)\right] + \sum_{i=1}^{n}\left[\left(1-\mu_A^{\alpha}(x_i)\right)\ln\left(1-\mu_A^{\alpha}(x_i)\right) - \left(1-\mu_A^{\beta}(x_i)\right)\ln\left(1-\mu_A^{\beta}(x_i)\right)\right]\right\}$$

$$= \frac{1}{\beta-\alpha}\left\{\left[\sum_{i=1}^{n}\mu_A^{\alpha}(x_i)\ln\mu_A^{\alpha}(x_i)-\sum_{i=1}^{n}\mu_A^{\beta}(x_i)\ln\mu_A^{\beta}(x_i)\right]\right.$$

$$+\sum_{i=1}^{n}\left[\ln\left(1-\mu_A^{\alpha}(x_i)\right)-\ln\left(1-\mu_A^{\beta}(x_i)\right)\right]$$

$$\left.-\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\ln\left(1-\mu_A^{\alpha}(x_i)\right)+\mu_A^{\beta}(x_i)\ln\left(1-\mu_A^{\beta}(x_i)\right)\right]\right\}$$

$$= \frac{1}{\beta-\alpha}\left\{\sum_{i=1}^{n}\mu_A^{\alpha}(x_i)\ln\frac{\mu_A^{\alpha}(x_i)}{1-\mu_A^{\alpha}(x_i)}-\sum_{i=1}^{n}\mu_A^{\beta}(x_i)\ln\frac{\mu_A^{\beta}(x_i)}{1-\mu_A^{\beta}(x_i)}+\sum_{i=1}^{n}\ln\frac{\left(1-\mu_A^{\alpha}(x_i)\right)}{\left(1-\mu_A^{\beta}(x_i)\right)}\right\}$$

(3)

According to fuzzy set theory, entropy is a fuzziness metric that indicates the average degree of ambiguity or difficulty in determining whether an element is a member of a set or not. $H(A)$, a fuzziness measure

a fuzzy set A must possess at least the four qualities listed below.

(1) $H(A)$ is at least A crisp set is one in which $\mu_A(x_i)$ = 0 or 1 for every x.

(2)The greatest $H(A)$, if A, where $\mu_A(x_i)$ = 0.5 for all x, is the most fuzzy set.

(3) $H(A) \geq H(A^*)$), where A* is A modified by sharpening. Where x is A's complement set,

(4)$H(A) = H(\overline{A})$, Where x is $\overline{A}$ complement set,

Different expressions for the entropy of a fuzzy set have been provided by various authors. If Xand Y are two fuzzy subsets of U, we can write using Kosko's fuzzy conditioning or subset hood measure.

Since $S(X,Y) = \frac{\sum_x \mu_{X\cap Y}(x)}{\sum_x \mu_X(x)}$

Therefore

$$\frac{S(Y,X)}{S(X,Y)} = \frac{\frac{\sum_x \mu_{Y\cap X}(x)}{\sum_x \mu_Y(x)}}{\frac{\sum_x \mu_{X\cap Y}(x)}{\sum_x \mu_X(x)}}$$

$$= \frac{\sum_x \mu_X(x)}{\sum_x \mu_Y(x)}$$

Taking logarithm both sides ,we have

$$\log\frac{S(Y,X)}{S(X,Y)} = \log\frac{\sum_x \mu_X(x)}{\sum_x \mu_Y(x)}$$

Now for, $\log\frac{S(Y,X)}{S(X,Y)} = \log\frac{\sum_i \mu_X(x_i)}{\sum_i \mu_Y(x_i)}$

Currently, the amount of information for discriminating for a specific fit value for $\mu_A(x_i)\, for\, x_i, i = 1,2,\ldots\ldots.., n$ is provided by

$$I'(X,Y;x_i) = \log\frac{\mu_X(x_i)}{\mu_Y(x_i)}$$

Therefore, it is possible to write the fuzzy anticipated information for discrimination in favor of X against Y as

$$I'(X,Y) = \sum_i \mu_X(x_i)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)}$$

Likewise, the quantity of fuzzy information needed to distinguish x from $\overline{Y}$ is provided by $\overline{X}$

$$I'(\overline{X},\overline{Y}) = \sum_i \big(1-\mu_X(x_i)\big)\log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)}$$

However, in general $I'(X,Y) \neq I'(\overline{X},\overline{Y})$,
the discriminability between (X,Y) and $(\overline{X},\overline{Y})$should be uniform. To do so, the fuzzy data for discrimination against Y in favor of X can be described as

$$I(X,Y) = I'(X,Y) + I'(\overline{X},\overline{Y})$$
$$= \sum_i \mu_X(x_i)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)} + \sum_i\big(1-\mu_X(x_i)\big)\log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)}$$
(4)

Equation (4) has previously shown us the fuzzy information needed to distinguish between A and B. Likewise, we may have as

$$I(Y,X) = \sum_i \mu_Y(x_i)\log\frac{\mu_{Y\ (x_i)}}{\mu_X\ (x_i)} + \sum_i\big(1-\mu_Y(x_i)\big)\log\frac{\big(1-\mu_{Y\ (x_i)}\big)}{\big(1-\mu_X\ (x_i)\big)}$$
$$-I(Y,X) = \sum_i \mu_Y(x_i)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)} + \sum_i\big(1-\mu_Y(x_i)\big)\log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)}$$
(5)

Let us now define the fuzzy divergence $D(X,Y)$between *X and Y as*

$$D(X,Y) = I(X,Y) + I(Y,X)$$

$$= \sum_i\big(\mu_X(x_i)-\mu_Y(x_i)\big)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)} + \sum_i \log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)} - \sum_i \mu_X(x_i)\log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)}$$
$$-\sum_i \log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)} + \sum_i \mu_Y(x_i)\log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)}$$

$$= \sum_i\big(\mu_X(x_i)-\mu_Y(x_i)\big)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)} + \sum_i\big(\mu_Y(x_i)-\mu_X(x_i)\big)\log\frac{\big(1-\mu_{X\ (x_i)}\big)}{\big(1-\mu_Y\ (x_i)\big)}$$
(6)

Two fuzzy sets can be distinguished by this measure. In relation toμ_X andμ_Y, take note that is $D(X,Y)$symmetric. Additionally, it meets the following requirements.
(1) If $X=Y$, then$D(X,Y)=0, D(X,Y)>0$
(2). $D(X,Y)=D(Y,X)$

III. PROPOSITION

Let *X* and *Y* be two fuzzy subsets of S then

$$D(X\cup Y, X\cap Y) = D(X,Y)$$
Proof.Let μ_X andμ_Y be the fuzzy membership functions of X and *Y*, respectively.
Let $S^+ = \{x\in S\ /\ \mu_X(x) \geq \ \mu_Y(x)\}$ and $S^- = \{x\in S\ /\ \mu_X(x) < \mu_Y(x)\}$

Therefore

$$D(X\cup Y,X\cap Y) = \sum_{x_i\in S^+}\left(\mu_X(x_i)-\mu_Y(x_i)\right)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)}+$$

$$\sum_{x_i\in S^+}\left(\mu_Y(x_i)-\ \mu_X(x_i)\right)\log\frac{\left(1-\mu_{X\ (x_i)}\right)}{\left(1-\mu_Y\ (x_i)\right)}\ +$$

$\sum_{x_i\in S^-}\left(\mu_Y(x_i)-\mu_X(x_i)\right)\log\frac{\mu_{Y\ (x_i)}}{\mu_X\ (x_i)}+$

$$\sum_{x_i\in S^-}\left(\mu_X(x_i)-\mu_Y(x_i)\right)\log\frac{\left(1-\mu_{Y\ (x_i)}\right)}{\left(1-\mu_X\ (x_i)\right)}$$

$$=\sum_{x_i\in S^+}\left(\mu_X(x_i)-\mu_Y(x_i)\right)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)}+$$

$$\sum_{x_i\in S^+}\left(\mu_Y(x_i)-\ \mu_X(x_i)\right)\log\frac{\left(1-\mu_{X\ (x_i)}\right)}{\left(1-\mu_Y\ (x_i)\right)}\ +$$

$\sum_{x_i\in S^-}\left(\mu_X(x_i)-\mu_Y(x_i)\right)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)}+$

$$\sum_{x_i\in S^-}\left(\mu_Y(x_i)-\mu_X(x_i)\right)\log\frac{\left(1-\mu_{X\ (x_i)}\right)}{\left(1-\mu_Y\ (x_i)\right)}$$

$$=\sum_{i=1}^{n}\left\{\left(\mu_X(x_i)-\mu_Y(x_i)\right)\log\frac{\mu_{X\ (x_i)}}{\mu_Y\ (x_i)}+\left(\mu_Y(x_i)-\ \mu_X(x_i)\right)\log\frac{\left(1-\mu_{X\ (x_i)}\right)}{\left(1-\mu_Y\ (x_i)\right)}\right\}=D(X,Y)$$

IV. ESSENTIAL PROPERTIES OF FUZZY SET

(1)$\mathrm{H}_\alpha(\mathrm{A})\geq 0$
(2) $\frac{\partial^2\mathrm{H}_\alpha(\mathrm{A})}{\partial\mu_A{}^2(x_i)}<0$.Thus $\mathrm{H}_\alpha(\mathrm{A})$ is a concave function of $\mu_A(x_i)\ \forall\ i$
(3)$\mathrm{H}_\alpha(\mathrm{A})$ does not change when $\mu_A(x_i)$ is replaced by $1-\mu_A(x_i)$
(4)$\mathrm{H}_\alpha(\mathrm{A})$ is an increasing function of $0\leq\ \mu_A(x_i)\leq\frac{1}{2}$
$[\mathrm{H}_\alpha(\mathrm{A})/\mu_A(x_i)=0]=0$ and $\left[\mathrm{H}_\alpha(\mathrm{A})/\mu_A(x_i)=\frac{1}{2}\right]=n\log 2>0$
(5)$\mathrm{H}_\alpha(\mathrm{A})$ is decreasing function of $\mu_A(x_i)\ for\frac{1}{2}\leq\ \mu_A(x_i)\leq 1$

$$\left[\mathrm{H}_\alpha(\mathrm{A})/\mu_A(x_i)=\frac{1}{2}\right]=n\log 2$$
$$[\mathrm{H}_\alpha(\mathrm{A})/\mu_A(x_i)=1]=0$$

(6)$[\mathrm{H}_\alpha(\mathrm{A})=0]\ for\ \mu_A(x_i)=0\ or\ 1$

Under these conditions ,the measures $\mathrm{H}_\alpha(\mathrm{A})$ is a valid measures of fuzzy entropy.
Applying Havrda and Charvat Concept to Fuzzy Entropy. The entropy of order (α) of a probability distribution $(x_1,x_2,\ldots\ldots,x_n)$ was defined by Havrda and Charvat as

$$\mathrm{H}_\alpha(\mathrm{A})=\ \tfrac{1}{1-\alpha}\left\{\textstyle\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha+\left(\left(1-\mu_A(x_i)\right)\right)^\alpha-1\right]\right\}$$

(7)
(1)$\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{A})=\mathbf{0}$
First let $\mathrm{H}_\alpha(\mathrm{A})=0,\forall\ A\in P(X)$
Then

$$\frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha+\left(\left(1-\mu_A(x_i)\right)\right)^\alpha-1\right]\right\}=0$$

Or

$$\left(\mu_A(x_i)\right)^\alpha+\left(\left(1-\mu_A(x_i)\right)\right)^\alpha=1$$

Now for $\alpha>0$,when either $\mu_A(x_i)=0\ or\ 1\forall\ i=1,2,\ldots.,n$ and so $A\in P(X)$

Conversely ,

If $A \in P(X)$therefore either $\mu_A(x_i) = 0$ *or* $1 \forall\, i = 1,2,\ldots,n$

It gives

$\big(\mu_A(x_i)\big)^{\alpha} + \Big(\big(1-\mu_A(x_i)\big)\Big)^{\alpha} = 1$ for all $\alpha > 0$

i,e. $\mathrm{H}_{\alpha}(\mathrm{A}) = 0, \forall\, A \in P(X)$

(2) $\frac{\partial^2 \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A{}^2(x_i)} < 0$ **.** Thus $\mathrm{H}_{\alpha}(\mathrm{A})$ is a concave function

Differentiating (7) with respect to $\mu_A(x_i)$ and then putting

$\frac{\partial \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A(x_i)} = 0$,we get

Or $\alpha\,\big(\mu_A(x_i)\big)^{\alpha-1} + \alpha\,\big(1-\mu_A(x_i)\big)^{\alpha-1}(0-1) - 0 = 0$

Or $\alpha\Big[\big(\mu_A(x_i)\big)^{\alpha-1} - \big(1-\mu_A(x_i)\big)^{\alpha-1}\Big] = 0$

Or $\big(\mu_A(x_i)\big)^{\alpha-1} - 1 + \big(\mu_A(x_i)\big)^{\alpha-1} = 0$

Or $2\big(\mu_A(x_i)\big)^{\alpha-1} = 1$

Or $\big(\mu_A(x_i)\big)^{\alpha-1} = \frac{1}{2}$

i.e.

$$\frac{\partial \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A(x_i)} = \big(\mu_A(x_i)\big)^{\alpha-1} - \frac{1}{2} \quad (8)$$

Clearly $\frac{\partial \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A(x_i)} \geq 0$ as $\alpha > 1$

Which implies that function is increasing function in

$$0 \leq \big(\mu_A(x_i)\big) < \frac{1}{2}$$

Similarly, from the symmetry of function, it is a decreasing function in $\frac{1}{2} \leq \mu_A(x_i) \leq 1$. Again we have

$$\mu_A(x_i) = 0 \text{ and } \mu_A(x_i) = 1$$

Differentiating (8) with respect to $\mu_A(x_i)$ and taking $\frac{\partial \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A(x_i)} = 0$,we get

$$\mu_A(x_i) = \frac{1}{2}$$

Again ,we get

$$\frac{\partial^2 \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A{}^2(x_i)} = (\alpha - 1)\big(\mu_A(x_i)\big)^{\alpha-2}$$

When $\alpha < 1$,we have $\frac{\partial^2 \mathrm{H}_{\alpha}(\mathrm{A})}{\partial \mu_A{}^2(x_i)} < 0$

Clearly ,we see that $\mathrm{H}_{\alpha}(\mathrm{A})$ is a concave function of $\mu_A(x_i)\ \forall\, i$.

(3)$\mathrm{H}_{\alpha}(\mathrm{A})$ does not change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$

From (7) ,we have

$$\mathrm{H}_\alpha(\mathrm{A}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\}$$

Replace $\mu_A(x_i)$ by $1-\mu_A(x_i)$

$$\mathrm{H}_\alpha(\mathrm{A}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(1-\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\}$$

$$= \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[2\left(1-\mu_A(x_i)\right)^\alpha - 1\right]\right\}$$

$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} = 0$,we get

Therefore $\frac{\partial}{\partial \mu_A(x_i)}\left\{\sum_{i=1}^{n}\left[2\left(1-\mu_A(x_i)\right)^\alpha - 1\right]\right\} = 0$

Or $\frac{\partial}{\partial \mu_A(x_i)}\left[2\left(1-\mu_A(x_i)\right)^\alpha - 1\right] = 0$

Or $2\alpha\left(\left(1-\mu_A(x_i)\right)^{\alpha-1}\right)(-1) = 0$

Or $\left(\mu_A(x_i)\right)^{\alpha-1} = 1$

Or $\mu_A(x_i) = 1^{1/\alpha-1}$

As $\alpha > 1$,we get , $\mu_A(x_i) = 1$

Therefore $\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} > 0$

Hence ,$\mathrm{H}_\alpha(\mathrm{A})$ does not change when $\mu_A(x_i)$ is replaced by $1-\mu_A(x_i)$.

(4)$\mathrm{H}_\alpha(\mathrm{A})$ is increasing on $0 \le \mu_A(x_i) \le \frac{1}{2}$and decreasing on$\frac{1}{2} \le \mu_A(x_i) \le 1$

Let $\mathrm{H}_\alpha(\mathrm{A}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\}$

then

$$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} = \left(\mu_A(x_i)\right)^{\alpha-1} - \frac{1}{2}$$

(8)

For ,$0 \le \mu_A(x_i) \le \frac{1}{2}$,we have $0.5 \le 1-\mu_A(x_i) \le 1$, if $0 < \alpha < 1$

Then

$$\left(\mu_A(x_i)\right)^{\alpha-1} - \frac{1}{2} \ge 0$$

$$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} \ge 0$$

If $\alpha > 1$,then

$$\left(\mu_A(x_i)\right)^{\alpha-1} - \frac{1}{2} \le 0$$

Thus

$$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} \ge 0$$

This means that $\mathrm{H}_\alpha(\mathrm{A})$ is increasing function in $0 \le \mu_A(x_i) \le \frac{1}{2}$.

Similarly ,we have to be $\mathrm{H}_\alpha(\mathrm{A})$ is decreasing function in$\frac{1}{2} \le \mu_A(x_i) \le 1$.

(5)$\mathrm{H}_\alpha(\mathrm{A}) \ge 0$

Since $\mathrm{H}_\alpha(\mathrm{A}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\}$

Therefore

$$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} = \left(\mu_A(x_i)\right)^{\alpha-1} - \frac{1}{2}$$

Now ,
$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} > 0 \Rightarrow \mu_A(x_i) < 1 - \mu_A(x_i)$for $t \in 0 \le \mu_A(x_i) \le \frac{1}{2}$
(9)

$\frac{\partial \mathrm{H}_\alpha(\mathrm{A})}{\partial \mu_A(x_i)} < 0 \Rightarrow \mu_A(x_i) > 1 - \mu_A(x_i)$ for $0.5 \le 1 - \mu_A(x_i) \le 1$
(10)

Combining (9) and (10)

$$\mathrm{H}_\alpha(\mathrm{A}) \ge 0$$

A mapping $H : P(X) \to \mathbb{R}$ that quantifies the level of fuzziness in A is a measure of fuzziness for a discrete fuzzy set. According to Ebanks [7], fuzziness measures should meet the following requirements for $A, B \in P(X)$

(P_1)Sharpnes :

$$\mathrm{H}_\alpha(\mathrm{A}) = 0 \Leftrightarrow A \ is\ crisp\ set\ \forall\, x \in X$$

(P_2)Maximally:
$\mathrm{H}_\alpha(\mathrm{A})$is maximum $\Leftrightarrow \mu_A(x_i) = 0.5 \ \forall\, x \in X$

(P_3)Resolution:
$\mathrm{H}_\alpha(\mathrm{A}) \ge \mathrm{H}_\alpha(A^*)$, where A^* is a sharped version of A.

(P_4)Symmetry:
if$\mathrm{H}_\alpha(\mathrm{A}) = \mathrm{H}_\alpha(1 - \mathrm{A})$,then $\mu_{1-A}(x) = 1 - \mu_A(x) \ \forall\, x \in X$

(P_5)Valuation:

$$H(A \cup B) + H(A \cap B) = 2\, H(A)$$

Ebanks provided the necessary and sufficient conditions listed below for functions that meet P1–P5 requirements for discrete fuzzy sets:

V. SOME PROPOSITION RELATED TO ABOVE PROPERTY FOR MEASURES OF FUZZINESS

$(\mathrm{P}_1)\mathrm{H}_\alpha(\mathrm{A}) = 0 \Leftrightarrow \mathrm{A}$ is crisp set $\forall$ x ∈ X

Proof: Let A be a crisp set

Since $S = S_{near}$

$$\Rightarrow D(S, S_{near}) = 0$$

Hence $S(A) = 0$

If $S = A$

Therefore $\mathrm{H}_\alpha = 0$

Let$\mathrm{H}_\alpha = 0$

So,

$$D(S, S_{near}) = 0$$
$$\Rightarrow S = S_{near}$$

Hence A is a crisp set.

(P_2)Maximally

Proof: Let A be the most fuzzy set, $\mu_A(x) = 0.5\ \forall\ x$
Hence $\mathrm{H}_\alpha(\mathrm{A}) - 1$,which is maximum.
Let $\mathrm{H}_\alpha(\mathrm{A}) - 1$ or $D(S, S_{near}) - D(S, S_{far})$
Simplifying we have

$$\bigcup_{i=1}^{n} F(\mu_A(x_i)) = \bigcup_{\mu_A(x_i)<0.5} F(\mu_A(x_i)) - \bigcup_{\mu_A(x_i)>0.5} F(\mu_A(x_i)) = 0$$

Where

$$F(\mu_A(x_i)) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[(\mu_A(x_i))^\alpha + \left((1-\mu_A(x_i))\right)^\alpha - 1\right]\right\}$$

$$F(\mu_A(x_i)) = \begin{cases} F(\mu_A(x_i)) < 0, & if\ \ \mu_A(x_i) < 0.5 \\ F(\mu_A(x_i)) > 0, & if\ \ \mu_A(x_i) > 0.5 \\ F(\mu_A(x_i)) = 0, & if\ \ \mu_A(x_i) = 0.5 \end{cases}$$

$$F(\mu_A(x_i)) = 0 \Longrightarrow \mu_A(x_i) = 0.5\ \forall\ x$$

This implies $\mathrm{H}_\alpha(\mathrm{A})$ is maximum iff A is most fuzzy set, that is $\mu_A(x) = 0.5\ \forall\ i = 1,2, \dots \dots, n$

(P_3)Resolution

$\mathrm{H}_\alpha(\mathrm{A}) \geq \mathrm{H}_\alpha(A^*)$, where A^* is a sharped version of A.

Proof: It is clear that $\mathrm{H}_\alpha(\mathrm{A})$ is an increasing function of $\mu_A(x_i)$,whenever $0 \leq \mu_A(x_i) \leq \frac{1}{2}$

And is a decreasing function of $\mu_A(x_i)$,whenever $\frac{\mathbf{1}}{\mathbf{2}} \leq \mu_A(x_i) \leq 1$
Therefore $\quad \mu_{A^*}(x_i) \leq \mu_A(x_i)$
$\Rightarrow\ \mathrm{H}_\alpha(\mathrm{A}) \geq \mathrm{H}_\alpha(A^*)$ in $0 \leq \mu_A(x_i) \leq \frac{1}{2}$

$$\Longrightarrow D(S, S_{near}) \geq D(S^*, \quad S^*_{near})$$

$\Rightarrow if\ \ S = A, therfore\ D(A, A_{near}) \geq D(A^*, A^*_{near})$
(11)

Again $\quad \mu_{A^*}(x_i) \geq \mu_A(x_i)$
$\Rightarrow\ \mathrm{H}_\alpha(\mathrm{A}) \leq \mathrm{H}_\alpha(A^*)$ in $\frac{\mathbf{1}}{\mathbf{2}} \leq \mu_A(x_i) \leq 1$

$$\Longrightarrow D(S, S_{near}) \leq D(S^*, \quad S^*_{near})$$

$\Rightarrow if\ \ S = A, therfore\ D(A, A_{near}) \leq D(A^*, A^*_{near})$
(12)

From (11) and (12) taking together,

$$\mathrm{H}_\alpha(\mathrm{A}) \geq \mathrm{H}_\alpha(A^*)$$

(P_4)Symmetry :
if $\mathrm{H}_\alpha(\mathrm{A}) = \mathrm{H}_\alpha(1 - \mathrm{A})$,then $\mu_A(x) = 1 - \mu_A(x)\ \forall\ x \in X$

Proof: We have from the definition of divergence

Since $D(S, S_{near}) = D(\bar{S}, \bar{S}_{near}) = D(\bar{S}, S_{far})$

And $D(S, S_{far}) = D(\bar{S}, \bar{S}_{far}) = D(\bar{S}, S_{near})$

if $\mathrm{H}_\alpha(\mathrm{S}) = \mathrm{H}_\alpha(1-\mathrm{S})$,

since $\mathrm{H}_\alpha(s) = \frac{\mathrm{D(S,S_{near})}}{\mathrm{D(S,S_{far})}}$

$$= \frac{D(\bar{S}, S_{far})}{D(\bar{S}, S_{near})}$$

$$= \frac{D(\bar{S}, \bar{S}_{near})}{D(\bar{S}, \bar{S}_{far})}$$

$$= \mathrm{H}_\alpha(\bar{s})$$

(13)

$$= 1 - \mathrm{H}_\alpha(s)$$

Therefor if $S = A$ and $\mu_A(x) = \mathrm{H}_\alpha(A)$

By substituting $1 - \mu_A(x)$instead of $\mu_A(x)$in Equation (13),we get
That $\mu_A(x) = 1 - \mu_A(x) \forall\, i = 1,2, \dots \dots .., n$

(P_5)Valuation
$\mathrm{H}(\mathrm{A} \cup \mathrm{B}) + \mathrm{H}(\mathrm{A} \cap \mathrm{B}) = 2\,\mathrm{H}(\mathrm{A})LetX_+ = \{x \in X, \mu_A(x_i) \geq \mu_B(x_i)\}$ and $X_- = \{x \in X, \mu_A(x_i) < \mu_B(x_i)\}$
Where

$$\mu_A(x_i) = \mu_B(x_i) = fuzzy\ member\ of\ \ A\ and\ B$$

Since from equation (7) we have

$$\mathrm{H}(\mathrm{A}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\}$$

Therefore , $\mathrm{H}(\mathrm{A} \cup \mathrm{B}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_{\mathrm{A}\cup\mathrm{B}}(x_i)\right)^\alpha + \left(\left(1-\mu_{\mathrm{A}\cup\mathrm{B}}(x_i)\right)\right)^\alpha - 1\right]\right\}$

$$= \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\} + \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_B(x_i)\right)^\alpha + \left(\left(1-\mu_B(x_i)\right)\right)^\alpha - 1\right]\right\}$$

$$= \frac{1}{1-\alpha}\left[\sum_{i=1}^{n}\left\{\left(\mu_A(x_i)\right)^\alpha + \left(\mu_B(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha + \left(\left(1-\mu_B(x_i)\right)\right)^\alpha - 2\right\}\right]$$

(14)

and $\mathrm{H}(\mathrm{A} \cap \mathrm{B}) = \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_{\mathrm{A}\cap\mathrm{B}}(x_i)\right)^\alpha + \left(\left(1-\mu_{\mathrm{A}\cap\mathrm{B}}(x_i)\right)\right)^\alpha - 1\right]\right\}$

$$= \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_A(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - 1\right]\right\}$$

$$- \frac{1}{1-\alpha}\left\{\sum_{i=1}^{n}\left[\left(\mu_B(x_i)\right)^\alpha + \left(\left(1-\mu_B(x_i)\right)\right)^\alpha - 1\right]\right\}$$

$$= \frac{1}{1-\alpha}\left[\sum_{i=1}^{n}\left\{\left(\mu_A(x_i)\right)^\alpha - \left(\mu_B(x_i)\right)^\alpha + \left(\left(1-\mu_A(x_i)\right)\right)^\alpha - \left(\left(1-\mu_B(x_i)\right)\right)^\alpha\right\}\right]$$

(15)

Adding (14) and (15), we get

$$\mathrm{H}(\mathrm{A} \cup \mathrm{B}) + \mathrm{H}(\mathrm{A} \cap \mathrm{B}) = \frac{1}{1-\alpha}\left[\sum_{i=1}^{n}\left\{2\left(\mu_A(x_i)\right)^{\alpha} + 2\left(\left(1-\mu_A(x_i)\right)\right)^{\alpha} - 2\right\}\right]$$
$$= 2\left[\frac{1}{1-\alpha}\sum_{i=1}^{n}\left\{\left[\left(\mu_A(x_i)\right)^{\alpha} + \left(\left(1-\mu_A(x_i)\right)\right)^{\alpha} - 1\right]\right\}\right]$$
$$= 2\ \mathrm{H}(\mathrm{A})$$

V. CONCLUSION

There is a new set of fuzzy set measures. Under certain conditions, the fuzzy entropy of Deluca and Termini [7] can be used to establish an information theoretic discrimination measure. It has since been applied to define divergence, D(A, B), a pseudo-distance measure between two sets A and B.

We have examined a few popular fuzziness metrics, and each of these classes' fuzzy-ness metrics meet five well-known Ebanks axioms. Specifying a Given the limited constraints of the underlying functional form, measuring fuzziness under either of these classes is simple. Non-negative, monotone-ly rising concave functions form the foundation of the multiplicative class. Only non-negative concave functions are needed for the additive class, which is more expansive. Numerous instances of

Additionally, we showed how these classes connect to a number of fuzziness measurements currently in use. Theoretical characteristics Investigated were a few new and current measures. The idea of fuzzy entropy was also presented, and its potential to include subjectivity in the fuzzy-ness metric was examined.

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