

Generalized Siamese Primes

M.Lewinter and S.Reyman

Abstract

The odd primes, p and q , are called *Siamese primes*, if there exists a natural number, n , such that $p = n^2 - 2$ and $q = n^2 + 2$. Letting a be a natural number which is not a square, we present properties of p and q , that satisfy the equations, $p = n^2 - a$ and $q = n^2 + a$.

Date of Submission: 08-09-2025

Date of acceptance: 19-09-2025

I. Siamese Primes

The odd primes, p and q , are called *Siamese primes*, if there exists a natural number, n , such that

$$p = n^2 - 2 \quad \text{and} \quad q = n^2 + 2$$

Examples:

n	p, q
3	7, 11
9	79, 83
15	223, 227
21	439, 443

n^2 is odd, implying that n is odd, so $n^2 \equiv 1 \pmod{4}$. Since $q - p = 4$, $q \equiv p \pmod{4}$. As odd numbers are 1 or $3 \pmod{4}$, $q \equiv p \pmod{4}$, either $p \equiv q \equiv 1 \pmod{4}$ or $p \equiv q \equiv 3 \pmod{4}$. Since $q = n^2 + 2$, we see that $q \equiv 3 \pmod{4}$, so $p \equiv q \equiv 3 \pmod{4}$, as exemplified by the table above.

Theorem 1: $3 \mid n$.

Proof: Observe that $n^2 \equiv 0$ or $1 \pmod{3}$. Also, $q \equiv 1$ or $2 \pmod{3}$. We have two cases.

Case 1: $q \equiv 1 \pmod{3}$. Since $n^2 = q - 2$, $n^2 \equiv -1 \equiv 2 \pmod{3}$, which is impossible.

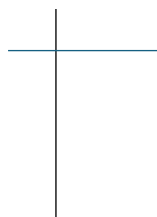
Case 2: $q \equiv 2 \pmod{3}$. Since $n^2 = q - 2$, $n^2 \equiv 0 \pmod{3}$, implying that $n \equiv 0 \pmod{3}$, so $3 \mid n$. ■

II. Generalized Siamese Primes

(a) We consider the system, $p = n^2 - 3$ and $q = n^2 + 3$.

Examples:

n	p, q
4	13, 19
8	61, 67
10	97, 103
14	193, 199



Since n^2 is even, $n^2 \equiv 0 \pmod{4}$. Then $p \equiv 0 - 3 \equiv 1 \pmod{4}$ and $q \equiv 0 + 3 \equiv 3 \pmod{4}$. Note, too, that $n^2 \equiv 0 \pmod{4}$ implies that $n \equiv 0$ or $2 \pmod{4}$. See the above table.

We bypass the system $p = n^2 - 4$ and $q = n^2 + 4$, since $n^2 - 4 = (n - 2)(n + 2)$ which is composite for $n > 3$. When $n = 3$, we have the only solution, $p = 5, q = 13$.

(b) We consider the system, $p = n^2 - 5$ and $q = n^2 + 5$.

Examples:

n	p, q
6	31, 41
12	139, 149
36	1291, 1301
72	5179, 5189

Since n^2 is even, we have $n^2 = 0 \pmod{4}$. Then $p = 0 - 5 = 3 \pmod{4}$ and $q = 0 + 5 = 1 \pmod{4}$. **Theorem 2:** $3 \mid n$.

Proof: Observe that $n^2 = 0$ or $1 \pmod{3}$. Also, $q = 1$ or $2 \pmod{3}$. If $n^2 = 1 \pmod{3}$, we find that $n^2 = q - 5$ becomes $1 = q - 5 \pmod{3}$, in which case $q = 6 = 0 \pmod{3}$, which is impossible. It follows that $n^2 = 0 \pmod{3}$, so $n = 0 \pmod{3}$. ■

There are two kinds of odd primes, p , and they have different properties. If $p = 1 \pmod{4}$, then p can be written in only one way as $a^2 + b^2$. If $p = 3 \pmod{4}$, then p cannot be written as the sum of two squares. This makes the following theorem noteworthy.

Theorem 3: Let $p = n^2 - a$ and let $q = n^2 + a$. If a is even, then $p = q \pmod{4}$. If a is odd, then either (a) $p = 1 \pmod{4}$ and $q = 3 \pmod{4}$, or (b) $p = 3 \pmod{4}$ and $q = 1 \pmod{4}$.

Proof: $q - p = 2a$. (a) If a is even, we have $q - p = 0 \pmod{4}$, so $p = q \pmod{4}$. (b) If a is odd, let $a = 2k + 1$, so $2a = 4k + 2$. Then $q - p = 2 \pmod{4}$, so $p = 1 \pmod{4}$ and $q = 3 \pmod{4}$, or $p = 3 \pmod{4}$ and $q = 1 \pmod{4}$. ■

Bibliography

- [1] David Wells, *Prime Numbers*, Wiley, 2005.
- [2] M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.
- [3] D. Burton, *Elementary Number Theory*, McGraw-Hill, 2005.