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Generalized Siamese Primes

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Abstract

The odd primes, p and q, are called Siamese primes, if there exists a natural number, n, such that $p = n^2 - 2$ *and* $q = n^2 + 2$*. Letting a be a natural number which is not a square, we present properties ofp and q, that satisfy the equations,*$p = n^2 - a$ *and* $q = n^2 + a$*.*

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I. Siamese Primes

The odd primes, p and q, are called *Siamese primes*, if there exists a natural number, n, such that

$$p = n^2 - 2 \quad \text{and} \quad q = n^2 + 2$$

Examples:

n	p, q
3	7, 11
9	79, 83
15	223, 227
21	439, 443

n^2 is odd, implying that n is odd,so$n^2 = 1$ (mod 4). Since $q - p = 4$, $q = p$ (mod 4). As odd numbers are 1 or 3 (mod 4), $q = p$ (mod 4), either $p = q = 1$ (mod 4)or$p = q = 3$ (mod 4). Since $q = n^2 + 2$, we see that $q = 3$ (mod 4), so $p = q = 3$ (mod 4), as exemplified by the table above.

Theorem 1: $3 \mid n$.

Proof: Observe that $n^2 = 0$ or 1 (**mod3**). Also, $q = 1$ or 2 (mod 3). We have two cases.

Case 1:$q = 1$ (mod 3). Since $n^2 = q - 2$, $n^2 = -1 = 2$ (mod 3), which is impossible.

Case 2:$q = 2$ (mod 3). Since $n^2 = q - 2$, $n^2 = 0$ (mod 3), implying that $n = 0$ (mod 3), so $3 \mid n$. ■

II. Generalized Siamese Primes

(a) We consider the system,$p = n^2 - 3$ and $q = n^2 + 3$.

Examples:

n	p, q
4	13,19
8	61,67
10	97,103
14	193,199

Since n^2 is even, $n^2 = 0$ (mod 4). Then $p = 0 - 3 = 1$ (mod 4) and $q = 0 + 3 = 3$ (mod 4). Note, too, that $n^2 = 0$ (mod 4) implies that $n = 0$ or 2 (mod 4). See the above table.

We bypass the system $p = n^2 - 4$ and $q = n^2 + 4$, since $n^2 - 4 = (n - 2)(n + 2)$ which is composite for $n> 3$. When $n = 3$, we have the only solution, $p = 5$, $q = 13$.

(b) We consider the system, $p = n^2 - 5$ and $q = n^2 + 5$.

Examples:

n	p,q
6	31,41
12	139,149
36	1291,1301
72	5179,5189

Since n^2 is even, we have $n^2 = 0$ (mod 4). Then $p = 0 - 5 = 3$ (mod 4) and $q = 0 + 5 = 1$ (mod 4). **Theorem 2:** $3 | n$.

Proof: Observe that $n^2 = 0$ or 1 (**mod 3**). Also, $q = 1$ or 2 (mod 3). If $n^2 = 1$ (mod 3), we find that $n^2 = q - 5$ becomes $1 = q - 5 =$ (mod 3), in which case $q = 6 = 0$ (mod 3), which is impossible. It follows that $n^2 = 0$ (mod 3), so $n = 0$ (mod 3). ■

There are two kinds of odd primes, p, and they have different properties. If $p = 1$ (mod 4), then p can be written in only one way as $a^2 + b^2$. If $p = 3$ (mod 4), then p cannot be written as the sum of two squares. This makes the following theorem noteworthy.

Theorem 3: Let $p = n^2 - a$ and let $q = n^2 + a$. If a is even, then $p = q$ (mod 4). If a is odd, then either **(a)** $p = 1$ (mod 4) and $q = 3$ (mod 4), or **(b)** $p = 3$ (mod 4) and $q = 1$ (mod 4).

Proof: $q - p = 2a$. **(a)** If a is even, we have $q - p = 0$ (mod 4), so $p = q$ (mod 4). **(b)** If a is odd, let $a = 2k + 1$, so $2a = 4k + 2$. Then $q - p = 2$ (mod 4), so $p = 1$ (mod 4) and $q = 3$ (mod 4), or $p = 3$ (mod 4) and $q = 1$ (mod 4). ■

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