International Journal of Mathematics and Statistics Invention (IJMSI)
e-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 13 Issue 5 || Sep. – Oct. 2025 || PP: 22-31

Existence and Stability Results for Nonlocal Boundary Value Problem of Nonlinear Fractional Differential Equations

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ABSTRACT: *The aim of this paper is to establish sufficient conditions for the existence, uniqueness and Ulam-type stability of solutions for a class of nonlocal boundary value problem for nonlinear fractional differential equation with positive constant coefficient. Finally, an example is given to demonstrate applicability of our results.*

KEYWORDS: *Boundary value conditions, Caputo's fractional derivative, Existence of solution, Fixed point, Stability.*

Date of Submission: 11-09-2025 Date of acceptance: 23-09-2025

I. INTRODUCTION

The concept of fractional differentiation was introduced in the century by Riemann and Liouville. It is the generalization of integral order differentiation and integration to arbitrary non-integer order. For detailed study, see the books such as [16, 17] and the survey papers [2, 3] and the references therein.

Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields such as control theory, signal processing, rheology, fractals, chaotic dynamics, modelling, bioengineering and biomedical applications and so on. For example, see the books such as [14, 22] and the references therein. Due to its importance in different fields, researcher in this area has grown significantly all around the world.

The stability problem of functional equations was introduced by Ulam [23, 24] and Hyers [9] which is known as Hyers-Ulam stability. Rassias [18] studied the Hyers-Ulam stability of linear and nonlinear mapping. Jung [11, 12] established Hyers-Ulam stability for more general mapping on restricted domain. Obloza [15] was the first author who studied the Hyer-Ulam stability of linear differential equations. For detailed study of Ulam-type stability with different approaches, we refer the reader to the papers [1, 10, 18, 21, 21, 26, 27, 28] and the books [8, 19, 20].

In [4], Benchohra and Bouriahi studied existence and stability of solutions for a class of boundary value problem for implicit fractional differential equations of the type:

$$
\begin{gathered}
{}^cD^\alpha y(t) = f(t, y(t), \quad {}^cD^\alpha y(t)), t \in J := [0, T], T > 0, \\
ay(0) + by(T) = c
\end{gathered}
$$

where ${}^cD^\alpha (0 < \alpha \leq 1)$ denotes the caputo fractional derivative, $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$is a given continuous function, and $\mathrm{a, b, c}$ are real constants with $a + b \neq 0$, and

$$
\begin{gathered}
{}^cD^\alpha y(t) = f(t, y(t), \quad {}^cD^\alpha y(t)), t \in J := [0, T], T > 0, \\
y(0) + g(y) = y_0
\end{gathered}
$$

where $g: C([0, T], \mathbb{R}) \to \mathbb{R}$ is a continuous function and y_0 is a real constant.

In [5], Benchohra et al. studied the existence of solutions for a class of boundary value problem for fractional differential equations and nonlocal boundary condition of the form:

$$^cD^\alpha y(t) = f(t, y(t)), t \in J := [0, T], T > 0$$
$$y(0) = g(y), y(T) = y_T$$

where $^cD^\alpha (1 < \alpha \leq 2)$ denotes the caputo fractional derivative, $f: J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and y_T is a real constant.

Motivated by the above-mentioned work, in this paper, we establish sufficient conditions for the existence, uniqueness and four types of Ulam stability, namely Ulam- Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability for the following nonlinear fractional differential equation with constant coefficient $\lambda > 0$ of the type:

$$^cD^\alpha y(t) = \lambda y(t) + f(t, y(t)), t \in J := [0, T], T > 0, \tag{1.1}$$
$$y(0) + g(y) = y_0 \tag{1.2}$$

where $^cD^\alpha (1 < \alpha \leq 2)$ denotes the caputo fractional derivative, $f: J \times \mathbb{R} \to \mathbb{R}$ is a given continuous and$g: C([0, T], \mathbb{R}) \to \mathbb{R}$ is a continuous function and y_0 is a real constant.

This type of non-local Cauchy problem was introduced by Byszewski [6, 7]. The nonlocal condition can be more useful than the classical initial condition to describe some physical phenomenons [6, 7]. We take an example of non-local conditions as follows:

$$g(y) = \sum_{i=1}^{p} c_i y(t_i) \tag{1.3}$$

where c_i, i $= 1, \ldots\ldots,$ p are constants and $0 < t_1 < \cdots \ldots . < t_p \leq T$.

II. PRELIMINARIES AND NOTATIONS

In this section, we will introduce some definitions, notations and results which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}$$

By $L^1(J)$ we denote the space of Lebesgue-integrable functions $y: J \to \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)| dt$$

Definition 2.1. [17] The Riemann-Liouville fractional (arbitrary) order integral of the function $h \in L^1([0, T], \mathbb{R}_+)$of order $\alpha \in \mathbb{R}_+$is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt, \alpha > 0$.

Definition 2.2. [14] The Caputo fractional derivative of order $\alpha > 0$ of a function $h \in L^1([0, T], \mathbf{R}_+)$is given by

$$(\ ^cD^\alpha h)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} h^{(n)}(s) ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1. [14] Let $\alpha > 0$ and $n = [\alpha] + 1$, then

$$I^\alpha(\ ^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k$$

where $f^k(t)$ is the usual derivative of $f(t)$ of order k.

Lemma 2.2. [17] Let $\alpha > 0$. Then the fractional differential equation

$$^cD^\alpha h(t) = 0,$$

has a solutions$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots \ldots . + c_{n-1} t^{n-1}$, where $c_i, i = 0,1,2, \ldots \ldots, n$ are constants and

$n = [\alpha] + 1$.

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 2.3. [25] Let v $:[0, T] \to [0, +\infty)$ be a real function and $w()$ is an on negative, locally integrable function on $[0, T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha \leq 1$

$$v(t) \leq w(t) + a \int_0^t (t-s)^{-\alpha} v(s) ds$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t (t-s)^{-\alpha} w(s) ds$$

for every $t \in [0, T]$.

To study the stability results we use following definitions adopted in [4, 21] .

Definition 2.3. The equation (1.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$| \ ^cD^\alpha z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon, t \in J$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_f \varepsilon, t \in J$$

Definition 2.4. The equation (1.1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+), \psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$| \ ^cD^\alpha z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon, t \in J$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq \psi_f(\varepsilon), t \in J$$

Definition 2.5. The equation (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R}_+)$if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$| \ ^cD^\alpha z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon \varphi(t), t \in J$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_f \varepsilon \varphi(t), t \in J$$

Definition 2.6. The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R}_+)$if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$| \ ^cD^\alpha z(t) - \lambda z(t) - f(t, z(t))| \leq \varphi(t), t \in J$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_{f,\varphi} \varphi(t), t \in J$$

Remark 2.1. A function $z \in C^1(J, \mathbb{R})$ is a solution of the inequality

$$| \ {}^{c}D^{\alpha}z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon, t \in J$$

if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depends on solution y) such that
i) $|g(t)| \leq \varepsilon, \forall t \in J$.
ii) ${}^{c}D^{\alpha}z(t) = \lambda z(t) + f(t, z(t)) + g(t), t \in J$.

Remark 2.2. Clearly,
i) Definition (2.3) ⇒ Definition (2.4)
ii) Definition (2.5) ⇒ Definition (2.6).

Remark 2.3. A solution of the fractional differential inequality

$$| \ {}^{c}D^{\alpha}z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon, t \in J$$

is called an fractional ε-solution of the fractional differential equation (1.1).

III. EXISTENCE AND ULAM-HYERS STABILITY OF THE NONLOCAL BOUNDARY VALUE PROBLEM

This section is devoted to some existence and uniqueness results for the nonlocal problem (1.1)-(1.2). Let us introduce the following set of conditions:

(H1): There exists a constant $L > 0$ such that $|f(t, x) - f(t, \bar{x})| \leq L|x - \bar{x}|$, for each $t \in J$.

(H2) The function $f: J \times \mathbb{R} \to \mathbb{R}$ is continuous.
(H3) There exists a constant $M > 0$ such that $|f(t, x)| \leq M$, for each $t \in J$ and for all $x \in \mathbb{R}$.
(H4) There exists a constant $\bar{M} > 0$ such that $|g(y)| \leq \bar{M}$, for each $y \in C(J, \mathbb{R})$.
(H5) There exists a constant $\bar{K} > 0$ such that $|g(y) - g(\bar{y})| \leq \bar{K}|y - \bar{y}|$, for each $y, \bar{y} \in C(J, \mathbb{R})$.

Lemma 3.1. [11] Let $0 < \alpha \leq 1$ and $h: [0, T] \to \mathbf{R}$ is a continuous function. Then the linear problem

$$ {}^{c}D^{\alpha}x(t) = h(t), t \in [0, T], T > 0, $$
$$ x(0) + g(x) = x_0 $$

has a unique solution which is given by:

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds$$

As a consequence of Lemma (3.1) we have,
Lemma 3.2. Let $f: J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, then the problem (1.1)-(1.2) is equivalent to the following problem

$$y(t) = y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s)) ds, t \in J \quad (3.1)$$

Theorem 3.1. Assume that (H1), (H5) hold. If

$$\left[\bar{K} + \frac{(\lambda + L)T^{\alpha}}{\Gamma(\alpha + 1)}\right] < 1 \quad (3.2)$$

then, the nonlocal problem (1.1)-(1.2) has a unique solution on J.
Proof. We transform problem (1.1)-(1.2) into a fixed point problem. For this, consider the operator $\bar{F}: C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by

$$\bar{F}(y)(t) = y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s)) ds \quad (3.3)$$

Let $x, y \in C(J, \mathbb{R})$. Then for each $t \in J$ we have

$$
\begin{aligned}
|\bar{F}(x)(t)-\bar{F}(y)(t)| &\leq |g(x)-g(y)| + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds \\
&\leq \bar{K}|x(t)-y(t)| + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds \\
&+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds \\
&\leq \left[\bar{K}+\frac{(\lambda+L)T^{\alpha}}{\Gamma(\alpha+1)}\right]\|x(s)-y(s)\|_{\infty}.
\end{aligned}
$$

Thus

$$
\|\bar{F}(x)-\bar{F}(y)\|_{\infty} \leq \left[\bar{K}+\frac{(\lambda+L)T^{\alpha}}{\Gamma(\alpha+1)}\right]\|x-y\|_{\infty}
$$

Thus, F is a contraction due to the (3.2).
By Banach contraction principle, we deduce that $\bar{F}$ has an unique fixed point which is just the unique solution of the problem (1.1)-(1.2) .

The next result is based on Schaefer's fixed point theorem.
Theorem 3.2. Assume that (H2), (H3), (H4) hold. Then the BVP (1.1)-(1.2) has at least one solution on J.
Proof. We shall use Schaefer's fixed point theorem to prove that $\bar{F}$ defined by (3.3) has a fixed point. The proof will be given in several steps.

Step 1: $\bar{F}$ is continuous.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$. Then for each $t \in J$, we have

$$
\begin{aligned}
|\bar{F}(y_n)(t)-\bar{F}(y)(t)| &\leq |g(y_n)-g(y)| + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y_n(s)-y(s)|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{t\in J}|f(s,y_n(s))-f(s,y(s))|ds
\end{aligned}
$$

Since f and g are continuous functions, then we have $\|\bar{F}(y_n)(t)-\bar{F}(y)(t)\|_{\infty} \to 0$ as $n \to \infty$.
Step 2: $\bar{F}$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$.
Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant l such that for each $y \in B_{\eta^*} = \{y \in C(J,\mathbb{R}): \|y\|_{\infty} \leq \eta^*\}$, we have $\|\bar{F}(y)\|_{\infty} \leq l$.

By (H3) and (H4) we have for each $t \in J$,

$$
\begin{aligned}
|\bar{F}(y)(t)| &\leq |y_0| + |g(y)| + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)|ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds \\
&\leq |y_0| + \bar{M} + \frac{\lambda\eta^* T^{\alpha}}{\Gamma(\alpha+1)} + \frac{MT^{\alpha}}{\Gamma(\alpha+1)} := l
\end{aligned}
$$

Thus

$$
\|\bar{F}(y)\|_{\infty} \leq |y_0| + \bar{M} + \frac{\lambda\eta^* T^{\alpha}}{\Gamma(\alpha+1)} + \frac{MT^{\alpha}}{\Gamma(\alpha+1)} := l
$$

Step 3: $\bar{F}$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1, t_2 \in (0,T], t_1 < t_2, B_{\eta^*}$ be a bounded set of $C(J,\mathbb{R})$ as in step 2 , and let $y \in B_{\eta^*}$. Then

$$|\bar{F}(y)(t_1) - \bar{F}(y)(t_2)| = \left| \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s)) ds \right.$$
$$\left. - \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) ds \right|$$
$$\leq \frac{\lambda\eta^*}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} ds + \frac{M}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} ds$$
$$+ \frac{\lambda\eta^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + \frac{M}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds$$
$$\leq \frac{(\lambda\eta^* + M)}{\Gamma(\alpha+1)} \{2(t_2 - t_1)^\alpha + (t_1^\alpha - t_2^\alpha)\}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $\bar{F}: C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous and completely continuous.

Step 4: A priori bounds.
Now it remains to show that the set

$$\mathcal{E} = \{y \in C(J, \mathbb{R}): y = \beta\bar{F}(y), \text{ for some } \beta \in (0,1)\}$$

is bounded.

Let $y \in \mathcal{E}$, then $y = \beta\bar{F}(y)$, for some $\beta \in (0,1)$. Thus, for each $t \in J$ we have

$$y(t) = \beta \left\{ y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, y(s)) ds \right\}. \qquad (3.4)$$

This implies by (H3), (H4) that for each $t \in J$ we have

$$|\bar{F}(y)(t)| \leq |y_0| + |g(y)| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |y(s)| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s, y(s))| ds$$
$$\leq |y_0| + \bar{M} + \frac{\lambda\eta^* T^\alpha}{\Gamma(\alpha+1)} + \frac{M T^\alpha}{\Gamma(\alpha+1)}$$

Thus for every $t \in J$. We have

$$\|\bar{F}(y)\|_\infty \leq |y_0| + \bar{M} + \frac{\lambda\eta^* T^\alpha}{\Gamma(\alpha+1)} + \frac{M T^\alpha}{\Gamma(\alpha+1)} := l$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\bar{F}$ has a fixed point which is a solution of the problem (1.1)-(1.2).

Theorem 3.3. Assume that (H1), (H5), inequality (4.19) are satisfied then the problem (1.1)-(1.2) is Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$ and let $z \in C^1(J, \mathbb{R})$ be a function which satisfies the inequality:

$$| \ {}^cD^\alpha z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon, \text{ for every } t \in J \qquad (3.5)$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of the following Cauchy problem

$${}^cD^\alpha y(t) = \lambda y(t) + f(t, y(t)), t \in J, 0 < \alpha \leq 1$$
$$z(0) + g(y) = y_0$$

by Lemma (3.2),

$$y(t) = y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, y(s)) ds$$

by integration of the inequality (3.5), we obtain

$$\left|z(t) - y_0 + g(z) - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,z(s))ds\right| \leq \frac{\varepsilon t^\alpha}{\Gamma(\alpha+1)} \leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)} \quad (3.6)$$

For every $t \in J$, we have

$$\begin{aligned}|z(t) - y(t)| &\leq \left|z(t) - y_0 + g(y) - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds\right| \\ &\leq \left|z(t) - y_0 + g(z) - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,z(s))ds\right| + |g(z) - g(y)| \\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,z(s)) - f(s,y(s))|ds \\ &\leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)} + \bar{K}|z(t) - y(t)| + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|ds \\ &+ \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|ds\end{aligned}$$

Thus

$$|z(t) - y(t)| \leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)(1-\bar{K})} + \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}\int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|ds$$

Using Gronwall's lemma, we obtain for every $t \in J$

$$|z(t) - y(t)| \leq \frac{\varepsilon T^\alpha}{\Gamma(\alpha+1)(1-\bar{K})}\left[1 + \frac{K(\lambda+L)T^\alpha}{\Gamma(\alpha+1)(1-\bar{K})}\right] := c\varepsilon$$

where $K = K(\alpha)$ is a constant, so the problem (1.1)-(1.2) is Ulam-Hyers stable. Moreover, if we set $\psi(\varepsilon) = c\varepsilon$; $\psi(0) = 0$, then the problem (1.1) -(1.2) is generalized Ulam-Hyers stable.

Theorem 3.4. Assume that (H1), (H5), (4.19) and (H6): there exists an increasing function $\varphi \in C(J, \mathbb{R}_+)$and there exists $\kappa_\varphi > 0$ such that for any $t \in J$

$$I^\alpha \varphi(t) \leq \kappa_\varphi \varphi(t)$$

are satisfied, then, the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable.
Proof. Let $z \in C^1(J, \mathbb{R})$ be solution of the following inequality:

$$|\ {}^cD^\alpha z(t) - \lambda z(t) - f(t, z(t))| \leq \varepsilon\varphi(t), \text{ for any } t \in J, \varepsilon > 0 \quad (3.7)$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of the following Cauchy problem

$$\begin{gathered}{}^cD^\alpha y(t) = \lambda y(t) + f(t, y(t)), t \in J; 0 < \alpha \leq 1 \\ z(0) + g(y) = y_0\end{gathered}$$

by Lemma (3.2), we have

$$y(t) = y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds$$

By integration of (3.7), we obtain

$$\left|z(t) - y_0 + g(y) - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,z(s))ds\right| \leq \varepsilon\kappa_\varphi\varphi(t) \quad (3.8)$$

We have for any $t \in J$

$$|z(t)-y(t)|\le \left|z(t)-y_0+g(y)-\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds\right|$$
$$\le \left|z(t)-y_0+g(z)-\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,z(s))ds\right|+|g(z)-g(y)|$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,z(s))-f(s,y(s))|ds$$

Using equation (3.8), (H1) and (H5) we obtain

$$|z(t)-y(t)|\le \varepsilon\kappa_\varphi\varphi(t)+\bar{K}|z(t)-y(t)|+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds$$
$$+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds$$

Thus

$$|z(t)-y(t)|\le \frac{\varepsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}+\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds$$

applying Gronwall's lemma, we get for any $t\in J$:

$$|z(t)-y(t)|\le \frac{\varepsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}+\frac{K_1(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})^2}\int_0^t (t-s)^{\alpha-1}\varepsilon\kappa_\varphi\varphi(s)ds$$
$$=\frac{\varepsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}+\frac{\varepsilon K_1\kappa_\varphi(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})^2}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds$$

where $K_1=K_1(\alpha)$ is a constant, and by (H6), we have:

$$|z(t)-y(t)|\le \frac{\varepsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}+\frac{\varepsilon K_1\kappa_\varphi^2(\lambda+L)\varphi(t)}{(1-\bar{K})^2}$$
$$=\left[\frac{1}{(1-\bar{K})}+\frac{K_1\kappa_\varphi(\lambda+L)}{(1-\bar{K})^2}\right]\varepsilon\kappa_\varphi\varphi(t)$$

Then for any $\in J$:

$$|z(t)-y(t)|\le c\varepsilon\varphi(t)$$

which completes the proof of Theorem (3.4).

IV. EXAMPLE

Example 4.1. Consider the boundary value problem:

$${}^cD^{\frac{1}{2}}y(t)=\frac{1}{10}y(t)+\frac{e^{-t}}{(9+e^t)}\left[\frac{|y(t)|}{1+|y(t)|}\right], t\in J=[0,1] \quad (4.1)$$

$$y(0)+\sum_{i=1}^{n} c_i y(t_i)=1 \quad (4.2)$$

where $0<t_1<<t_n<1$ and $c_i, i=1,\ldots\ldots,n$ are positive constants with

$$\sum_{i=1}^{n} c_i\le\frac{1}{5} \quad (4.3)$$

Define $f(t,y)=\frac{e^{-t}}{(9+e^t)}\left[\frac{y(t)}{1+y(t)}\right], t\in[0,1], \alpha=\frac{1}{2}, \lambda=\frac{1}{10}, y\in[0,\infty)$.
Clearly, the f is continuous. For each $y,\bar{y}\in\mathbb{R}$ and $t\in[0,1]$:

$$|f(t,y)-f(t,\bar{y})| \leq \frac{e^{-t}}{(9+e^{t})}\left|\frac{y}{1+y}-\frac{\bar{y}}{1+\bar{y}}\right|$$
$$\leq \frac{1}{10}|y-\bar{y}|$$

Also, we have

$$|g(x)-g(\bar{x})| \leq \left|\sum_{i=1}^{n} c_i x - \sum_{i=1}^{n} c_i \bar{x}\right|$$
$$\leq \sum_{i=1}^{n} c_i |x-\bar{x}|$$
$$\leq \frac{1}{5}|x-\bar{x}|.$$

Hence condition (H1) and (H5) is satisfied with $L=\frac{1}{10}, \bar{K}=\frac{1}{5}$ and $\lambda=\frac{1}{10}$. We have

$$\left[\bar{K}+\frac{(\lambda+L)T^{\alpha}}{\Gamma(\alpha+1)}\right]=\left[\frac{1}{5}+\frac{\left(\frac{1}{10}+\frac{1}{10}\right)}{\Gamma\left(\frac{3}{2}\right)}\right]<1$$

It follows from Theorem (3.1) that the problem (4.1)-(4.2) has a unique solution on [0,1] and by Theorem (3.3), the problem (4.1)-(4.2) is Ulam-Hyers stable.

ACKNOWLEDGEMENTS

The authors are grateful to the referees for their valuable comments and suggestions.

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