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The Alon-Tarsi Number of Two Kinds of Graph Operations

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ABSTRACT*: The Alon-Tarsi number $AT(G)$ of a graph G is the least k for which there is an orientation D of G with maximum outdegree $k-1$ such that the number of Eulerian subgraphs of G with an even number of edges differs from the number of Eulerian subgraphs with an odd number of edges. In this paper, we obtain the range of the Alon-Tarsi number of two kinds of graph operations of special graphs with any simple graph, namely, edge corona product and generalized edge corona product.*
KEYWORDS*: Alon-Tarsi number; choice number; chromatic number; edge corona product; generalized edge corona product*

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I. INTRODUCTION

All graphs considered in this article are finite and simple. One of the most popular topics in graph theory is graph coloring. In addition to classical coloring, list coloring is also a hot topic, it is a well-established generalization of graph coloring and has been widely studied. The study of list coloring problems, was obtained in the 1970s by Vizing [2] and independently by Erdos, Rubin and Taylor [3].

A k-list assignment of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of k permissible colors. Given a k-list assignment L of G, an L-coloring of G is a mapping ϕ which assigns to each vertex v a color $\phi(v)\in L(v)$ such that $\phi(u)\neq\phi(v)$ for every edge $e=uv$ of G. A graph G is k-choosable if G has an L-coloring for every k-list assignment L. The choice number of a graph G is the least positive integer k such that G is k-choosable, denoted by $ch(G)$.

In the classic article [4], an upper bound for the choice number and for some related parameters of graphs is obtained by applying algebraic techniques, which was later called the Alon-Tarsi number of G, and denoted by $AT(G)$ (see e.g. Jensen and Toft (1995) [5]). The Alon-Tarsi number of G, $AT(G)$, is the smallest k for which there is an orientation D of G with maximum outdegree $k-1$ such that the number of odd Eulerian subgraphs of G is not the same as the number of even Eulerian subgraphs of G. It was proposed by Alon and Tarsi [1], subsequently they used algebraic methods to prove that $\chi(G)\le ch(G)\le AT(G)$. A graph G is chromatic-AT choosable if $\chi(G)=AT(G)$.

The graph operation, especially the graph product, plays a significant role not only in pure and applied mathematics, but also in computer science. For graph operations, we are familiar with the Kronecker, Cartesian, and Corona products. There are some results concerning the Alon-Tarsi number of Cartesian products and Corona products.

Kaul and Mudrock proved that the Cartesian product of cycle and path has $AT(C_k\square P_n)=3$ in [7] (k, n are integers). Suppose that G is a complete graph or an odd cycle with $|V(G)\ge 3|$ and suppose H is a graph on at least two vertices that contains a Hamilton path, $w_1,w_2,\ldots,w_m$, such that w_i has at most k neighbours among $w_1,w_2,\ldots,w_{i-1}$. Then $AT(G\square H)\le\Delta(G)+k$. Li, Shao, Petrov and Gordeev in [8] proved that the Cartesian product of a cycle and a cycle has

$$AT(C_m \square C_n) = \begin{cases} 4, & n \ and \ m \ are \ both \ odd \ numbers; \\ 3, & otherwise. \end{cases}$$

Li, Gai and Shao in [15] proved that the Cartesian product of a n-cube Q_n and a tree T_m has

$$AT(Q_n \square T_m) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1, & n \ is \ odd \ and \ m = 2; \\ \left\lceil \frac{n}{2} \right\rceil + 2, & otherwise. \end{cases}$$

and the corona product of Q_n and T_m has

$$AT(Q_n \circ T_m) = \begin{cases} 3, & n \le 2; \\ \left\lceil \frac{n}{2} \right\rceil + 1, & n > 2. \end{cases}$$

In this paper, we are interested in the Alon-Tarsi number of two types of graph operation. One kind of graph operation is edge corona product [1], which is defined as following: G is a simple graph with m edges, taking m copies of the graph H, and connecting the two endpoints of the i-th edge of G to all the vertices of the i-th copy of H. The graph obtained in this way is called the edge corona product $G \lozenge H$ of G and H (See Figure 1).

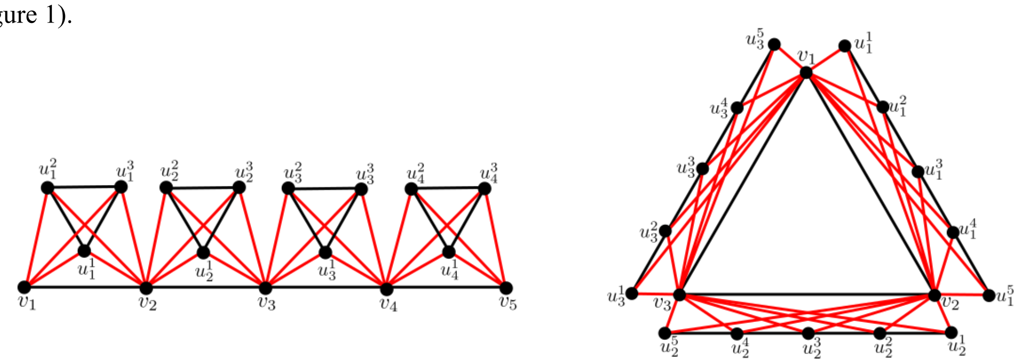

Figure 1. The edge corona product graph $P_5 \lozenge K_3$ (left) and $K_3 \lozenge P_5$ (right)

Another kind of graph operation is generalized edge corona produc [14] which is denoted by $G \lozenge (H_1, H_2, \ldots, H_m)$ (See Figure 2), its vertex set and edge set are respectively:

(a) $V(G \lozenge (H_1, H_2, \ldots, H_m)) = V(G) \cup \bigcup_{i=1}^{m} V(H_i)$;

(b) $E(G \lozenge (H_1, H_2, \ldots, H_m)) = \{e_i \in E(G) : 1 \le i \le m\} \cup \bigcup_{i=1}^{m} E(H_i) \cup A$

where A denotes the cross edge between each $e_i \in E(G)$ and $V(H_i)$.

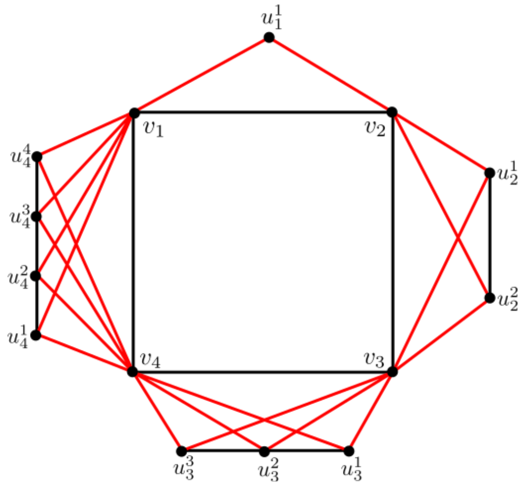

Figure 2. The generalized edge corona product graph $C_4\Diamond(P_1, P_2, P_3, P_4)$

Throughout the paper, we denote by P_n, C_n, S_n, K_n the path, cycle, star and complete graph with n vertices, respectively. By using the relationship between chromatic number and Alon-Tarsi number and the AT-orientation method, we obtain the range of the Alon-Tarsi number of the edge corona product $G\Diamond H$ of $G \in \{P_n, S_n, C_n, K_n\}$ and any simple graph H in Section 3. Similarly, in Section 4, we get that the generalized edge corona product $G\Diamond(H_1, H_2, \ldots, H_{|E(G)|})$ of $G \in \{P_n, S_n, C_n, K_n\}$ and chromatic-AT choosable graphs $H_1, H_2, \ldots, H_{|E(G)|}$ is also chromatic-AT choosable.

II. PRELIMINARIES

Definition 2.1. [4] A subdigraph H of a directed graph D is called Eulerian if $V(H) = V(G)$ and the indegree $d_H^-(v)$ of every vertex v of H in H is equal to its outdegree $d_H^+(v)$. Note that H might not be connected. For a digraph D, we denote by $\varepsilon(D)$ the family of Eulerian subdigraphs of D. H is even if it has an even number of edges, otherwise, it is odd. Let $\varepsilon_e(D)$ and $\varepsilon_o(D)$ denote the numbers of even and odd Eulerian subgraphs of D, respectively. Let $\mathrm{diff}(D) = |\varepsilon_e(D)| - |\varepsilon_o(D)|$. We say that D is Alon-Tarsi if $\mathrm{diff}(D) \neq 0$. If an orientation D of G yields an Alon-Tarsi digraph, then we say D is an Alon-Tarsi orientation (or an AT-orientation, for short) of G.

Generally, it is difficult to determine whether an orientation D of a graph G is an AT-orientation. However, in certain cases, this task becomes straightforward. It's worth noting that every digraph D contains at least one even Eulerian subdigraph, specifically, the empty subgraph. If D has no odd directed cycle, then D has no odd Eulerian subdigraph, so D is an AT-orientation.

Lemma 2.1. [9] Assume that D is a digraph and $V(D) = X_1 \dot{\cup} X_2$. For $i = 1, 2$, let $D_i = D[X_i]$ be the subdigraph of D induced by X_i. If all the arcs between X_1 and X_2 are from X_1 to X_2, then D is Alon-Tarsi if and only if D_1 and D_2 are both Alon-Tarsi.

III. RESULTS FOR THE EDGE CORONA PRODUCT

Theorem 3.1. For any simple graph H of order m and a path P_n of order at least 3, we have

$$\chi(P_n\Diamond H) = \chi(H) + 2;$$

$$\chi(H) + 2 \leq AT(P_n\Diamond H) \leq AT(H) + 2.$$

Proof. Let $P_n = v_1v_2\ldots v_n$. In $P_n\Diamond H$, H_i is the copy of H corresponding to the edge v_iv_{i+1} of P_n, which is the graph with set vertices $V_i = \{u_i^1, u_i^2, \ldots, u_i^m\}$, where $i \in \{1, 2, \ldots, n-1\}$. According to the definition of $P_n\Diamond H$, we can get $\chi(P_n\Diamond H) \geq \chi(H) + 2$.

Assume that $\pi: V(H) \to \{1, 2, \dots, \chi(H)\}$ is a proper coloring of H. In order to show that $\chi(P_n \lozenge H) \le \chi(H) + 2$, we construct the following coloring π of $P_n \lozenge H$:

(a) The restriction of π' on H_i is π;

(b) $\pi'(v_i) = \chi(H) + 1 + i(mod\ 2)$.

It is easy to know that π' is a proper coloring of $P_n \lozenge H$, so $\chi(P_n \lozenge H) \le \chi(H) + 2$. Therefore $\chi(P_n \lozenge H) = \chi(H) + 2$.

H is a subgraph of $P_n \lozenge H$, hence $AT(P_n \lozenge H) \ge \max\{AT(H), \chi(H) + 2\}$. Next, we will show that $AT(P_n \lozenge H) \le AT(H) + 2$.

Suppose D is an AT-orientation of H with $\Delta^+(D) = AT(H) - 1$, we denote L_i as the set of edges connecting H_i and v_i, v_{i+1} ($i \in [n-1]$), let D' is an orientation of $P_n \lozenge H$. The rules of D' are as follows:

R_1: For the copy H_i of H, the edges belonging to H_i are oriented like D;

R_2: The edges of P_n are oriented from v_i to v_{i+1} ($i \in [n-1]$);

R_3: The edges in L_i are oriented from H_i to v_i and v_{i+1}.

By Lemma 2.1, the orientation D' of $P_n \lozenge H$ is an AT-orientation, and satisfying that $\Delta^+(D') = \Delta^+(D) + 2$ $= AT(H) + 1$, hence $AT(P_n \lozenge H) \le \Delta^+(D') + 1 = AT(H) + 2$.

Using the same method we can obtain the following theorem.

Theorem 3.2. For any simple graph H of order m and a star S_n of order at least 3, we have

$$\chi(S_n \lozenge H) = \chi(H) + 2;$$

$$\chi(H) + 2 \le AT(S_n \lozenge H) \le AT(H) + 2.$$

The chromatic number and Alon-Tarsi number of the edge corona product of a cycle C_n and any simple graph H yield the following results:

Theorem 3.3. For any simple graph H of order m and a cycle C_n of order at least 3, we have

$$\chi(C_n \lozenge H) = \chi(H) + 2;$$

$$\chi(H) + 2 \le AT(C_n \lozenge H) \le AT(H) + 2.$$

Proof. Let $C_n = v_1 v_2 \dots v_n v$, In $C_n \lozenge H$, H_i is the copy of H corresponding to the edge $v_i v_{i+1}$ of C_n, which is the graph with set vertices $V_i = \{u_i^1, u_i^2, \dots, u_i^m\}$, where $i \in \{1, 2, \dots, n\}$. According to the definition of $C_n \lozenge H$, we can get $\chi(C_n \lozenge H) \ge \chi(H) + 2$. Assume that π: $V(H) \to \{1, 2, \dots, \chi(H)\}$ is a proper coloring of H. There are two cases are discussed, as follows:

Case 1. n is odd.

The coloring π' of $C_n \lozenge H$ satisfies the following conditions:

(a) $\pi'(v_i) = \chi(H) + 1 + i(mod\ 2)$, $i \in [n-1]$, $\pi'(v_n) = \chi(H)$;

(b) For $1 \le i \le n-2$, the restriction of π' on H_i is exactly π;

(c) For the vertex u_{n-1}^j of H_{n-1}, if its corresponding vertex u in H is colored $\chi(H)$ under π, then $\pi'(u_{n-1}^j) = \chi(H) + 2$, $j = 1, 2, \dots, m$. Similarly, for the vertex u_n^j of H_n, if its corresponding vertex u in H is colored $\chi(H)$ under π, then $\pi'(u_{n-1}^j) = \chi(H) + 1$, $j = 1, 2, \dots, m$. The coloring π' for the remaining vertices in H_{n-1} and H_n is the same as π.

It is obvious that $\pi' : V(C_n \Diamond H) \to \{1, 2, \ldots, \chi(H)+2\}$ is a proper coloring of $C_n \Diamond H$, then $\chi(C_n \Diamond H) \le \chi(H)+2$. Hence $\chi(C_n \Diamond H) = \chi(H)+2$.

Case 2. n is even.

The construction of coloring π' of $C_n \Diamond H$ is the same as in the case of $P_n \Diamond H$, therefore $\chi(C_n \Diamond H) \le \chi(H)+2$.

It is easy to know that $AT(C_n \Diamond H) \ge \max\{AT(H), \chi(H)+2\}$. Next, we will show that $AT(C_n \Diamond H) \le AT(H)+2$ by constructing an AT-orientation of $C_n \Diamond H$.

Since $C_n \Diamond H$ is consists of $P_n \Diamond H$ with one additional edge $v_n v_1$ and one additional copy H_n of H, the orientation D'' of $C_n \Diamond H$ is based on the AT-orientation D' of $P_n \Diamond H$, with all arcs between H_n and C_n oriented from H_n to C_n, and the orientation of the edge $v_n v_1$ is as follows: if n is even, it is oriented from v_n to v_1; if n is odd, the orientation is opposite. Then according to Lemma 2.1 we can get that D'' is an an AT-orientation of $C_n \Diamond H$ and $\Delta^+(D'') = AT(H)+1$. Therefore

$$AT(C_n \Diamond H) \le \Delta^+(D'')+1 = AT(H)+2.$$

Particularly, if H is chromatic-AT choosable, namely $AT(H) = \chi(H)$, $G \in \{P_n, S_n, C_n\}$ where n is an integer, then $AT(G \Diamond H) = \chi(G \Diamond H) = AT(H)+2$. That is to say $G \Diamond H$ is also chromatic-AT choosable.

The chromatic number and Alon-Tarsi number of the edge corona product of a complete graph K_{n} and any simple graph *H* yields the following results.

Theorem 3.4. For any simple graph H of order m and a complete graph K_n of order at least 3, we have

$$\chi(K_n \Diamond H) = \max\{n, \chi(H)+2\};$$

$$\max\{\chi(H)+2; n\} \le AT(C_n \Diamond H) \le \max\{AT(H)+2, n\}.$$

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, In $K_n \Diamond H$, H_i is the copy of H corresponding to the edge of K_n, which is the graph with set vertices $V_i = \{u_i^1, u_i^2, \ldots, u_i^m\}$, where $i \in \{1, 2, \ldots, \frac{n(n-1)}{2}\}$. Let $M = \max\{n, \chi(H)+2\}$. Since K_n is a subgraph of $K_n \Diamond H$, and from the definition of $K_n \Diamond H$, it follows that $\chi(K_n \Diamond H) \ge M$. Assume that $\pi : V(H) \to \{1, 2, \ldots, \chi(H)\}$ is a proper coloring of H. The coloring π' of $K_n \Diamond H$ is defined as follows:

(a) $\pi'(v_i) = i$, $i \in [n]$;

(b) For the vertex u_i^j of H_i ($1 \le i \le \frac{n(n-1)}{2}$) which is adjacent to the edge $v_k v_t$ of K_n, if its corresponding vertex u in H is colored k or t under π, then $\pi'(u_i^j) \in \{\chi(H)+1, \chi(H)+2, \ldots, M\} - \{k, t\}$. The restriction of π' on the remaining vertices of H_i is π. An example of the coloring of $K_4 \Diamond K_3$ is illustrated in Figure 3.

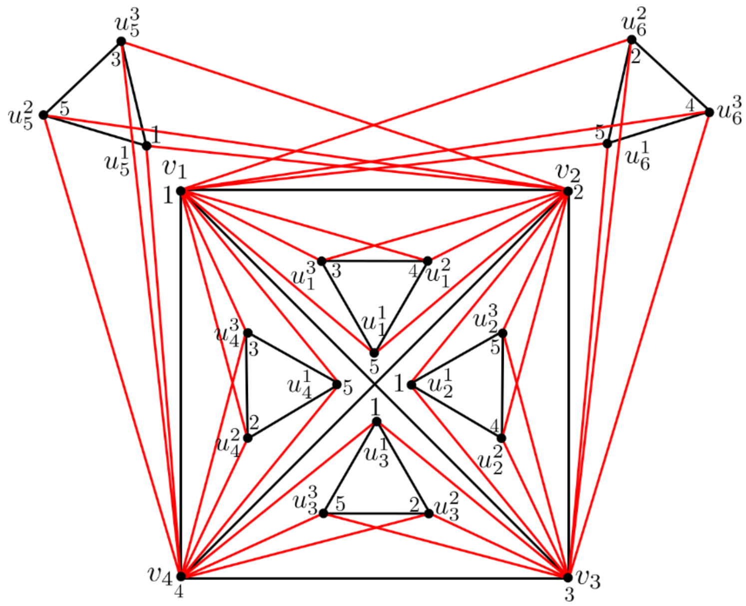

Figure 3. A proper coloring of $K_4\Diamond K_3$

Then $\pi': V(K_n\Diamond H)\rightarrow\{1,2,\dots,n\}$ is a proper coloring of $K_n\Diamond H$, then $\chi(K_n\Diamond H)\le M$. Hence $\chi(K_n\Diamond H)=M$.

Since $AT(K_n\Diamond H)\ge\chi(K_n\Diamond H)=M$. Next we will show the upper bound of the Alon-Tarsi number of $K_n\Diamond H$ using the AT-orientation method. The orientation D' of $K_n\Diamond H$ is similar to Theorem 3.1, with the change being that R_2 is replaced by "the edges of K_n are oriented from v_i to v_j where $1\le i<j\le n$ ". Then D' is an AT-orientation of $K_n\Diamond H$ with $\Delta^+(D')=\max\{n-1, AT(H)+1\}$. Therefore $AT(K_n\Diamond H)\le\Delta^+(D')+1=\max\{n, AT(H)+2\}$.

In particular, if H is chromatic- AT choosable, then $AT(K_n\Diamond H)=\chi(K_n\Diamond H)$ $=\max\{n, AT(H)+2\}$. $K_n\Diamond H$ is chromatic-AT choosable as well.

IV. RESULTS FOR THE GENERALIZED EDGE CORONA PRODUCT

Using the same method as for the coloring and orientation of the edge corona product we can obtain the following results:

Theorem 4.1. If $H_1, H_2,\dots,H_{n-1}$ are chromatic- AT choosable with $AT(H_i)=\chi(H_i)=k_i$ ($1\le i\le n-1$), G is a path P_n or a star S_n of order n, we have

$$AT(G\Diamond(H_1,H_2,\dots,H_{n-1}))=\chi(G\Diamond(H_1,H_2,\dots,H_{n-1}))=\max\{k_1,k_2,\dots,k_{n-1}\}+2.$$

Namely, the generalized edge corona product graph $G\Diamond(H_1,H_2,\dots,H_{n-1})$ is chromatic-AT choosable.

Theorem 4.2. If $H_1,H_2,\dots,H_n$ are chromatic-AT choosable with $AT(H_i)=\chi(H_i)=k_i$ ($1\le i\le n$), the generalized edge product of a cycle C_n and $H_1,H_2,\dots,H_n$ has

$$AT(C_n\Diamond(H_1,H_2,\dots,H_n))=\chi(C_n\Diamond(H_1,H_2,\dots,H_n))=\max\{k_1,k_2,\dots,k_n\}+2.$$

Namely, the generalized edge corona product graph $C_n\Diamond(H_1,H_2,\dots,H_n)$ is chromatic-AT choosable.

Theorem 4.3. If $H_1,H_2,\dots,H_{\frac{n(n-1)}{2}}$ are chromatic- AT choosable with $AT(H_i)=\chi(H_i)=k_i$ ($1\le i\le\frac{n(n-1)}{2}$), the generalized edge product of a complete graph K_n and $H_1,H_2,\dots,H_{\frac{n(n-1)}{2}}$ has

$$AT(K_n\Diamond(H_1,H_2,\dots,H_{\frac{n(n-1)}{2}}))=\chi(K_n\Diamond(H_1,H_2,\dots,H_{\frac{n(n-1)}{2}}))=\max\{k_1,k_2,\dots,k_{\frac{n(n-1)}{2}}\}+2.$$

Namely, the generalized edge corona product graph $K_n \Diamond (H_1, H_2, \ldots, H_{\frac{n(n-1)}{2}})$ is chromatic-AT choosable.

V. CONCLUSIONS

In this work, we obtain the range of Alon-Tarsi number of two types of graph operation by using the AT-orientation skill, namely the edge corona product of $G \in \{P_n, S_n, C_n, K_n\}$ and any simple graph H, and the generalize edge corona product of $G \in \{P_n, S_n, C_n, K_n\}$ and $H_1, H_2, \ldots, H_t$ where t represents the edge number of G. Furthermore, we get that if H is chromatic-AT choosable then $G \Diamond H$ is chromatic-AT choosable as well; if $H_1, H_2, \ldots, H_t$ are all chromatic-AT choosable, then $G \Diamond (H_1, H_2, \ldots, H_t)$ is also chromatic-AT choosable.

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