*International Journal of Mathematics and Statistics Invention (IJMSI)*
*e-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759*
*www.ijmsi.org Volume 14 Issue 1 || Jan. – Feb. 2026 || PP: 13-17*

# Interpolative Contraction in Distance Mappings and Unique Fixed Point Results in the Frame of Partial Metric Space

Rohit Kumar Verma
*Associate Professor,*
*Department of Mathematics,*
*Bharti Vishwavidyalaya, Durg, Chhattisgarh, India.*

***Abstract***
*In the framework of metric space, this communication aims to investigate various unique fixed-point theorems that meet rational contractive conditions and demonstrate the uniqueness of fixed points for a pair of mappings.*

***Keywords*** *- Cauchy sequence, metric space, Lipschitz mapping, continuous mapping, contraction etc.*

## I. Introduction

A subfield of mathematics known as functional analysis examines vector spaces that have a concept of size or distance, with an emphasis on spaces that are complete in terms of this measure. Building on ideas from linear algebra and calculus, it began with the study of infinite-dimensional spaces and linear operators. The study of normed vector spaces—where a norm offers a means of measuring the length or magnitude of vectors—is fundamental to functional analysis.

The Banach space is one of the most important kinds of normed spaces. It is a full normed space, which means that every Cauchy sequence—a sequence in which the elements go arbitrarily near to one another—converges to a limit inside the space. Banach spaces are guaranteed to behave properly and be appropriate for a variety of analytical methods because to this characteristic.

By tackling issues in infinite dimensions, such as solving differential equations, optimising functions, and researching multiple kinds of convergence, functional analysis expands on classical analysis. The field is closely related to other branches of mathematics, such as probability, measure theory, and topology. It offers fundamental tools and theorems that are necessary for comprehending and using mathematical ideas in both theoretical and practical situations, such as the Hahn-Banach theorem, the Banach-Steinhaus theorem, and the Closed Graph theorem.

If A is a real Banach space where the multiplication operation is defined, then $\forall\ x, y, z \in A$ and $a \in R$.

1. $(xy)z = x(yz)$,
2. $x(y + z = xy + xz)$ and $(x + y)z = xz + yz$,
3. $a(xy) = (ax)y = x(ay)$,
4. $\|xy\| \leq \|x\|\,\|y\|$

For the purposes of this study, we will assume that a Banach algebra has a unit e such that $= xe = x\ \forall x \in A$. If there is an inverse element x within A such that $xy = yx = e$, then the element x within A is said to be invertible. The symbol $x^{-1}$ represents the inverse of x.

**Preposition** Assume that x belongs to A and that A is a Banach algebra with a unit e. $(e - x)$ is invertible if the spectral radius $\sigma(x)$ of x is less than 1, i.e. $\sigma(x) = \lim_{n\to\infty}\|x^n\|^{\frac{1}{n}} = \inf\|x^n\|^{\frac{1}{n}} < 1$. In actuality, $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Matthews [6] first proposed the concept of a partial metric space in 1994. The self-distance of any point in space may not be zero, which is a unique property of the partial metric that replaces the standard metric in this space. Huang and Zhang [4] first proposed the idea of cone metric spaces in 2007 as a generalization of metric spaces. In the context of cone metric spaces over a normal cone, they also demonstrated the Banach contraction principle.

The results in the setting of cone metric spaces are only appropriate if the underlying cone is not necessarily normal, as demonstrated by Rezapour and Hamlbarani's [7] omission of the assumption of normality of cone given in [4] and presentation of a few examples to support the existence of non-normal cones.

The concept of cone metric space over Banach algebra was then introduced by Liu and Xu [5] in 2013 by substituting Banach algebra A for Banach space E. This clearly shows that the existence of fixed points of the mappings in cone metric spaces over Banach algebra is not equivalent to metric spaces. Additionally, they provided several instances to clarify their findings. As can be seen in [2, 3, 8, 9, 10, 11] numerous authors have focused on generalizing cone metric spaces. The notion of an M-metric space, a generalization of a partial metric space, was first presented in 2014 by Asadi et al. [1], who also produced some fixed point results for generalized contractions in the new context.

As a generalization of both M-metric space and cone metric space over Banach algebra, we show the structure of M-cone metric spaces over Banach algebra in this paper.

**Definition** Let X be a set that is not empty. Next, a function $\mathrm{T}: \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{A}$ with the following requirements

1. $\mathrm{T}(\mathrm{x}, \mathrm{x})=\mathrm{T}(\mathrm{y}, \mathrm{y})=\mathrm{T}(\mathrm{x}, \mathrm{y}) \Leftrightarrow \mathrm{x}=\mathrm{y}$,
2. $\mathrm{T}_{\mathrm{xy}} \leq \mathrm{T}(\mathrm{x}, \mathrm{y})$,
3. $\mathrm{T}(\mathrm{x}, \mathrm{y})=\mathrm{T}(\mathrm{y}, \mathrm{x})$,
4. $\left(\mathrm{T}(\mathrm{xy})-\mathrm{T}_{\mathrm{xy}}\right) \leq\left(\mathrm{T}(\mathrm{xy})-\mathrm{T}_{\mathrm{xy}}\right)+\left(\mathrm{T}(\mathrm{yz})-\mathrm{T}_{\mathrm{yz}}\right)$

The pair $(\mathrm{X}, \mathrm{T})$ is thus referred to as an M-metric space.

**Definition** Let X be a non-empty set, and let $\mathrm{T}: \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{A}$ be a mapping that fully satisfies the requirements of metric space˜ $\mathrm{x}, \mathrm{y}, \mathrm{z}$ in X, where $\leq$ is a partial order defined on A [4]. The complex valued metric space is thus denoted by $(\mathrm{X}, \mathrm{T})$.

**Definition** Let $T: X \times X \rightarrow A$ satisfy the following, and let $X$ be a non-empty set with $s \geq 1$.

1. $0 \leq T(x, y)\ \forall\, x, y \in X$ with $T(x, y)=0$ if and only if $x=y$
2. $T(x, y)=T(y, x)\ \forall\, x, y \in X$
3. $T(x, y) \leq s(T(x, z)+T(z, y))\ \forall\, x, y, z \in X$.

The complex valued b-metric space is thus defined as $(X, T)$.

Additionally, we introduce the concept of generalized Lipschitz mapping in the context of M-cone metric spaces over Banach algebra and look into whether such mappings have fixed points.

**Definition** Let $(X, T)$ be a metric space. If there is a constant $a \geq 0$ such that a mapping $T_1: X \rightarrow X$ is Lipschitz continuous, then

$$T\left(T_1\left(x_1\right), T_1\left(x_2\right)\right) \leq a T(x, y)\ \forall\, x, y \in X$$

1. If $a=1$, then $T$ is said to be non-expansive.
2. If $a \in(0,1)$, then $T$ is said to be contraction.
3. If $\left(T_1\left(x_1\right), T_1\left(x_2\right)\right)<a T(x, y)\ \forall\, x \neq y$. The integer $a$ is known as the Lipschitz constant of $T$, and $T$ is said to be contractive.

## 2. New Results

**Theorem 2.1** Let $T_1: X \rightarrow X$ be a Lipschitz mapping with $\varphi(\lambda)<1$, and let $(X, T)$ be a complete metric space over Banach algebra $A$. Then, under the following circumstances, $T$ admits a unique fixed point.

(a) $\frac{T\left(T_1 x, T_1 y\right)}{T\left(x, T_1 x\right)+T\left(y, T_1 y\right)} \leq \lambda\ \forall\, x, y \in X$

(b) $\frac{\emptyset(h)}{\emptyset\left(\frac{\lambda}{1-\lambda}\right)}=1$ and

(c) $\frac{T\left(x, T_1 x\right)-T_{x, T_1 x}}{T\left(x, x_n\right)-T_{x, x_n}+T\left(x_n, T_1 x\right)-T_{x_n, T_1 x}} \leq 1\ \forall\, n \in N$.

**Proof:** Given a sequence $\left\{x_n\right\}$ in $X$ such that $x_{n+1}=T_1 x_n \forall\, n \in N$, let $x_0 \in X$. Next,

$$\begin{aligned} T\left(x_{n+2}, x_{n+1}\right) &= T\left(T_1 x_{n+1}, T_1 x_n\right) \\ &\leq \lambda\left(T\left(x_{n+1}, T_1 x_{n+1}\right)+T\left(x_n, T_1 x_n\right)\right) \\ &= \lambda\left(T\left(x_{n+1}, x_{n+2}\right)+T\left(x_n, x_{n+1}\right)\right) \end{aligned}$$

Hence, $\quad (1-\lambda) T\left(x_{n+2}, x_{n+1}\right) \leq \lambda T\left(x_n, x_{n+1}\right)$

Since $\theta(\lambda)<1$ and $A$ is Banach algebra. Hence, $(1-\lambda)$ is invertible.

So,

$$T(x_{n+2},x_{n+1}) \leq \lambda(1-\lambda)^{-1}T(x_n,x_{n+1})$$

On assuming $h = \lambda(1-\lambda)^{-1}$

Hence, $T(x_{n+2},x_{n+1}) \leq hT(x_n,x_{n+1})$.

Continuing in the same argument, we will get

$$T(x_{n+1},x_{n+2}) \leq h^{n+1}T(x_0,x_1)$$

Moreover, $\forall\ n,m \in N, n > m$, we have

$$\begin{aligned}
T(x_{n+1},x_m) &\leq T(x_{n+1},x_{n+2}) - T_{x_{n+1},x_{n+2}} + T(x_{n+2},x_m) - T_{x_{n+2},x_n}\\
&\leq T(x_{n+1},x_{n+2}) + T(x_{n+2},x_m)\\
&\leq T(x_{n+1},x_{n+2}) + T(x_{n+2},x_{n+3}) - T_{x_{n+2},x_{n+3}} + T(x_{n+3},x_m) - T_{x_{n+3},x_m}\\
&\leq T(x_{n+1},x_{n+2}) + T(x_{n+2},x_{n+3}) + T(x_{n+3},x_m)\\
&\leq T(x_{n+1},x_{n+2}) + T(x_{n+2},x_{n+3}) + T(x_{n+3},x_{n+2}) + \cdots + T(x_{m-1},x_m)\\
&\leq h^{n+1}T(x_0,x_1) + h^{n+2}T(x_0,x_1) + h^{n+3}T(x_0,x_1) + \cdots + h^{m-1}T(x_0,x_1)\\
&= h^n(h + h^2 + h^3 + \cdots + h^{m-n-1})T(x_0,x_1)\\
&= h^n\big((1 + h + h^2 + h^3 + \cdots + h^{m-n-1}) - 1\big)T(x_0,x_1)\\
&= h^n\left(\frac{1}{1-h} - 1\right)T(x_0,x_1)\\
&= \frac{h^{n+1}}{1-h}T(x_0,x_1)
\end{aligned}$$

Ofcourse, $\frac{\|T(x_0,x_1)\|\|h^{n+1}\|}{\|h^{n+1}T(x_0,x_1)\|} \geq 1$ and its approaches to zero as $n$ approaches to $\infty$ and hence $\{x_n\}$ is a Cauchy sequence in $X$. So, by completeness of $X$, there exists $x \in X$ such that $lim_{n\to\infty} T(x_n,x) = lim_{n,m\to\infty} T(x_n,x_m) = T(x,x) = \delta$ as $x_n$ approaches to $x$ as $n$ approaches to $\infty$ for some $x$. So, $T(x_n,x) - T_{x_n,x}$ approaches to $\delta$ as $n$ approaches to $\infty$ and hence, $T_{x_n,x} = min\{T(x_n,x_n) - T(x,x)\}$ approaches to $\delta$ also, $T_{x_n,T_1x} = min\{T(x_n,x_n) - T(T_1x,T_1x)\} = \delta$. Now, since $T(x,T_1x) - T_{x,T_1x} \leq T(x,x_n) - T_{x,x_n} + T(x_n,T_1x) - T_{x_n,T_1x}$. So that,

$$\begin{aligned}
T(x,T_1x) - T_{x,T_1x} &\leq T(x,x_n) + d(x_n,T_1x)\\
&= T(x,x_n) + T(T_1x_{n-1},T_1x)\\
&\leq T(x,x_n) + K\big(d(x_{n-1},T_1x_{n-1}) + T(x,T_1x)\big) = T(x,x_n) + kd(x_{n-1},x_n) + kd(x,T_1x)
\end{aligned}$$

$$(1-\lambda)T(x,T_1x) - T_{x,T_1x} \leq T(x,x_n) + T(x_{n-1},x_n)$$

$$(1-\lambda)T(x,T_1x) \leq \delta$$

If we multiply by $(1-\lambda)\sum_{i=0}^{\infty}\lambda^i = 1 \geq 0$ yields that $d(x,T_1x) \leq \delta$. Now, by given condition (a), we get

$$\begin{aligned}
T(T_1x,T_1x) &\leq \lambda\big(T(x,T_1x) + T(x,T_1x)\big)\\
&= 2\lambda T(x,T_1x)
\end{aligned}$$

So that, $T(T_1x,T_1x) = \delta$

From (6.2), (6.5) and (6.6), we obtain,

$$d(x,x) = T(T_1x,T_1x) = T(x,T_1x)$$

We get $T_1x = x$. Finally, we will show that $T_1$ has a unique fixed. Assume that $y$ is an another fixed point of $T_1$ from (6.1), we get

$$\begin{aligned}
T(x,y) &= T(T_1x,T_1y)\\
&\leq \lambda\big(T(x,T_1x) + T(y,T_1y)\big)\\
&= K\big(T(x,x) + T(y,y)\big)
\end{aligned}$$

From (6.2), $T(x,y) = \theta$ and hence $x = y$. Therefore, $x$ is a unique point of $T_1$. This finishes the proof.

**Theorem 2.2** Assume that in a complex valued partial complete metric space $(X,T)$ with two self mappings $T_1: X \to X$ and $T_2: X \to X$, where the numbers are higher and equal to 1, there exists a function $\lambda: X \to \left[0, \frac{1}{2}\right)$ such that $\forall\ x,y \in X$ satisfies the following:

(a) $0 \leq \lambda(T_1x_n) < \frac{1}{2}$,

(b) $0 \leq \lambda(T_2x_n) < \frac{1}{2}$,

(c) $\frac{T(x,T_2y)\big(1-T(x,T_1y)\big)+T(y,T_1x)\big(1-T(y,T_2y)\big)}{T(x,T_1y)T(x,T_2y)+T(y,T_2y)T(y,T_1x)} \leq \frac{\lambda(x)-T(T_1x,T_2y)}{T(T_1x,T_2y)}$

(d) $T(T_1x,T_2y) \leq \frac{\lambda(x)\big(T(x,T_1y)T(x,T_2y)+T(y,T_2y)T(y,T_1x)\big)}{T(x,T_2y)+T(y,T_1x)}$

**Proof.** Since $T_1(X)$ and $T_2(X)$ are subsets of $X$, if we create the sequence $\{x_n\}$ for any point $x_0$ in order to $x_{n-1} = T_1x_{n-2}, x_n = T_2x_{n-1} \forall\ n \geq 0$.

$T(x_{n-1},x_n) = T(T_1x_{n-2},T_2x_{n-1})$

$$\leq \frac{\lambda(x_{n-2})\big(T(x_{n-2},T_1x_{n-2})T(x_{n-2},T_2x_{n-1})+T(x_{n-1},T_2x_{n-1})T(x_{n-1},T_1x_n-2)\big)}{T(x_n,T_2x_{n+1})+T(x_{n+1},T_1x_n)}$$

Thus, $T(x_{n-1},x_n)\leq\lambda(x_0)T(x_{n-2},x_{n-1})$
Setting, $\tau=\lambda(x_0)$
Then, $|T(x_{n-1},x_n)|\leq\tau^{n+1}|T(x_0,x_1)|$ (2.2.1)
Now, for $m>n\ \forall\ m,n\in N$
$|T(x_n,x_m)|\leq p|T(x_{n-1},x_n)|+p^2|T(x_n,x_{n+1})|+\cdots+p^{m-n}|T(x_{m-1},x_m)|$
From (2.2.1) we have
As $|T(x_n,x_m)|\leq\frac{(p\tau)^nT(x_0,x_1)}{1-p\tau}\to 0$ as $m,n\to 0$ as $\tau\in\left[0,\frac{1}{2}\right)$ and $p\geq\frac{1}{2}$. Since $X$ is complete, there exists $x\in X$ such that $x_n\to x$ as $n\to\infty$. Consequently, $\{x_n\}$ is a Cauchy sequence. Additionally, we demonstrate that $px=x$. If this isn't the case, then there is a $y$ such that $|T(x,px)|=|y|>0$ and $y=T(x,T_1x)\leq pT(x,x_n)+pT(T_1x,T_2x_{n-1})$
This means that $T(x,T_1x)\leq pT(x,x_n)+p\left(\frac{\lambda(x)(|T(x,T_1x)||T(x,x_n)|+|T(x_{n-1},x_n)||T(x_{n-1},T_1x)|)}{|T(x,x_n)|+|T(x_{n-1},T_1x)|}\right)$
Since $n\to\infty, |y|\leq 0$, this is logically impossible. Consequently, $|y|=0$. Thus, $T_1x=x$ similarly $T_2x=x$. In order to determine the uniqueness of $x$, let us assume that $z$ is an additional fixed point in both mappings. However, since

$$\frac{T(x,T_2y)\big(1-T(x,T_1y)\big)+T(y,T_1x)\big(1-T(y,T_2y)\big)}{T(x,T_1y)T(x,T_2y)+T(y,T_2y)T(y,T_1x)}\leq\frac{\lambda(x)-T(T_1x,T_2y)}{T(T_1x,T_2y)}$$

so, $$T(x,z)=T(T_1x,T_2z)\leq\frac{\lambda(x)\big(T(x,T_1x)T(x,T_2z)+T(z,T_2z)T(z,T_1z)\big)}{T(x,T_2z)+T(z,T_1x)}$$

This suggests that $x=z$ since $T(x,z)\leq 0$. As a result, $T(x,z)\leq 0$ is distinct from both $T_1$ and $T_2$. This completes the proof.

**Theorem 2.3** Let $T_1$ be a continuous mapping such that $T_1: X\to X$ satisfies the following requirements, and let $(X,T)$ be a complete rectangular M-metric space.

(a) $\frac{\epsilon}{\lambda}\leq S(x_{n+1},x_{n+2})<\frac{\epsilon+\delta}{\lambda}$,

(b) $T(T_1x_{n+1},T_1x_{n+1})<\epsilon$,

(c) $\frac{T(x_{n+1},x_{n+2})}{T(x_n,x_{n+1})}\leq 1$,

(d) $\lambda<\frac{1}{2}$ for some $\lambda\in\left]0,\frac{1}{4}\right[$.

Then, $x\in X$ is the unique fixed point of $T_1$. Furthermore, the sequence $\{T_nx\}$ converges to $x$ for every $x$ in $X$.

**Proof.** We note that $T$ is a rigorous contraction.

$$S(x_n,x_{n+1})=\frac{T(x_n,x_{n+1})+\big(T(x_n,x_{n+1})\big)^2+T(x_{n+1},x_{n+2})+T(x_n,x_{n+1})T(x_{n+1},x_{n+2})}{1+T(x_n,x_{n+1})}+\frac{T(x_n,x_{n+1})T(x_{n+1},x_{n+2})}{T(x_n,x_{n+1})}$$
$$\leq\lambda T(x_{n+1},x_{n+2})\left(\frac{T(x_n,x_{n+1})}{T(x_{n+1},x_{n+2})}+2\right)$$
$$T(x_{n+1},x_{n+2})-\frac{3}{4}T(T_1x_n,Tx_{n+1})=\frac{1}{4}T(T_1x_n,Tx_{n+1})$$
$$\leq\lambda S(x_n,x_{n+1})$$

$i.e.\ \frac{T(x_n,x_{n+1})-\frac{3}{4}T(x_n,x_{n+1})}{T(x_n,x_{n+1})+2T(x_n,x_{n+1})}\leq\lambda$ due to $\frac{T(x_{n+1},x_{n+2})}{T(x_n,x_{n+1})}\leq 1$
However, if $\lambda<\frac{1}{2}\ i.e.\ \frac{\lambda}{1-\lambda}<1$ and $\{x_n\}$ is a $T$-Cauchy sequence, then $T_1x_n=x_{n+1}\to x^*$ in $T$ for some $x^*\in X$ due to the completeness of $X$. $x_{n+1}=T_1x_n\to T_1x^*$ in $T$ since $T_1$ is a continuous mapping. Obviously

$$T(x^*,T_1x^*)=T_{x^*T_1x^*}$$
$$0=\lim_{n\to\infty}\big(T(x_{n+1},T_1x_{n+1})-T_{x_{n+1},T_1x_{n+1}}\big)$$
$$=T(x^*,x^*)-T_{x^*T_1x^*}$$
$$=T(T_1x^*,T_1x^*)-T_{x^*T_1x^*}$$
$$T(x^*,T_1x^*)=T_{x^*T_1x^*}=T(T_1x^*,T_1x^*)=T(x^*,x^*)$$

Thus, $x^*=T_1x^*$. Therefore, the uniqueness element is evident by contraction $T_1$.

## II. Concluding Remarks

This communication reveals that the rational contractive condition plays a crucial role in the existence of common fixed points. Furthermore, our fixed point theorems in cone metric spaces over Banach algebras are not equivalent to their counterparts in metric spaces, even when assuming that the cone is normal. In other words, our results are not simply copies of the classical results in metric spaces. Based on these observations, cone metric spaces over Banach algebras provide a more general framework compared to ordinary metric spaces.

## References:


[1]. **Asadi, M., Karapmar, E., Salimi, P. (2014):** New extension of P-metric spaces with some fixed-point results on M-metric spaces, Journal of Inequalities and Appl.

[2]. **Fernandez, J., Malviya, N. (2016):** Partial cone metric spaces over Banach algebra and Generalized Lipschitz mappings with applications, Selected for Young Scientist Award, M.P., India (accepted).

[3]. **Fernandez, J., Malviya, N., Djekic-Dolićanin, D. and Pučić, Dz. (2020):** The ps-cone metric spaces over Banach algebra with applications, Filomat, vol. 34, no. 3.

[4]. **Huang, L. G. and X. Zhang, X. (2007):** Cone metric spaces and fixed point theorems for contractive mappings, J. Math. Anal. Appl. 332(2) 1468-1476.

[5]. **Liu, H., Xu, S. (2013):** Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory Appl. 320.

[6]. **Matthews, S. G. (1994):** Partial metric topology, 8th Summer Conference on General topology and Appl. (1994)183-197.

[7]. **Rezapour, Sh., Hamlbarani, R. (2008):** Some notes on the paper, "Cone metric spaces and fixed point theorems of contractive mappings" J. Math. Anal. Appl. 345 (2008) 719-724.

[8]. **Verma, R. K. (2022):** Some Common Fixed Point Theorems in Metric and Menger Spaces Under Self and Compatible Mappings of Type (A) with Special Condition, Asian Journal of Pure and Applied Mathematics (AJPAM), 2022, 4(3), 616-623.

[9]. **Verma, R. K. (2023):** Fixed Point Theorems in Bi Two Metric Spaces and Quasi Partial Metric Spaces, International Journal of Pure and Applied Mathematical Sciences (IJPAMS), 2023, Vol. 16(1), pp. 9-15.

[10]. **Verma, R. K. (2023):** Fixed Point Theory for Self and Ciric Type Weak Quasi Contraction Mappings in Different Metric Spaces, Research Highlights in Mathematics and Computer Science (RHMCS), 2023, Vol. 6, pp. 118-126 DOI: 10.9734/bpi/rhmcs/v6/4641C.

[11]. **Verma, R. K. (2023):** On Some Fixed Point Results in Generalized Metric Space with Self mappings Under the Bounds, International Research Journal of Engineering and Technology (IRJET), 2023, Vol.10, Issue 08, pp. 35-39.