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Hilbert Graceful Labeling on Some Pendant Graphs

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ABSTRACT: *This paper investigates Hilbert graceful labeling, a structured variant of classical graceful labeling in which vertices are assigned Hilbert numbers of the form $H_n = 4(n-1)+1, n \geq 1$, and edge labels are induced by $H^*(uv) = \frac{1}{4}|H(u) - H(v)|$, producing a bijection onto $1,2, \ldots, q\}$. While graceful labeling has been widely studied, Hilbert graceful labeling, particularly for graphs with pendant vertices, remains relatively unexplored. We present constructive labeling methods for obtaining Hilbert graceful labeling of graphs containing pendant structures and demonstrate their applicability to several graph families, including cyclic, regular, complete, octahedron, tetrahedron, and square pyramid graphs. A case study is provided to illustrate the effectiveness of the proposed approach. This work contributes to the advancement of Hilbert graceful graph theory and expands the class of known Hilbert graceful graphs.*
Keywords: *Hilbert numbers, Hilbert graceful labeling, pendant graphs and graph labeling.*

I. Introduction

Graph theory is a fundamental branch of mathematics that provides essential tools for modeling structural relationships in complex systems. Among the various topics in graph theory, graph labeling has emerged as an important and dynamic research area. In particular, graceful labeling, introduced by Alexander Rosa in 1967, has attracted considerable attention due to its elegant combinatorial structure and wide range of applications. Over time, several variations of graceful labeling have been developed to address additional structural and arithmetic constraints.

One such structured variation is Hilbert graceful labeling, which is based on the sequence of Hilbert numbers defined by $H_n = 4(n-1)+1, n \geq 1$. Unlike classical graceful labeling, where vertex labels are taken from consecutive integers, Hilbert graceful labeling restricts vertex labels to Hilbert numbers. The induced edge labels are obtained by $H^*(uv) = \frac{1}{4}|H(u) - H(v)|$ where $uv \in E(G), u, v \in V(G)$, producing a bijection onto the set $\{1,2, \ldots, q\}$, where q denotes the number of edges of the graph. This additional arithmetic structure introduces new combinatorial challenges and enriches the theory of graph labeling.

Graph labeling is generally classified into vertex labeling and edge labeling. For a graph $G = (V, E)$, a vertex labeling is a function from $V(G)$ to a set of integers satisfying prescribed conditions, while an edge labeling is a function from $E(G)$ to a set of integers. In Hilbert graceful labeling, the vertex labeling determines the edge labeling uniquely through the scaled difference condition, ensuring distinct edge labels.

Among the numerous graph structures studied in labeling theory, graphs with pendant vertices-vertices of degree one is of particular interest. Pendant graphs arise naturally in both theoretical constructions and applied network models. While classical graceful labeling for pendant graphs has been widely investigated, the study of Hilbert graceful labeling for such graphs remains comparatively limited.

The present work focuses on Hilbert graceful labeling of certain pendant graph families. In particular, we examine cyclic graphs, tetrahedron graphs, and the complete graph K_4 with pendant extensions under the Hilbert graceful framework. By constructing explicit Hilbert graceful labeling for these graphs, we extend the class of known Hilbert graceful graphs and contribute to the growing body of research in structured graph labeling theory.

II. Preliminaries

Definition 1: A Cyclic graph C_3 is a cycle graph consisting of three vertices connected in a closed chain. Each vertex has degree 2, and the graph contains exactly three edges forming a triangle. In general, a cycle graph C_n is a connected graph in which every vertex has degree 2 and forms a single cycle [3] [16].

Definition 2: A Tetrahedron graph is the graph formed by the vertices and edges of a tetrahedron. It consists of four vertices, each adjacent to the other three vertices. The tetrahedron graph is isomorphic to the complete graph K_4 and is one of the Platonic graphs [16].

Definition 3: A graph is called a regular graph if every vertex has the same degree. If each vertex has degree r. the graph is said to be an r-regular graph. For example, a cycle graph C_n is 2-regular. Regular graphs are fundamental structures in graph theory due to their uniform degree distribution [3].

Definition 4: An Octahedron graph is the graph corresponding to the edges and vertices of a regular octahedron. It contains six vertices and twelve edges, with each vertex having degree 4. The Octahedron graph is also a Platonic graph and is 4-regular [16].

Definition 5: The Complete graph K_4 is a simple graph on four vertices in which every pair of distinct vertices is connected by an edge. It contains 6 edges, and each vertex has degree 3 [3].

Definition 6: A Square pyramid graph is the graph formed by the vertices and edges of a square pyramid. It consists of five vertices: four vertices forming a square base and one apex vertex connected to each base vertex. The graph contains eight edges and is not regular, since the apex has degree 4 while each base vertex has degree 3. It is a polyhedral graph derived from a convex polyhedron [16].

III. Results

Theorem 1: A Cyclic graph C_3 with $3n -$ pendent vertices is Hilbert graceful.

Proof: Let G be a Cyclic graph C_3 with $3n -$ pendent vertices.

$V(G) = \{v_i : 1 \le i \le 3\} \cup \{u_{i,k} : 1 \le i \le 3 ; 1 \le k \le n\}$

$E(G) = \{v_i v_{i+1} : 1 \le i \le 2\} \cup \{v_3 v_1\} \cup \{v_i u_{i,k} : 1 \le i \le 3 ; 1 \le k \le n\}$

$|V(G)| = 3n + 3$ and $|E(G)| = 3n + 3$.

We define a vertex labeling function $f : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ as follows:

$f(v_1) = 1$

$f(v_2) = 12n + 13$

$f(v_3) = 5$

$f(u_{1,k}) = 5 + 4k$ for $1 \le k \le n$

$f(u_{2,k}) = 4n + 5 + 4k$ for $1 \le k \le n$

$f(u_{3,k}) = 8n + 9 + 4k$ for $1 \le k \le n$

The above labeling pattern, we find that the function f is injective (one-to-one)

The induced function $f^* : E(G) \to \{1, 2, 3, \ldots, q\}$ defined by $f^*(uv) = \frac{1}{4}|f(u) - f(v)|$.

$f^*(v_1 v_2) = 3n + 3$

$f^*(v_2 v_3) = 3n + 2$

$f^*(v_3 v_1) = 1$

$f^*(v_1 u_{1,k}) = 1 + 4k$ for $1 \le k \le n$

$f^*(v_2 u_{1,k}) = 2n + 2 - k$ for $1 \le k \le n$

$f^*(v_3 u_{1,k}) = 2n + 1 + k$ for $1 \le k \le n$

By Above we can observe that f^* is bijective.

Hence, f is a Hilbert graceful labeling, and therefore, the Cyclic graph C_3 with $3n -$ pendent vertices is Hilbert graceful graph.

Theorem 2: A Tetrahedron graph with $3n -$ pendent vertices is Hilbert graceful.

Proof: Let G be a Tetrahedron graph with $3n -$ pendent vertices.

$V(G) = \{v_i : 1 \le i \le 4\} \cup \{u_{i,k} : 1 \le i \le 3 ; 1 \le k \le n\}$

$E(G) = \{v_i v_{i+1} : 1 \le i \le 2\} \cup \{v_3 v_1\} \cup \{v_i v_4 : 1 \le i \le 3\} \cup \{v_i u_{i,k} : 1 \le i \le 3 ; 1 \le k \le n\}$

$|V(G)| = 3n + 4$ and $|E(G)| = 3n + 6$.

We define a vertex labeling function $f : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ as follows:

$f(v_1) = 1$

$f(v_2) = 12n + 21$

$f(v_3) = 9$

$f(v_4) = 12n + 25$

$f(u_{1,k}) = 9 + 4k$ for $1 \le k \le n$

$f(u_{2,k}) = 4n + 13 + 4k$ for $1 \le k \le n$

$f(u_{3,k}) = 8n + 25 + 4k$ for $1 \leq k \leq n$

The above labeling pattern, we find that the function f is injective (one-to-one)

The induced function $f^*: E(G) \rightarrow \{1, 2, 3, \ldots, q\}$ defined by $f^*(uv) = \frac{1}{4}|f(u) - f(v)|$.

$f^*(v_1v_2) = 3n + 5$
$f^*(v_2v_3) = 3n + 3$
$f^*(v_3v_1) = 2$
$f^*(v_1v_4) = 3n + 6$
$f^*(v_2v_4) = 1$
$f^*(v_3v_4) = 3n + 4$
$f^*(v_1u_{1,k}) = 2 + k$ for $1 \leq k \leq n$
$f^*(v_2u_{1,k}) = 2n + 2 - k$ for $1 \leq k \leq n$
$f^*(v_3u_{1,k}) = 2n + 4 + k$ for $1 \leq k \leq n$

By Above we can observe that f^* is bijective.

Hence, f is a Hilbert graceful labeling, and therefore, the Tetrahedron graph with $3n -$ pendent vertices is Hilbert graceful graph.

Theorem 3: A 3-regular graph with $3n -$ pendent vertices is Hilbert graceful.

Proof: Let G be a 3-regular graph with $3n -$ pendent vertices.

$V(G) = \{v_i : 1 \leq i \leq 6\} \cup \{u_{i,k} : 1 \leq i \leq 3\, ; 1 \leq k \leq n\}$
$E(G) = \{v_iv_{i+1} : 1 \leq i \leq 2\} \cup \{v_3v_1\} \cup \{v_{i+3}v_{i+4} : 1 \leq i \leq 2\}$
$\cup \{v_6v_4\} \cup \{v_1v_4\} \cup \{v_2v_6\} \cup \{v_3v_5\} \cup \{v_iu_{i,k} : 1 \leq i \leq 3\, ; 1 \leq k \leq n\}$
$|V(G)| = 3n + 6$ and $|E(G)| = 3n + 9$.

We define a vertex labeling function $f: V(G) \rightarrow \{x : x = 4(i-1) + 1, 1 \leq i \leq 2q\}$ as follows:

$f(v_1) = 12n + 29$
$f(v_2) = 21$
$f(v_3) = 9$
$f(v_4) = 1$
$f(v_5) = 12n + 33$
$f(v_6) = 12n + 37$
$f(u_{1,k}) = 13 + 12k$ for $1 \leq k \leq n$
$f(u_{2,k}) = 17 + 12k$ for $1 \leq k \leq n$
$f(u_{3,k}) = 21 + 12k$ for $1 \leq k \leq n$

The above labeling pattern, we find that the function f is injective (one-to-one)

The induced function $f^*: E(G) \rightarrow \{1, 2, 3, \ldots, q\}$ defined by $f^*(uv) = \frac{1}{4}|f(u) - f(v)|$.

$f^*(v_1v_2) = 3n + 2$
$f^*(v_2v_3) = 3$
$f^*(v_3v_1) = 3n + 5$
$f^*(v_4v_5) = 3n + 8$
$f^*(v_5v_6) = 1$
$f^*(v_6v_4) = 3n + 9$
$f^*(v_1v_4) = 3n + 7$
$f^*(v_2v_6) = 3n + 4$
$f^*(v_3v_5) = 3n + 6$
$f^*(v_1u_{1,k}) = 3n + 4 - 3k$ for $1 \leq k \leq n$
$f^*(v_2u_{1,k}) = 3k - 1$ for $1 \leq k \leq n$
$f^*(v_3u_{1,k}) = 3k - 3$ for $1 \leq k \leq n$

By Above we can observe that f^* is bijective.

Hence, f is a Hilbert graceful labeling, and therefore, the 3-regular graph with $3n -$ pendent vertices is Hilbert graceful graph.

Theorem 4: An Octahedron graph with $3n-$ pendent vertices is Hilbert graceful.
Proof: Let G be a Octahedron graph with $3n-$ pendent vertices.
$V(G)=\{v_i: 1\leq i\leq 6\}\cup\{u_{i,k}: 1\leq i\leq 3\,; 1\leq k\leq n\}$
$E(G)=\{v_iv_{i+1}: 1\leq i\leq 2\}\cup\{v_3v_1\}\cup\{v_{i+3}v_{i+4}: 1\leq i\leq 2\}\cup\{v_6v_1\}$
$\cup\{v_1v_4\}\cup\{v_1v_6\}\cup\{v_2v_4\}\cup\{v_2v_5\}\cup\{v_3v_5\}\cup\{v_3v_6\}$
$|V(G)|=3n+6$ and $|E(G)|=3n+12$.
We define a vertex labeling function $f: V(G)\to\{x: x=4(i-1)\ +1, 1\leq i\leq 2q\}$ as follows:
$f(v_1)=12n+29$
$f(v_2)=12n+49$
$f(v_3)=17$
$f(v_4)=5$
$f(v_5)=12n+53$
$f(v_6)=12n+45$
$f(u_{1,k})=13+12k$ for $1\leq k\leq n$
$f(u_{2,k})=21+12k$ for $1\leq k\leq n$
$f(u_{3,k})=37+12k$ for $1\leq k\leq n$
The above labeling pattern, we find that the function f is injective (one-to-one)
The induced function $f^*: E(G)\to\{1,2,3,\ldots,q\}$ defined by $f^*(uv)\ =\frac{1}{4}|f(u)-f(v)|$.
$f^*(v_1v_2)=5$
$f^*(v_2v_3)=3n+8$
$f^*(v_3v_1)=3n+3$
$f^*(v_4v_5)=3n+12$
$f^*(v_5v_6)=2$
$f^*(v_6v_4)=3n+10$
$f^*(v_1v_4)=3n+6$
$f^*(v_1v_6)=4$
$f^*(v_2v_4)=3n+11$
$f^*(v_2v_5)=1$
$f^*(v_3v_5)=3n+9$
$f^*(v_3v_6)=3n+7$
$f^*(v_1u_{1,k})=3n+4-12k$ for $1\leq k\leq n$
$f^*(v_2u_{1,k})=3n+7-12k$ for $1\leq k\leq n$
$f^*(v_3u_{1,k})=5+12k$ for $1\leq k\leq n$
By Above we can observe that f^* is bijective.
Hence, f is a Hilbert graceful labeling, and therefore, the Octahedron graph with $3n-$ pendent vertices is Hilbert graceful graph.

Theorem 5: A Complete graph K_4 with $4n-$ pendent vertices is Hilbert graceful.
Proof: Let G be a Complete graph K_4 with $4n-$ pendent vertices.
$V(G)=\{v_i: 1\leq i\leq 4\}\cup\{u_{i,k}: 1\leq i\leq 4\,; 1\leq k\leq n\}$
$E(G)=\{v_iv_{i+1}: 1\leq i\leq 3\}\cup\{v_4v_1\}\cup\{v_iv_{i+2}: 1\leq i\leq 2\}$
$|V(G)|=4n+4$ and $|E(G)|=4n+6$.
We define a vertex labeling function $f: V(G)\to\{x: x=4(i-1)\ +1, 1\leq i\leq 2q\}$ as follows:
$f(v_1)=5$
$f(v_2)=16n+25$
$f(v_3)=16n+29$
$f(v_4)=13$
$f(u_{1,k})=13+4k$ for $1\leq k\leq n$

$f(u_{2,k}) = 4n + 13 + 4k$ for $1 \le k \le n$
$f(u_{3,k}) = 8n + 17 + 4k$ for $1 \le k \le n$
$f(u_{4,k}) = 12n + 21 + 4k$ for $1 \le k \le n$
The above labeling pattern, we find that the function f is injective (one-to-one)
The induced function $f^*: E(G) \to \{1, 2, 3, \dots, q\}$ defined by $f^*(uv) = \frac{1}{4}|f(u) - f(v)|$.
$f^*(v_1v_2) = 4n + 5$
$f^*(v_2v_3) = 1$
$f^*(v_3v_4) = 4n + 4$
$f^*(v_4v_1) = 2$
$f^*(v_1v_3) = 4n + 6$
$f^*(v_2v_4) = 4n + 3$
$f^*(v_1u_{1,k}) = 2 + k$ for $1 \le k \le n$
$f^*(v_2u_{2,k}) = 3n + 3 - k$ for $1 \le k \le n$
$f^*(v_3u_{3,k}) = 2n + 3 - k$ for $1 \le k \le n$
$f^*(v_4u_{4,k}) = 3n + 2 + k$ for $1 \le k \le n$
By Above we can observe that f^* is bijective.
Hence, f is a Hilbert graceful labeling, and therefore, the Complete graph K_4 with $4n-$ pendent vertices is Hilbert graceful graph.

Theorem 6: A Square pyramid graph with $4n-$ pendent vertices is Hilbert graceful.
Proof: Let G be a Square pyramid graph with $4n-$ pendent vertices.
$V(G) = \{v_i : 1 \le i \le 5\} \cup \{u_{i,k} : 1 \le i \le 4\,; 1 \le k \le n\}$
$E(G) = \{v_iv_{i+1} : 1 \le i \le 3\} \cup \{v_4v_1\} \cup \{v_iv_5 : 1 \le i \le 4\}$
$|V(G)| = 4n + 5$ and $|E(G)| = 4n + 8$.
We define a vertex labeling function $f: V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ as follows:
$f(v_1) = 16n + 29$
$f(v_2) = 16n + 25$
$f(v_3) = 16n + 33$
$f(v_4) = 13$
$f(v_5) = 1$
$f(u_{1,k}) = 1 + 16k$ for $1 \le k \le n$
$f(u_{2,k}) = 5 + 16k$ for $1 \le k \le n$
$f(u_{3,k}) = 9 + 16k$ for $1 \le k \le n$
$f(u_{4,k}) = 13 + 16k$ for $1 \le k \le n$
The above labeling pattern, we find that the function f is injective (one-to-one)
The induced function $f^*: E(G) \to \{1, 2, 3, \dots, q\}$ defined by $f^*(uv) = \frac{1}{4}|f(u) - f(v)|$.
$f^*(v_1v_2) = 1$
$f^*(v_2v_3) = 2$
$f^*(v_3v_4) = 5$
$f^*(v_4v_1) = 4n + 4$
$f^*(v_1v_5) = 4n + 7$
$f^*(v_2v_5) = 4n + 6$
$f^*(v_3v_5) = 4n + 8$
$f^*(v_4v_5) = 3$
$f^*(v_1u_{1,k}) = 4n + 7 - 4k$ for $1 \le k \le n$
$f^*(v_2u_{2,k}) = 4n + 5 - 4k$ for $1 \le k \le n$
$f^*(v_3u_{3,k}) = 4n + 6 - 4k$ for $1 \le k \le n$
$f^*(v_4u_{4,k}) = 4k$ for $1 \le k \le n$

By Above we can observe that f^* is bijective.

Hence, f is a Hilbert graceful labeling, and therefore, the Square pyramid graph with $4n-$ pendent vertices is Hilbert graceful graph.

IV. Conclusion

In this paper, we have investigated the concept of Hilbert graceful labeling in the context of certain pendant graph families. By restricting vertex labels to Hilbert numbers of the form $H_n = 4(n-1)+1, n \geq 1$ and defining the induced edge labeling through a scaled difference condition, we established a structured framework that extends classical graceful labeling.

Explicit Hilbert graceful labeling were constructed for selected graphs, including cyclic graphs, tetrahedron graphs, regular graphs, octahedron graphs, the complete graph K_4, and square pyramid graphs with pendant extensions. The results demonstrate that these graph families admit Hilbert graceful labeling under suitable constructions, thereby enlarging the class of known Hilbert graceful graphs. The presence of pendant vertices, often considered structurally restrictive, was shown to be compatible with the Hilbert labeling scheme.

This study contributes to the growing body of research on structured graph labeling by introducing systematic labeling methods tailored to Hilbert numbers. The findings not only strengthen the theoretical foundation of Hilbert graceful labeling but also open new directions for further investigation. Future work may focus on extending these results to broader classes of graphs, exploring necessary and sufficient conditions for Hilbert gracefulness, and examining potential applications in combinatorial design and network modeling.

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