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An Exploration of Prime Cordial Labeling of Octopus and its Related Graphs

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ABSTRACT*:A prime cordial labeling of G with vertex set V(G) is a bijection f:V(G)→{1,2,…,|V(G)|} such that each edge uv is assigned the label 1 if gcd(f(u),f(v))=1 and 0 if gcd(f(u),f(v))>1, then the number of edges labelled with 1 and the number of edges labelled with 0 differ by atmost 1. A graph which admits prime cordial labeling is called prime cordial graph. In this paper, we prove that the octopus graph and its related graphs such as switching any vertex in octopus, joining two copies of octopus graph with path graph of arbitrary length, corona product of octopus with K_1, duplication of any edge by a vertex in octopus are prime cordial graph.*

KEYWORDS*: Prime cordial labeling, Octopus graph, Switching a vertex, Corona product, Duplication of an edge by a vertex.*

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I. INTRODUCTION

Graph labeling is a significant area of graph theory, with elegant labeling this will being one of its most impactful variants. For a graph G with q edges, a graph labeling is an assignment of labels to edges, vertices or both. Cahit. I [2] introduced the concept of cordial labeling in 1987. Then after cordial labeling was extended to divisor cordial labeling, sum divisor cordial labeling, prime cordial labeling, total cordial labeling, etc., The present work is focused on prime corial labeling. Which was introduced by Sundaram et al [14]. Sugumaran and Prakash [13] proved that one point union of path of Theta graphs, open star of Theta graphs and path union of even copies of Theta graphs are prime cordial graphs. Sugumaran and Mohan [10] have proved prime cordial labeling of the graphs such as $W-$ graph, butterfly graph, $H-$ graph and duplication of edges of an$H-$ graph. Also the Same author [11] have proved that the graph such as $H \odot K_1$, path union of r copies of H- graph, cycle union of r copies of H- graph and open star of r copies of $H-$ graph are prime cordial graphs.

An excellent survey of graph labeling and various types of graph labeling can be found in Gallian [3].

In this paper, we discuss the prime cordial labeling of octopus graphs and some graphs obtained by applying some operations like joining two copies of octopus with path graph of arbitrary length, switching any vertex of an octopus graph, corona product of octopus graph with complete graph K_1, duplication of any edge by a vertex in octopus graph.

II. PRELIMINARIES

We consider simple, connected and undirected graph $G=(V(G),E(G))$ with p vertices and q edges.
For all undefined terminologies, readers may refer to [3, 6, 7]. We will provide a brief summary of definitions, which are necessary for the present investigation.

Definition 1.1. A mapping $f:V(G)\rightarrow\{0,1\}$ is called binary vertex labeling of graph G and $f(v)$ is called the label of the vertex v of Gunder$f$.

Definition 1.2. A binary vertex labeling f of a graph G is called a cordial labeling if$\left|v_f(0)-v_f(1)\right|\leq 1$and $\left|e_f(0)-e_f(1)\right|\leq 1$, where $v_f(i)$ =number of vertices G having label i and $e_f(i)$ = number of edges G having label i.

Definition 1.3. A prime cordial labeling of G with vertex set $V(G)$ is a bijection $f:V(G)\rightarrow\{1,2,\dots,|V(G)|\}$ such that each edge uv is assigned the label 1 if $\gcd(f(u),f(v))=1$ and 0 if $\gcd(f(u),f(v))>1$, then the number of edges labelled with 1 and the number of edges labelled with 0 differ by atmost 1. A graph which admits prime cordial labeling is called prime cordial graph.

Definition 1.4. An octopus graph $O_n (n \geq 2)$ can be constructed by a fan graph $F_n (n \geq 2)$ joining a star graph $K_{1,n}$ with sharing a common vertex. i.e., $O_n = F_n + K_{1,n}$.

Definition 1.5.A vertex switching G_v of a graph G is obtained by taking a vertex v of G, removing all the entire edges incident with v and adding edges joining v to every edge which are not adjacent to v in G.

Definition 1.6.In graph theory, the corona product of graphs G and H, denoted $G \odot H$, can be obtained by taking one copy of G, and a number of copies of H equal to the order of G. Then, each copy of H is assigned a vertex in G, and that one vertex is attached to each vertex in its corresponding H copy by an edge.

Definition 1.7. Duplication of an edge $e = uv$ by a vertex v' in a graph G is a new graph G' where $V(G') = V(G) \cup \{v'\}$ and $E(G') = E(G) \cup \{uv', v'v\}$.

III. MAIN RESULTS

Theorem 2.1.The octopus graph O_n is prime cordial graph.
Proof. Let $G = O_n$ be any octopus graph.
Let $V(G) = \{v_i \,/\, 0 \leq i \leq 2n\}$and $E(G) = \{v_i v_{i+1} \;/\; 1 \leq i \leq n-1\} \cup \{v_0 v_i \;/\; 1 \leq i \leq 2n\}$.
The vertex labeling function $f: V(G) \rightarrow \{1,2, \dots, |V(G)|\}$ defined as follows.
$f(v_o) = 2,$

$$f(v_i) = 2(i+1), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1$$

$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 3, \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq \left\lceil \frac{3n+3}{2} \right\rceil$

$f(v_i) = 2\left\lceil \frac{n}{2} \right\rceil + 2i - 2\left\lceil \frac{3n+5}{2} \right\rceil + 6, \left\lceil \frac{3n+5}{2} \right\rceil \leq i \leq 2n$

The labeling function $f$defined as above is one-one, as there is no repeated vertex labels. It is easy to check that the edge labeling function $f^*: E(G) \rightarrow\{0,1\}$ is onto. Also $f$satisfies the condition $\left|e_f(0) - e_f(1)\right| \leq 1$. Thus, f is a prime cordial labeling for given graph.
Therefore, the Octopus graph O_n is prime cordial graph.

Illustration 2.2. Prime Cordiallabeling ofO_6 is shown in the following Figure 1.

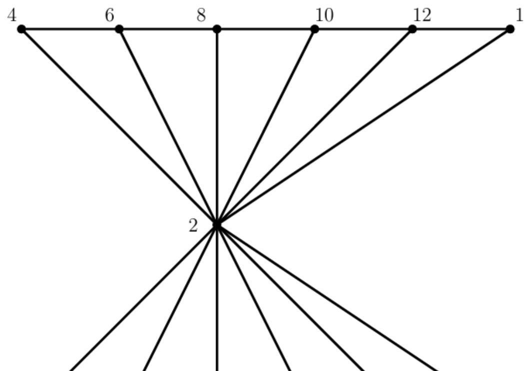

Figure 1. Prime cordial labeling of O_6

Theorem 2.3.The Graph G obtained by joining two copies of octopus graph O_n by a path of arbitrary length is prime cordial graph.
Proof. Let G be the graph obtained by joining two copies of any octopus graphO_n by a path P_k oflengh k-1. Let $u_0, u_1, \dots, u_{2n}$ be vertices of first copy and $w_0, w_1, \dots, w_{2n}$ be vertices of second copyof octopus graph respectively. Let $v_1, v_2, \dots, v_k$ be vertices of path graph P_k with $u_n = v_1$and $v_k = w_1$.
Therefore,$V(G) = \{u_i, w_i \,/\; 0 \leq i \leq 2n\} \cup \{v_j \;/1 \leq j \leq k\}$ and
$E(G) = \{u_i u_{i+1}, w_i w_{i+1} /1 \leq i \leq n-1\} \cup \{u_0 u_i, w_0 w_i \;/\; 1 \leq i \leq 2n\} \cup$

$\{v_i v_{i+1} \,/\, 1 \le i \le k-1, v_1 = u_n, v_k = w_1\}$.

The vertex labeling function $f: V(G) \to \{1, 2, \ldots, |V(G)|\}$ defined as follows.

$$f(v_0) = 2, f(w_0) = 1, f(u_1) = 4,$$

$f(u_i) = 2(i+2), 1 \le i \le n$

$f(u_i) = 2\left(\left\lfloor \frac{k}{2} \right\rfloor + i\right), n+1 \le i \le 2n$

$f(w_i) = 2\left(k - \left\lfloor \frac{k}{2} \right\rfloor + i\right) - 1, 1 \le i \le 2n$

$f(v_i) = 2(n+i+1), 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor - 1$

$f\left(v_{\left\lfloor \frac{k}{2} \right\rfloor}\right) = 6,$

$f(v_i) = 1 + 2i - 2\left\lfloor \frac{k}{2} \right\rfloor, \left\lfloor \frac{k}{2} \right\rfloor + 1 \le i \le k$

The labeling function *f*defined as above is one-one, as there is no repeated vertex labels. It is easy to check that the edge labeling function $f^* : E(G) \to \{0,1\}$ is onto. Also *f*satisfies the condition $|e_f(0) - e_f(1)| \le 1$. Thus, f is a prime cordial labeling for given graph.
Therefore, the graph obtained by joining two copies of octopus graph O_n by a path of arbitrary length is prime cordial graph.

Illustration 2.4. Prime cordial labeling of the graph obtained by joining two copies of the octopus graph O_5 by a path P_4 is shown in the following Figure 2.

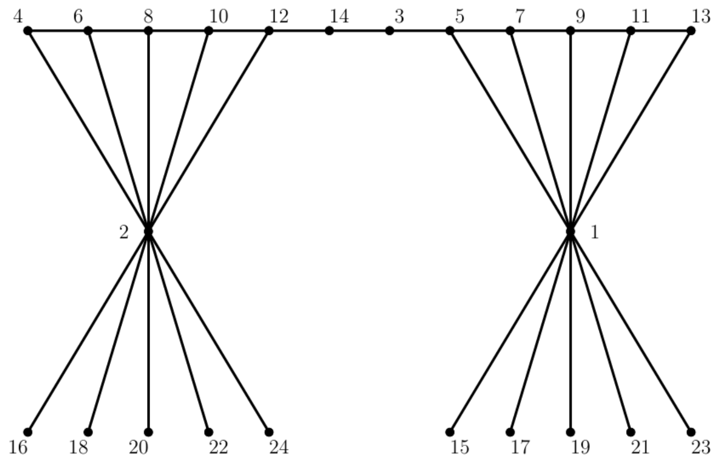

Figure 2. Prime cordial labeling of the graph obtained by joining two copies of the octopus graph O_5 by a path P_4

Theorem 2.5.The graph obtained by switching any vertex in octopus graph O_n is prime cordial graph ($\forall n \ge 4$).
Proof. Let $G = O_n$ be any octopus graph.
Let $V(G) = \{v_i \,/\, 0 \le i \le 2n\}$and $E(G) = \{v_0 v_i \,/\, 1 \le i \le 2n\} \cup \{v_i v_{i+1} \,/\, 1 \le i \le n-1\}$.
Let G_v denotes the graph obtained by switching of a vertex v.Observe that in any octopus graph O_n there are two vertices of deg 2 say $v_1$and v_n, $n-2$ vertices of deg 3 say $v_2, v_3, \ldots, v_{n-1}$, one vertex of deg 2n say v_0 and n pendent vertices. So, on the basis of degree of vertices we have following four cases.

(a) Switching a vertex of deg 2 in octopus graph.

Without loss of generality let the switched vertex be v_1. Therefore, $V(G_{v_1}) = \{ v_i \,/\, 0 \le i \le 2n\}$ and
$E(G_{v_1}) = \{v_0 v_i / \, 2 \le i \le 2n\} \cup \{v_i v_{i+1} \,/\, 2 \le i \le n-1\} \cup \{v_1 v_i \,/\, 3 \le i \le 2n\}$.
To obtain vertex labeling function f: $V(G_{v_1}) \to \{1, 2, \ldots, |V(G_{v_1})|\}$, we take following cases.

Case-1: $n = 2$ & 3

For n= 2 & 3 we can easily check that in all possible arrangement of vertex labels we have
$|e_f(0) - e_f(1)| > 1$. Thus, $G_{v_1}$is not a prime cordial graph.

Case-2: $n = 4$

$f(v_0) = 6, f(v_1) = 1, f(v_2) = 2, f(v_3) = 4, f(v_4) = 8, f(v_5) = 3, f(v_6) = 5, f(v_7) = 7, f(v_8) = 9$

Case-3: $\boldsymbol{n} = \mathbf{5}$ & 6
$f(v_0) = 6, f(v_1) = 2, f(v_2) = 1, f(v_3) = 4,$
$f(v_i) = 2i, 4 \leq i \leq n$
$f(v_i) = 2i - 2n + 1, n + 1 \leq i \leq 2n$
Case-4: $\boldsymbol{n} = \mathbf{7}, \mathbf{8}, \mathbf{9}$ & 10
$f(v_0) = 2, f(v_1) = 4, f(v_2) = 5,$
$f(v_i) = 2i, 3 \leq i \leq n$
$f(v_{n+1}) = 1, f(v_{n+2}) = 3,$
$f(v_i) = 2i - 2n + 1, n + 3 \leq i \leq 2n$
Case-5: $\boldsymbol{n} \geq \mathbf{11}$
$f(v_0) = 2,$
$f(v_i) = 2(i+1), 1 \leq i \leq \left\lceil \frac{n+9}{2} \right\rceil$
$f(v_i) = 1 + 2\left(i - \left\lceil \frac{n+11}{2} \right\rceil\right), \left\lceil \frac{n+11}{2} \right\rceil \leq i \leq \left\lceil \frac{3n+11}{2} \right\rceil$
$f(v_i) = 2(i-n), \left\lceil \frac{3n+13}{2} \right\rceil \leq i \leq 2n$
It is easy to check that *f* satisfies the condition $\left|e_f(0) - e_f(1)\right| \leq 1$ in above all cases except $n = 2$ & 3.

Illustration 2.6 Prime cordial labeling of graph obtained by switching v_1 vertex in O_9 is shown in the following Figure 3.

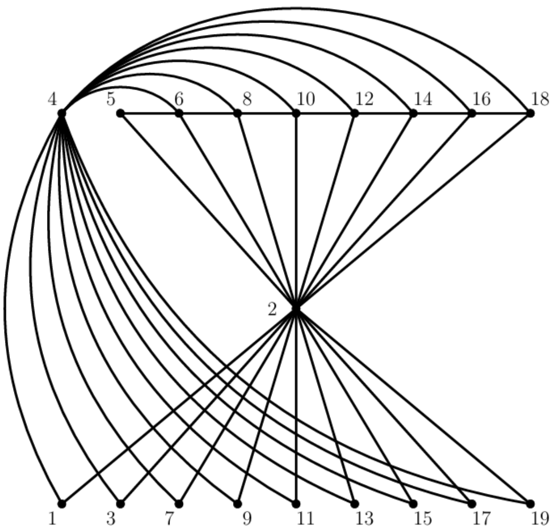

Figure 3. Prime cordial labeling of the graph obtained by switching v_1 vertex in O_9

(b) Switching a vertex of degree 2n in octopus graph
If we switch a vertex of degree 2n in octopus graph then the case become trivial. And in this case we simply give label as follows.
$f(v_i) = 2i, 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$
$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$
It is easy to check that *f* satisfies the condition $\left|e_f(0) - e_f(1)\right| \leq 1$.
(c) Switching any pendent vertex in octopus graph
Without loss of generality let the switched vertex be v_{n+1}. Therefore, $V(G_{v_{n+1}}) = \{v_i \,/\, 0 \leq i \leq 2n\}$ and $E(G_{v_{n+1}}) = \{\{v_0 v_i / 1 \leq i \leq 2n\} \setminus \{v_0 v_{n+1}\}\} \cup \{v_i v_{i+1} \,/\, 1 \leq i \leq n-1\} \cup \{v_{n+1} v_i \,/\, 1 \leq i \leq 2n\}$.
To obtain vertex labeling function $f: V(G_{v_{n+1}}) \rightarrow \{1, 2, \ldots, |V(G_{v_{n+1}})|\}$, we take following cases.
Case-1: $\boldsymbol{n} = \mathbf{2}$ & 3
For n = 2 & 3 we can easily check that in all possible arrangement of vertex labels we have $\left|e_f(0) - e_f(1)\right| > 1$. Thus, $G_{v_{n+1}}$ is not a prime cordial graph.
Case-2: $\boldsymbol{n} = \mathbf{4}$
$f(v_0) = 6, f(v_1) = 4, f(v_2) = 8, f(v_3) = 3, f(v_4) = 9, f(v_5) = 2, f(v_6) = 1, f(v_7) = 5, f(v_8) = 7$
Case-3: $\boldsymbol{n} = \mathbf{5}$

$f(v_0) = 6, f(v_1) = 4, f(v_2) = 8, f(v_3) = 10, f(v_4) = 3, f(v_5) = 9, f(v_6) = 2, f(v_7) = 1, f(v_8) = 5,$
$f(v_9) = 7, f(v_{10}) = 11$

Case-4: $n = 6$

$f(v_0) = 2, f(v_1) = 3, f(v_2) = 6, f(v_3) = 8, f(v_4) = 10, f(v_5) = 12, f(v_6) = 9, f(v_7) = 4, f(v_8) = 1,$
$f(v_9) = 5, f(v_{10}) = 7, f(v_{11}) = 11, f(v_{12}) = 13.$

Case-5: $n = 7$

$f(v_0) = 2, f(v_1) = 3, f(v_2) = 6, f(v_3) = 8, f(v_4) = 10, f(v_5) = 12, f(v_6) = 14, f(v_7) = 7, f(v_8) =$
$4, f(v_9) = 1, f(v_{10}) = 5, f(v_{11}) = 9, f(v_{12}) = 11, f(v_{13}) = 13, f(v_{14}) = 15.$

Case-6: $n = 8$ & 9

$f(v_0) = 2, f(v_1) = 3,$
$f(v_i) = 2(i + 1), 2 \le i \le n - 1$
$f(v_i) = 2i - 2n + 1, n + 2 \le i \le 2n$

Case-7: $n = 10, 11$ & 12

$f(v_0) = 2,$
$f(v_i) = 2(i + 2), 1 \le i \le n - 2$
$f(v_i) = 3 + 2i - 2n, n - 1 \le i \le 2n$

Case-8: $n \ge 13$

$f(v_0) = 2,$
$f(v_i) = 2(i + 2), 1 \le i \le \left\lceil \frac{n+7}{2} \right\rceil$
$f(v_i) = 1 + 2\left(i - \left\lceil \frac{n+9}{2} \right\rceil\right), \left\lceil \frac{n+9}{2} \right\rceil \le i \le \left\lceil \frac{3n+11}{2} \right\rceil$
$f(v_i) = 2(i - n), \left\lceil \frac{3n+13}{2} \right\rceil \le i \le 2n$

It is easy to check that *f*satisfies the condition $|e_f(0) - e_f(1)| \le 1$ in above all cases except $n = 2$ & 3.

Illustration 2.7Primecordial labeling of graph obtained by switching a pendent vertexv_{11} in $O_{10}$is shown in the following Figure 4.

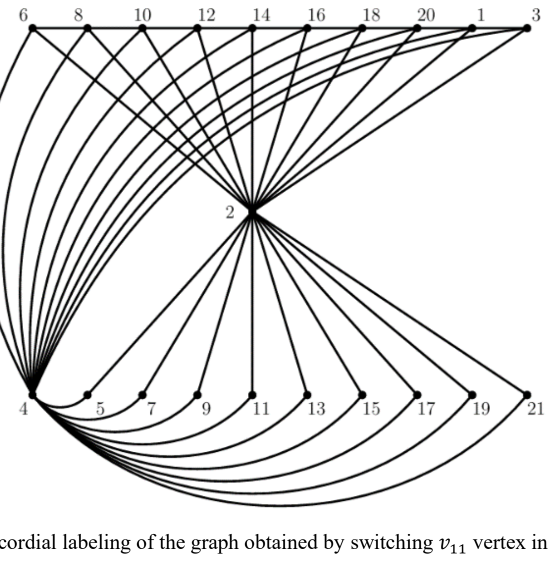

Figure 4. Prime cordial labeling of the graph obtained by switching v_{11} vertex in O_{10}

(d) Switching any vertex of degree 3 in octopus graph

Due function pattern we have following two subcases for the vertices of degree 3.

Subcase-I: Switching$v_2$vertex in octopus graph.

Without loss of generality let the switched vertex be v_2. Therefore, $V(G_{v_2}) = \{v_i \,/\, 0 \le i \le 2n\}$ and $E(G_{v_2}) = \{\{v_0v_i / 1 \le i \le 2n\} \setminus \{v_0v_2\}\} \cup \{v_iv_{i+1} \,/\, 3 \le i \le n-1\} \cup \{v_2v_i \,/\, 4 \le i \le 2n\}$.

Observe that by the symmetries v_2 simply repsresentative of v_{n-1}.

To obtain vertex labeling function f: $V(G_{v_2}) \rightarrow \{1, 2, ..., \ |V(G_{v_2})|\}$, we take following cases.

Case-1: $n = 2$ & 3

For n = 2 & 3 we can easily check that in all possible arrangement of vertex labels we have

$|e_f(0) - e_f(1)| > 1$. Thus, G_{v_2} is not a prime cordial graph.
Case-2: $\boldsymbol{n = 4}$ & 5
$f(v_0) = 6, f(v_1) = 9, f(v_2) = 2, f(v_3) = 4,$
$f(v_i) = 2i, 4 \leq i \leq n$
$f(v_i) = 2i - 2n - 1, n + 1 \leq i \leq 2n$
Case-3: $\boldsymbol{n = 6}$ & 7
$f(v_0) = 4, f(v_1) = 1, f(v_2) = 2, f(v_3) = 3,$
$f(v_i) = 2(i - 1), 4 \leq i \leq n$
$f(v_i) = 3 + 2i - 2n, n + 1 \leq i \leq 2n$
Case-4: $\boldsymbol{n = 8, 9}$ & 10
$f(v_0) = 2, f(v_1) = q,$
$f(v_i) = 2i, 2 \leq i \leq n$
$f(v_i) = 2i - 2n - 1, n + 1 \leq i \leq 2n$
Case-5: $\boldsymbol{n \geq 11}$
$f(v_0) = 2, f(v_1) = q,$
$f(v_i) = 2i, 2 \leq i \leq \left\lceil \frac{n+9}{2} \right\rceil$
$f(v_i) = 2i - 2n + 1, \left\lceil \frac{n+11}{2} \right\rceil \leq i \leq \left\lceil \frac{3n+9}{2} \right\rceil$
$f(v_i) = 2(i - n), \left\lceil \frac{3n+11}{2} \right\rceil \leq i \leq 2n$
It is easy to check that f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ in above all cases except $n = 2$ & 3.

Illustration 2.8. Prime cordial labeling of graph obtained by switching v_2 vertex in O_{11} is shown in the following Figure 5.

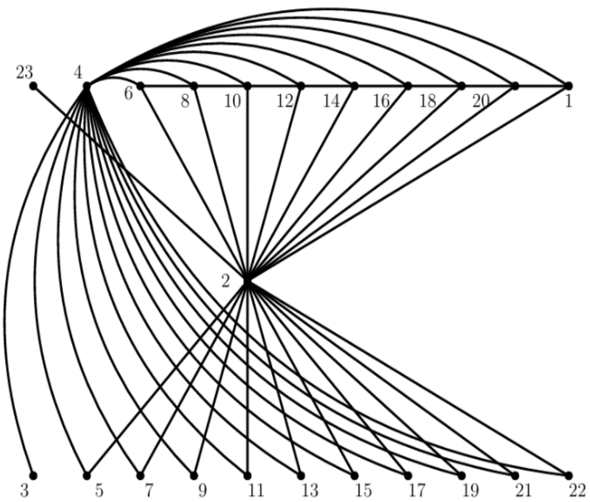

Figure 5. Prime cordial labeling of the graph obtained by switching v_2 vertex in O_{11}

Subcase-II: Switching v_3 vertex in octopus graph.
Without loss of generality let the switched vertex be v_3. Therefore, $V(G_{v_3}) = \{v_i \ / \ 0 \leq i \leq 2n\}$ and
$E(G_{v_3}) = \{\{v_0 v_i / 1 \leq i \leq 2n\} \setminus \{v_0 v_3\}\} \cup \{v_i v_{i+1} \ / \ 4 \leq i \leq n - 1\} \cup \{v_3 v_i \ / \ 5 \leq i \leq 2n\} \cup \{v_1 v_2\} \cup \{v_1 v_3\}$.
Observe that by the symmetries v_3 simply representative of $v_4, v_5, \dots, v_{n-2}$.
To obtain vertex labeling function $f: V(G_{v_3}) \rightarrow \{1, 2, \dots, \ |V(G_{v_3})|\}$, we take following cases.

Case-1: $\boldsymbol{n = 2}$ & 3
For n = 2 & 3 we can easily check that in all possible arrangement of vertex labels we have
$|e_f(0) - e_f(1)| > 1$. Thus, G_{v_3} is not a prime cordial graph.
Case-1: $\boldsymbol{n = 4}$
$f(v_0) = 6, f(v_1) = 4, f(v_2) = 8, f(v_3) = 2, f(v_4) = 9, f(v_5) = 1, f(v_6) = 3, f(v_7) = 5, f(v_8) = 7$
Case-2: $\boldsymbol{n = 5}$ & 6
$f(v_0) = 6, f(v_1) = 4, f(v_2) = 8, f(v_3) = 2, f(v_4) = 5, f(v_i) = 2i, 5 \leq i \leq n,$

$f(v_{n+1}) = 1, f(v_{n+2}) = 3, f(v_i) = 2i - 2n + 1, n + 3 \leq i \leq 2n$

Case-3: $\boldsymbol{n = 7}$

$f(v_0) = 6, f(v_1) = 2, f(v_2) = 3, f(v_3) = 4, f(v_4) = 8, f(v_5) = 10, f(v_6) = 12, f(v_7) = 15, f(v_8) = 1,$
$f(v_9) = 5, f(v_{10}) = 7, f(v_{11}) = 9, f(v_{12}) = 11, f(v_{13}) = 13, f(v_{14}) = 14$

Case-4: $\boldsymbol{n = 8, 9}$ & 10

$f(v_0) = 2, f(v_1) = 6, f(v_2) = 3, f(v_3) = 4,$
$f(v_i) = 2i, 4 \leq i \leq n,$
$f(v_{n+1}) = 1,$
$f(v_i) = 3 + 2i - 2n, n + 2 \leq i \leq 2n$

Case-5: $\boldsymbol{n = 11}$ & 12

$f(v_0) = 2, f(v_1) = 6, f(v_2) = 1, f(v_3) = 4,$
$f(v_i) = 2i, 4 \leq i \leq n,$
$f(v_i) = 1 + 2i - 2n, n + 1 \leq i \leq 2n$

Case-6: $\boldsymbol{n \geq 13}$

$f(v_0) = 2, f(v_3) = 4,$
$f(v_i) = 2(i + 2), 1, 2, 4 \leq i \leq \left\lceil \frac{n + 11}{2} \right\rceil$
$f(v_i) = 1 + 2\left(i - \left\lceil \frac{n + 13}{2} \right\rceil\right), \left\lceil \frac{n + 13}{2} \right\rceil \leq i \leq \left\lceil \frac{3n + 13}{2} \right\rceil$
$f(v_i) = 2(i - n), \left\lceil \frac{3n + 15}{2} \right\rceil \leq i \leq 2n$

It is easy to check that f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ in above all cases except $n = 2$ & 3.

Illustration 2.9. Prime cordial labeling of graph obtained by switching v_3 vertex in O_{11} is shown in the following Figure 6.

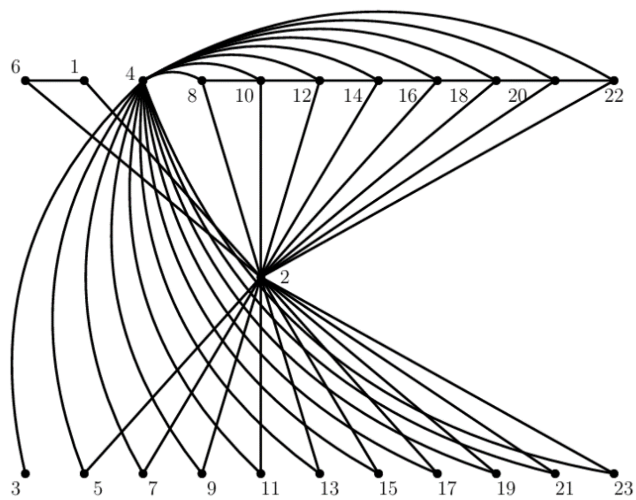

Figure 6. Prime cordial labeling of graph obtained by switching v_3 vertex in O_{11}

In all above cases the labeling function f defined as above is one-one, as there is no repeated vertex labels. Also it is easy to check that the edge labeling function $f^* : E(G) \rightarrow \{0,1\}$ is onto and f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. Thus, f is a prime cordial labeling for given graph for all $n \geq 4$.

Therefore, the graph obtained by switching any vertex in octopus graph O_n is prime cordial graph, $\forall n \geq 4$.

Theorem 2.10. The corona product of octopus graph O_n with complete graph K_1 (i.e., $O_n \odot K_1$) is prime cordial graph.

Proof. Let $G = O_n \odot K_1$ be a corona product of any octopus graph O_n with complete graph K_1.

Let $V(O_n) = \{v_i / 0 \leq i \leq 2n\}$ and $E(O_n) = \{v_0 v_i / 1 \leq i \leq 2n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n - 1\}$.

Therefore, $V(G) = \{v_i, v_i' / 0 \leq i \leq 2n\}$ and

$$E(G) = \{v_0 v_i / 1 \leq i \leq 2n\} \cup \{v_i v_i' / 0 \leq i \leq 2n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n - 1\}$$

The vertex labeling function $f: V(G) \rightarrow \{1, 2, \ldots, |V(G)| = p\}$ defined as follows.

$f(v_0) = 2, f(v_0') = p - 1,$

$$f(v_i) = 2(i+1), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1$$

$f(v_i) = 4i - 4\left\lceil \frac{n}{2} \right\rceil - 7, \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n$

$f(v_i) = 2\left\lceil \frac{n}{2} \right\rceil + 2i - 2n + 4, n + 1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor - 1$

$f(v_i) = 4i - 3 - 4n, n + \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq 2n$

$f(v_i') = 2(n + i + 1), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1$

$f(v_i') = 4i - 4\left\lceil \frac{n}{2} \right\rceil - 5, \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n$

$f(v_i') = 2(n + i - 5), n + 1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor - 1$

$f(v_i') = 4i - 4n - 1, n + \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq 2n$

The labeling function *f*defined as above is one-one, as there is no repeated vertex labels. It is easy to check that the edge labeling function $f^* : E(G) \rightarrow \{0,1\}$ is onto. Also *f*satisfies the condition $\left|e_f(0) - e_f(1)\right| \leq 1$. Thus, f is a prime cordial labeling for given graph.
Therefore, the graph $O_n \odot K_1$ is a prime cordial graph.

Illustration 2.11. Prime cordial labeling of $O_9 \odot K_1$ is shown in following Figure 7.

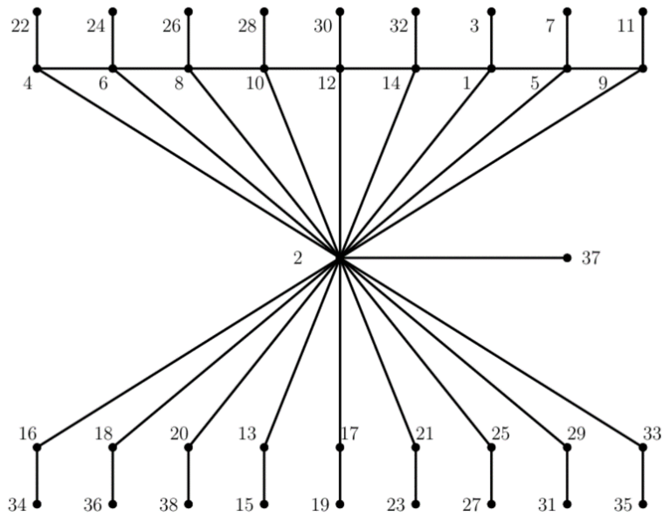

Figure 7.Prime cordial labeling of $O_9 \odot K_1$

Theorem 2.12.The graph obtained by duplication of any edge by a vertex in octopus graph O_n is prime cordial graph($n \geq 2$).
Proof. Let $G = O_n$ be any octopus graph.
Let $V(G) = \{v_i \ / \ 0 \leq i \leq 2n\}$ and $E(G) = \{v_0 v_i \ / \ 1 \leq i \leq 2n\} \cup \{v_i v_{i+1} \ / \ 1 \leq i \leq n-1\}$.
For simplicity we take edges as $e_i = (v_0, v_i)$where $i = 1, 2, .., n-1, n+1, ..., 2n$and $e_j' = (v_j, v_{j+1})$ where $j = 1, 2, ..., n-1$. For each edge e_i and e_j' we assume that the vertex formed by duplication is v'.We have following three cases to cover all edge duplication by a vertex in a graph G.

(a) Duplication of a pendent edge by a vertex in octopus graph.

We duplicate any of the $\{e_{n+1}, e_{n+2}, ..., e_{2n}\}$ pendent edges by a vertex in octopus then we have new graph say $G' = (V(G'), E(G'))$.Withoutloss of generality we assume that we duplicate edge $e_{n+1} = (v_0, v_{n+1})$.
Observe that $V(G') = \{v_0 \ / \ 0 \leq i \leq 2n\} \cup \{v'\}$ and
$E(G') = \{v_0 v_i \ / \ 1 \leq i \leq 2n\} \cup \{v_i v_{i+1} \ / \ 1 \leq i \leq n-1\} \} \cup \{v' v_{n+1}, v' v_0\}$.
To obtain vertex labeling function f: V(G')→{1,2, ..., |V(G')|}, we take following cases.

Case-1: $n = 2$ & 3
$f(v_0) = 2$,
$f(v_i) = 2(i+1), 1 \leq i \leq n$,

$$f(v_{n+1}) = 5, f(v_{n+2}) = 1, f(v_{n+3}) = 7$$

$f(v') = 3$

Case-2: $\boldsymbol{n} = \mathbf{4}$ & 5
$f(v_0) = 2,$
$f(v_i) = 2(i+1), 1 \leq i \leq n-1,$
$$f(v_n) = 1, f(v_{n+1}) = 5,$$
$f(v_i) = 2i - 2n + 3, n+2 \leq i \leq \left\lfloor \frac{3n+3}{2} \right\rfloor$
$f(v_{2n}) = 2(n+1)$
$f(v') = 3$
Case-3: $\boldsymbol{n} \geq \mathbf{6}$
$f(v_0) = 2, f(v') = 3,$
$f(v_i) = 2(i+1), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1,$
$$f\left(v_{\left\lceil \frac{n}{2} \right\rceil + 2}\right) = 1,$$
$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil + 1, \left\lceil \frac{n}{2} \right\rceil + 3 \leq i \leq n,$
$f(v_{n+1}) = 5,$
$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 1, n+2 \leq i \leq \left\lfloor \frac{3n+3}{2} \right\rfloor$
$f(v_i) = 2(i-n+1), \left\lfloor \frac{3n+5}{2} \right\rfloor \leq i \leq 2n$
It is easy to check that *f* satisfies the condition $\left| e_f(0) - e_f(1) \right| \leq 1$ in this case.

Illustration 2.13. Prime cordial labeling of graph obtained by duplication of edge e_9 by a vertex in 0_8 is shown in the following Figure 8.

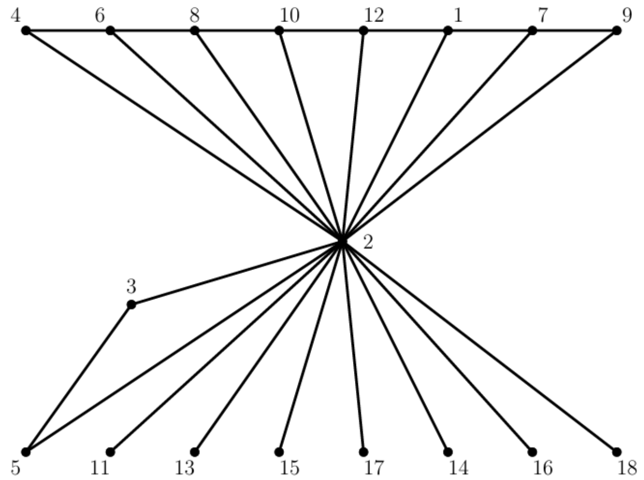

Figure 8. Prime cordial labeling of graph obtained by duplication of e_9 by a vertex in O_8

(b) Duplication of an edge $e_k = (v_o, v_k)$ $(1 \leq k \leq n)$ by a vertex in octopus graph.
We duplicate any of the $\{e_1, e_2, \dots, e_n\}$ edges by a vertex in octopus then we have new graph say $G' = (V(G'), E(G'))$.
Observe that $V(G') = \{v_0 \,/\, 0 \leq i \leq 2n\} \cup \{v'\}$ and
$E(G') = \{v_0 v_i \,/\, 1 \leq i \leq 2n\} \cup \{v_i v_{i+1} \,/\, 1 \leq i \leq n-1\} \} \cup \{v' v_k, v' v_0\}$
To obtain vertex labeling function f: V(G')→{1,2, ..., |V(G')|}, we take following cases.
Case-1: $\boldsymbol{n} = \mathbf{3}$
$f(v_0) = 2,$
$f(v_i) = 2(i+1), 1 \leq i \leq 3,$
$$f(v_4) = 1,$$
$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 1, if\ i = 5\ \&\ 6$
$f(v') = \begin{cases} 3 & \text{if we duplicate edge } e_1 \text{ or } e_3 \\ 5 & \text{if we duplicate edge } e_2 \end{cases}$
Case-2: $\boldsymbol{n} \neq \mathbf{3}$

$f(v_0) = 2, f(v') = 1,$
$f(v_i) = 2\left(\left\lceil\frac{n}{2}\right\rceil - i + 3\right), 1 \leq i \leq \left\lceil\frac{n}{2}\right\rceil + 1,$
$$f(v_i) = 2i - 2\left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil + 2 \leq i \leq \left\lfloor\frac{3n+3}{2}\right\rfloor$$

$f(v_i) = 2(i - n + 1), \left\lfloor\frac{3n+5}{2}\right\rfloor \leq i \leq 2n$

It is easy to check that *f*satisfies the condition $\left|e_f(0) - e_f(1)\right| \leq 1$ in this case.

Illustration 2.14. Prime cordial labeling of graph obtained by duplication of edge e_1 by a vertex in $O_7$is shown in the following Figure 9.

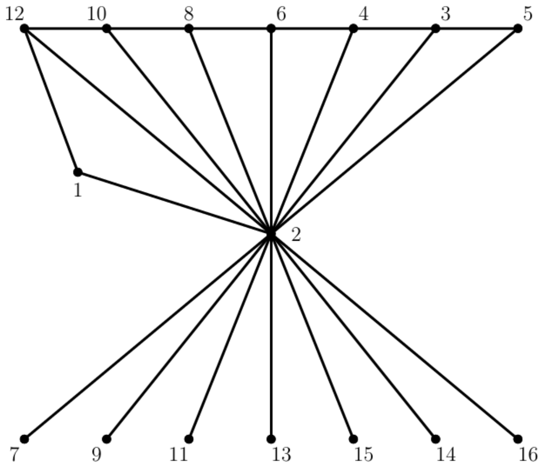

Figure 9. Prime cordial labeling of graph obtained by duplication of $e_1$by a vertex in O_7

(c) Duplication of an edge $e'_k = (v_o, v_k)$ $(1 \leq k \leq n-1)$ by a vertex in octopus graph.
In this case we duplicate any of the $\{e'_1, e'_2, \dots, e'_{n-1}\}$ edges by a vertex in octopus then we have new graph say $G' = \left(V(G'), E(G')\right)$.Therefore,$V(G') = \{v_0 \,/\, 0 \leq i \leq 2n\} \cup \{v'\}$ and
$$E(G') = \{v_0v_i \,/\, 1 \leq i \leq 2n\} \cup \{v_iv_{i+1} \,/\, 1 \leq i \leq n-1\} \cup \{v'v_k, v'v_{k+1}\}.$$
To obtain vertex labeling function f: V(G')→{1,2, ..., |V(G')|}, we take following cases.

Case-1: $n \leq 5$

$f(v_0) = 2, f(v') = 1,$
$f(v_i) = 2(i+1), 1 \leq i \leq \left\lceil\frac{n}{2}\right\rceil + 1$
$f(v_i) = 2i - 2\left\lceil\frac{n}{2}\right\rceil - 1, \left\lceil\frac{n}{2}\right\rceil + 2 \leq i \leq \left\lfloor\frac{3n+3}{2}\right\rfloor$
$$f(v_i) = 2(n+1), \left\lfloor\frac{3n+5}{2}\right\rfloor \leq i \leq 2n$$
Illustration 2.15. Prime cordial labeling of graph obtained by duplication of edge e'_4 by a vertex in $O_5$is shown in the following Figure 10.

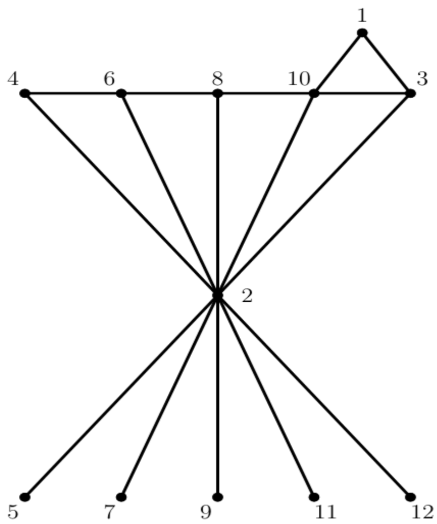

Figure 10. Prime cordial labeling of graph obtained by duplication of $e'_4$by a vertex in O_5

Case-2: $n \geq 6$
Due to function pattern we have following two subcases.
Subcase-2(I): Duplicate any edge $e'_k \left(1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil\right)$ by a vertex in O_n

$f(v_0) = 2, f(v') = 6, f(v_1) = 4,$

$f(v_i) = 4 + 2i, 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$

$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 1, \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq \left\lfloor \frac{3n+3}{2} \right\rfloor$

$$f(v_i) = 2(i - n + 1), \left\lfloor \frac{3n+5}{2} \right\rfloor \leq i \leq 2n$$

Subcase-2(II): Duplicate an edge $e'_{\left\lceil \frac{n}{2} \right\rceil+1}$ by a vertex in O_n

$f(v_0) = 2, f(v') = 6, f(v_1) = 4,$

$f(v_i) = 4 + 2i, 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil+1$

$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 3, \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq \left\lceil \frac{3n+3}{2} \right\rceil$

$$f(v_i) = 2(i - n + 1), \left\lceil \frac{3n+5}{2} \right\rceil \leq i \leq 2n$$

Subcase-2(III): Duplicate any edge $e'_k \left(\left\lceil \frac{n}{2} \right\rceil + 2 \leq k \leq n\right)$ by a vertex in O_n

$f(v_0) = 2,$

$f(v_i) = 2(i + 1), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1$

$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 3, \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq k$

$f(v') = 2k - 2\left\lceil \frac{n}{2} \right\rceil - 1$

$f(v_i) = 2i - 2\left\lceil \frac{n}{2} \right\rceil - 1, k + 1 \leq i \leq \left\lfloor \frac{3n+3}{2} \right\rfloor$

$$f(v_i) = 2(i - n + 1), \left\lfloor \frac{3n+5}{2} \right\rfloor \leq i \leq 2n$$

Illustration 2.16. Prime cordial labeling of graph obtained by duplication of edge e'_4 by a vertex in $O_6$is shown in the following Figure 11.

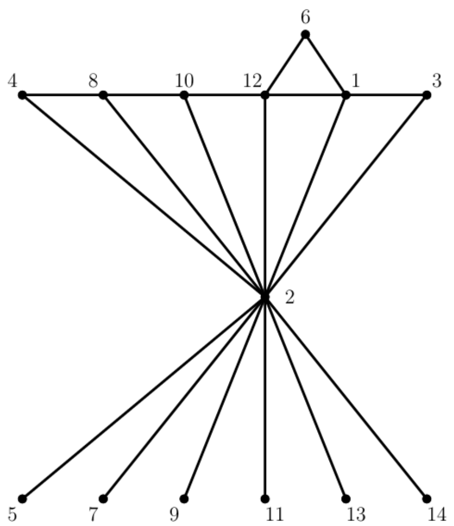

Figure 11. Prime cordial labeling of graph obtained by duplication of $e'_4$by a vertex in O_6

In all above cases the labeling function *f*defined as above is one-one, as there is no repeated vertex labels. Also it is easy to check that the edge labeling function $f^* : E(G) \rightarrow \{0,1\}$ is onto and *f*satisfies the condition $\left|e_f(0) - e_f(1)\right| \leq 1$. Thus, *f* is a prime cordial labeling for given graph for all $n \geq 2$.

Therefore, the graph obtained by duplication of any edge by vertex in octopus graph $O_n$is prime cordial graph, $\forall n \geq 2$.

IV. CONCLUSION

In this work, we investigate the octopus graph and its related graph such as joining two copies of octopus graph with path of arbitrary length, switching any vertex of an octopus graph, $O_n \odot K_1$, duplication of any edge by a vertexin $O_n$are prime cordial graphs. Extending our results to various other graph operations related to octopus graph is an open area of research.

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