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Inversion Formula of Generalized Aboodh-Finite Mellin Transform in the Distributional Sense

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***ABSTRACT**: In the field of Mathematics and Engineering, there are several integral transforms and many researchers worked on it. Integral transforms are broadly used in various areas of applied science and engineering for finding solution of problem. The Mellin transform is a most important integral transform to solve differential problem in mathematics and physics. A new transform Aboodh transform arises as an alternative to solve differential equation in recent days.*

This paper, presents the generalization of Aboodh-Finite Mellin Transform in the distributional sense. We first present definition of generalized Aboodh-Finite Mellin Transform, define some testing function spaces. The main aim of this paper is to prove Inversion theorem and Uniqueness theorem which will be useful to solve differential and integral equations.

***KEYWORDS**: Aboodh Transform, Finite Mellin Transform, Aboodh-Finite Mellin Transform, Generalized functions, Testing functions Space, Inversion theorem, Uniqueness theorem.*

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I. INTRODUCTION

The role of integral transform is well-established, particularly through their application in applied mathematics, mathematical physics and engineering science [1]. Mellin Transform is introduced by Robert Hjalmar Mellin in 1854-1933. Mellin Transform has many applications in medical field, agriculture and quantum calculus [2]. Also, important use in solution of fractional differential equation [3], to derive different properties in statistics and probability densities of single continues random variable [4]. Aboodh Transform (AT) is named in the honor of Khalid Suliman Aboodh in 2013 [5]. It is designed to simplify the process of solving ordinary and partial differential equation in time domain. Many authors studied on integral transforms extending double transformation. B.N. Bhosale and M.S. Choudhary [6] and S.M. Khairnar et.al.[7] has discussed double transform and their application. So, we have introduced a new combination of integral transform namely Aboodh-Finite Mellin Transform with definition and its analytical study in distributional generalized sense is already proved.

We present and prove the Inversion Theorem and Uniqueness Theorem for the Distributional Aboodh-Finite Mellin Transform.For the generalization of Aboodh-Finite Mellin Transform and for its inversion, various testing function spaces are required, which we have been defined and discussed in this paper by Gelfand-Shilov technique.

The outline of the paper is as follows: Testing function Spaces are defined in section 2. In section 3 definition of Distributional generalized Aboodh-Finite Mellin Transform is given. In section 4 main course of this paper that is Inversion Theorem is proved. In section 5 conclusions are given. The notations and terminology as per A. H. Zemanian [9], [10].

II. TESTING FUNCTION SPACES

2.1 The Space $AM_{f,a,b,c,d,\alpha}$

Let I be the open sets in $R_+ \times R_+$ and E_+ denote the class of infinitely differentiable function defined on I, the space $AM_{f,a,b,c,d,\alpha}$ is given by,

$$\mathrm{A}M_{f,a,b,c,d,\alpha} = \left\{ \phi : \phi \in E_+/\gamma_{a,b,c,d,q,l}\ \phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| K_{a,b}(t)\, \lambda_{c,d}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \phi(t,x) \right| \le C_{lq}\, A^a\, a^{a\alpha} \right\}$$

for each $l, q = 0,1,2,3, \ldots$

where $K_{a,b}(\mathrm{t}) = \begin{cases} e^{at}, & 0 \le t < \infty \\ e^{bt}, & -\infty < t < 0 \end{cases}$

$\lambda_{c,d}(\mathrm{x}) = \begin{cases} x^{+c}, & 0 < x < 1 \\ x^{+d}, & 1 < x < a \end{cases}$ are the kernels for testing function space of Aboodh-Finite Mellin Transform respectively.

Also, where the constants A and C_{lq} depend on the testing function ϕ.

2.2 The Space $\mathrm{A}M^{\beta}_{f,a,b,c,d,\alpha}$

The space $\mathrm{A}M^{\beta}_{f,a,b,c,d,\alpha}$ is given by,

$$\mathrm{A}M^{\beta}_{f,a,b,c,d,\alpha} = \left\{ \phi : \phi \in E_+/\rho_{a,b,c,d,q,l}\ \phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| K_{a,b}(t)\, \lambda_{c,d}(x)\, x^{q+1}\, D_t^l\, D_x^q\, \phi(t,x) \right| \le C A^a a^{a\alpha} B^l\, l^{l\beta} \right\}$$

where the constants A, B and C depend on the testing function ϕ.

III. DISTRIBUTIONAL GENERALIZED ABOODH-FINITE MELLIN TRANSFORM (AM_fT)

For $f(t,x) \in \mathrm{A}M^{*\beta}_{f,a,b,c,d,\alpha}$, where $\mathrm{A}M^{*\beta}_{f,a,b,c,d,\alpha}$ is the dual space of $\mathrm{A}M^{\beta}_{f,a,b,c,d,\alpha}$. It contains all distributions of compact support. The distributional Aboodh-Finite Mellin Transform is a function of $f(t,x)$ is defined as

$$\mathrm{A}M_f\{f(t,x)\} = F(s,p) = \left\langle f(t,x), \frac{1}{s} e^{-st} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \quad (3.1)$$

where, for each fixed $t\ (0 < t < \infty), x(0 < x < a), s > 0$ and $p > 0$, the right-hand side of (3.1) has a sense as an application of $f(t,x) \in \mathrm{A}M^{*\beta}_{f,a,b,c,d,\alpha}$ *to* $\frac{1}{s} e^{-st} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \in \mathrm{A}M^{\beta}_{f,a,b,c,d,\alpha}$

IV. INVERSION THEOREM FOR ABOODH-FINITE MELLIN TRANSFORM

4.1LEMMA 1

STATEMENT: Let $AM_f\{f(t,x)\} = F(s,p)$ and $\sup pf \subset S_A \cap S_B$,where $S_A = \{t : t \in R^n, |t| \le A, A > 0\}$ and $S_B = \{x : x \in R^n, |x| \le B, B > 0\}$, for $s > 0$ and $\rho_1 < Re\, p < \rho_2$.

Let $\phi \in D$ and $\psi(s,p) = \int_{-\infty}^{\infty} \int_0^a \phi(t,x)\, s e^{st} x^{-p} dt dx$

Then for any fixed real number τ *and* r with $-\infty < r < \infty$,$0 < \tau < a$,

$$\int_{-r}^{r} \int_{-\tau}^{\tau} \left\langle f(t,x), \frac{e^{-st}}{s} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \psi(s,p) ds dw = \left\langle f(t,x), \int_{-r}^{r} \int_{-\tau}^{\tau} \frac{e^{-st}}{s} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s,p) ds dw \right\rangle$$

Where $p = \rho + i\omega$, also s and p are fixed with $\sigma_1 < s < \sigma_2$ and $\rho_1 < p < \rho_2$.

Proof: - For $\phi(t,x) \equiv 0$, the result is trivial, so assume that $\phi(t,x) \neq 0$.

If $AM_f f(t,x) = F(s,p)$, then $F(s,p)$ is analytic for $s > 0$, $\rho_1 < Re\, p < \rho_2$ and $\psi(s,p)$ is an entire function. Therefore, above integrals certainly exist.

In order that right hand side is meaningful, we show that

$$\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)dsdw\in AM_{f,a,b,c,d,\alpha}$$

Consider,

$$\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l\,D_x^q\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)dsdw\right|$$

$$\leq\int_{-r}^{r}\int_{-\tau}^{\tau}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l\,D_x^q\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)\right|dsdw$$

$$\leq\int_{-r}^{r}\int_{-\tau}^{\tau}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}(-s)^l\frac{e^{-st}}{s}\{a^{2p}P(-p-q)x^{-p-q-1}-P(p-q)x^{p-q-1}\}\psi(s,p)\right|dsdw$$

$$\leq\int_{-r}^{r}\int_{-\tau}^{\tau}\left|K_{a,b}(t)\lambda_{c,d}(x)s^{l-1}e^{-st}\{a^{2p}P(-p)x^{-p}-P(p)x^{p}\}\psi(s,p)\right|dsdw$$

where P*(p)* is a polynomial in *p*

$$\leq\int_{-r}^{r}\int_{-\tau}^{\tau}\left|K_{a,b}(t)\lambda_{c,d}(x)s^{l-1}e^{-st}\left\{\left(\frac{a}{x}\right)^{2p}P(-p)x^{p}-P(p)x^{p}\right\}\psi(s,p)\right|dsdw$$

$$\leq\int_{-r}^{r}\int_{-\tau}^{\tau}\left|K_{a,b}(t)\lambda_{c,d}(x)s^{l-1}e^{-st}\left\{\left(\frac{a}{x}\right)^{2(\rho+i\omega)}P(-p)-P(p)\right\}x^{(\rho+i\omega)}\psi(s,p)\right|dsdw<\infty \quad (4.1.1)$$

$$\Rightarrow\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)dsdw\in AM_{\text{f,a,b,c,d,α}}$$

Partition the path of integration on the straight line from $s=-r\ to\ s=r$ into m-intervals, each of length $\frac{2r}{m}$

and from $p=\rho-i\tau$ to$p=\rho+i\tau$into n-intervals, each of length $\frac{2\tau}{n}$.

Let $s_\upsilon=\sigma$ be any point in υ^{th} interval and $p_\mu=\rho+i\omega$ be any point in μ^{th} interval.

Suppose,

$$\phi_{m,n}(t,x)=\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\frac{e^{-s_\upsilon t}}{s_\upsilon}\left(\frac{a^{2p_\mu}}{x^{p_\mu+1}}-x^{p_\mu-1}\right)\psi(s_\upsilon,p_\mu)\frac{2r}{m}\frac{2\tau}{n} \quad (4.1.2)$$

To show that $\phi_{m,n}(t,x)$ converges in $AM_{f,a,b,c,d,\alpha}$ to $\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)dsdw$

We have to show that $\phi_{m,n}(t,x)-\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)dsdw$, converges to zero in $AM_{f,a,b,c,d,\alpha}\ as\ m,n\to\infty$.

We write,

$$\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l\,D_x^q\left[\phi_{m,n}(t,x)-\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\psi(s,p)dsdw\right]\right|$$

$$= \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l\, D_x^q \left[\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n} \frac{e^{-s_\upsilon t}}{s_\upsilon}\left(\frac{a^{2p_\mu}}{x^{p_\mu+1}} - x^{p_\mu-1}\right)\psi(s_\upsilon,p_\mu)\frac{2r}{m}\frac{2\tau}{n} - \int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s,p)dsdw \right]\right|$$

$$= \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}(-1)^l(s_\upsilon)^l\frac{e^{-s_\upsilon t}}{s_\upsilon}\{a^{2p_\mu}P(-p_\mu-q)x^{-p_\mu-q-1} - P(-p_\mu-q)x^{p_\mu-q-1}\}\psi(s_\upsilon,p_\mu)\frac{2r}{m}\frac{2\tau}{n} - K_{a,b}(t)\lambda_{c,d}(x)x^{q+1} \int_{-r}^{r}\int_{-\tau}^{\tau}(s)^l\frac{e^{-st}}{s}\{a^{2p}P(-p-q)x^{-p-q-1} - P(-p-q)x^{p-q-1}\}\psi(s,p)dsdw \right|$$

where $P(-p_\mu - q)$is a polynomial in p_μ

$$= \left| K_{a,b}(t)\lambda_{c,d}(x)\left[\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}(s_\upsilon)^{l-1}e^{-s_\upsilon t}x^{p_\mu}\left\{\left(\frac{a}{x}\right)^{2p_\mu}P(-p_\mu) - P(-p_\mu)\right\}\psi(s_\upsilon,p_\mu)\frac{2r}{m}\frac{2\tau}{n}\right] - K_{a,b}(t)\lambda_{c,d}(x)\left[\int_{-r}^{r}\int_{-\tau}^{\tau}(s)^{l-1}e^{-st}\,x^p\left\{\left(\frac{a}{x}\right)^{2p}P(-p) - P(p)\right\}\psi\,(s,p)dsdw\right]\right| \quad (4.1.3)$$

Since $\int_{-r}^{r}\int_{-\tau}^{\tau}(s)^{l-1}\left\{\left(\frac{a}{x}\right)^{2p}P(-p) - P(p)\right\}\psi(s,p)dsdw$ *is finite and*

$\left|K_{a,b}(t)\lambda_{c,d}(x)e^{-st}x^p\right| \to 0$, *For sufficiently large values of x and t.*

Given any $\epsilon > 0$,we have chosen x_0 *and* t_0. So large that for $x>x_0$and $t>t_0$,

$$\left| K_{a,b}(t)\lambda_{c,d}(x)\int_{-r}^{r}\int_{-\tau}^{\tau}(s)^{l-1}\,e^{-st}x^p\left\{\left(\frac{a}{x}\right)^{2p}P(-p) - P(p)\right\}\psi(s,p)dsdw\right| < \frac{\epsilon}{3} \quad (4.1.4)$$

Now consider the first term of , choosing m_0 and n_0 so large that for $m > m_0$ and $n > n_0$,

$$\left| K_{a,b}(t)\lambda_{c,d}(x)\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}e^{-s_\upsilon t}x^{p_\mu}\left\{\left(\frac{a}{x}\right)^{2p_\mu}P(-p_\mu) - P(-p_\mu)\right\}\psi(s_\upsilon,p_\mu)\right| < \frac{2\epsilon}{3} \quad (4.1.5)$$

For all $x>x_0$, $t>t_0$

In view of above in equations (4.1.3), (4.1.4) and (4.1.5),

$$\left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l\, D_x^q\left[\phi_{m,n}(t,x) - \int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s,p)dsdw\right]\right| < \epsilon$$

$$\Rightarrow \phi_{m,n}(t,x)\ converges\ to \int_{-r}^{r}\int_{-\tau}^{\tau}\frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s,p)dsdw\ in\ AM_{f,a,b,c,d,\alpha}$$

$$Hence\ \langle f(t,x), \phi_{m,n}(t,x)\rangle = \left\langle f(t,x), \int_{-r}^{r}\int_{-\tau}^{\tau} \frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s,p)dsdw \right\rangle \quad (4.1.6)$$

Further left-hand side of (4.1.6)

$$\langle f(t,x), \phi_{m,n}(t,x)\rangle = \left\langle f(t,x), \int_{-r}^{r}\int_{-\tau}^{\tau} \frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s,p)dsdw \right\rangle$$

$$= \left\langle f(t,x), \sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n} \frac{e^{-s_\upsilon t}}{s_\upsilon}\left(\frac{a^{2p_\mu}}{x^{p_\mu+1}} - x^{p_\mu-1}\right)\psi(s_\upsilon, p_\mu)\frac{2r}{m}\frac{2\tau}{n} \right\rangle$$

$$= \sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n} \left\langle f(t,x), \frac{e^{-s_\upsilon t}}{s_\upsilon}\left(\frac{a^{2p_\mu}}{x^{p_\mu+1}} - x^{p_\mu-1}\right) \right\rangle \psi(s_\upsilon, p_\mu)\frac{2r}{m}\frac{2\tau}{n}$$

$$= \int_{-r}^{r}\int_{-\tau}^{\tau} \left\langle f(t,x), \frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right) \right\rangle \psi(s,p)dsdw$$

Since, $\left\langle f(t,x), \frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s_\upsilon, p_\mu)dsdw \right\rangle$ *is continuous function of s and w from* (4.1.6).

We have

$$\int_{-r}^{r}\int_{-\tau}^{\tau} \left\langle f(t,x), \frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right) \right\rangle \psi(s,p)dsdw = \left\langle f(t,x), \int_{-r}^{r}\int_{-\tau}^{\tau} \frac{e^{-st}}{s}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\psi(s,p)dsdw \right\rangle$$

Lemma 2

Let a, b, c, d, ρ, r *and* τ *be real numbers with* $c < s < d$, $a < p < b$. *Also let* $\phi \in D$ then

$$\frac{-1}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{a} \frac{\sinh(t-v)r}{(t-v)}\left[\frac{a^{2p}x^{-\rho+1}}{u^{\rho+1}}\frac{\sin(\tau(log(xu))}{x\,log(xu)} - \left(\frac{u}{x}\right)^{\rho+1}\frac{\sin(\tau(log\left(\frac{u}{x}\right))}{x\,log\left(\frac{u}{x}\right)}\right]\phi(t,x)dtdx = A(v,u)$$

Then $A(v,u)$ *converges in* $AM_{f,a,b,c,d,\alpha}$ *to* $\phi(t,x)$, $r \to \infty, \tau \to \infty$.

Proof: $-$*To prove*

$$\frac{-1}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{a} \frac{\sinh(t-v)r}{(t-v)}\left[\frac{a^{2p}x^{-\rho+1}}{u^{\rho+1}}\frac{\sin(\tau(log(xu))}{x\,log(xu)} - \left(\frac{u}{x}\right)^{\rho-1}\frac{\sin\left(\tau(log\left(\frac{u}{x}\right))\right)}{x\,log\left(\frac{u}{x}\right)}\right]\phi(t,x)dtdx \quad \to \phi(v,u)$$

We have to show that

$$\gamma_{l,q}[A(v,u) - \phi(v,u)] \to 0, where\ \gamma_{l,q}|A(v,u)| = sup\left|K_{a,b}(v)\,\lambda_{c,d}(u)\,u^{q+1}\,D_v^l\,D_u^q\,A(v,u)\right|$$

Consider,

$\gamma_{l,q}[A(v,u) - \phi(v,u)]$

$$= sup \left| K_{a,b}(v)\, \lambda_{c,d}(u)\, u^{q+1}\, D_v^l\, D_u^q \left\{ \frac{-1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sinh(t-v)r}{(t-v)} \left[\frac{a^{2p} x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau(log(xu))}{x\, log(xu)} - \left(\frac{u}{x}\right)^{\rho+1} \frac{\sin\left(\tau(log\left(\frac{u}{x}\right)\right)}{x\, log\left(\frac{u}{x}\right)} \right] \phi(t,x) dt dx - \phi(v,u) \right\} \right| \to 0 \ (4.2.1)$$

Because $\int_0^{\infty} \frac{\sinh(t-v)r}{(t-v)} dt = \int_0^{\infty} \frac{\sinh(rt_1)}{t_1} dt_1 = \pi$

$$\int_0^a \frac{\sin(\tau(log(xu))}{x\, log(xu)} dt = \int_{\infty}^{d} \frac{\sin(\tau x_1)}{x_1} dx_1 = \frac{\pi}{2}$$

$$\int_0^a \frac{\sin(\tau(log(xu))}{x\, log(xu)} dt = \int_{\infty}^{a} \frac{\sin(\tau x_1)}{x_1} dx_1 = \frac{\pi}{2}$$

$$\frac{-1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sinh(t-v)r}{(t-v)} \left[\frac{a^{2p} x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau(log(xu))}{x\, log(xu)} - \left(\frac{u}{x}\right)^{\rho+1} \frac{\sin\left(\tau(log\left(\frac{u}{x}\right)\right)}{x\, log\left(\frac{u}{x}\right)} \right] \phi(v,u) dt dx = \phi = \phi(v,u) (4.2.2)$$

In the view of 4.2.1 and 4.2.2, we have to prove

$$\gamma_{l,q}[A(v,u) - \phi(v,u)]$$

$$= sup \left| K_{a,b}(v)\, \lambda_{c,d}(u)\, u^{q+1}\, D_v^l\, D_u^q \left\{ \frac{-1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sinh(t-v)r}{(t-v)} \left[\frac{a^{2p} x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau(log(xu))}{x\, log(xu)} - \left(\frac{u}{x}\right)^{\rho+1} \frac{\sin\left(\tau(log\left(\frac{u}{x}\right)\right)}{x\, log\left(\frac{u}{x}\right)} \right] \phi(t,x) dt dx + \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sinh(t-v)r}{(t-v)} \left[\frac{a^{2p} x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau(log(xu))}{x\, log(xu)} - \left(\frac{u}{x}\right)^{\rho+1} \frac{\sin\left(\tau(log\left(\frac{u}{x}\right)\right)}{x\, log\left(\frac{u}{x}\right)} \right] \phi(v,u) dt dx \right\} \right|$$

$$= sup \left| K_{a,b}(v)\, \lambda_{c,d}(u)\, u^{q+1}\, D_v^l\, D_u^q \left\{ -\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sinh(t-v)r}{(t-v)} \left[\int_0^a \frac{a^{2p} x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau(log(xu))}{x\, log(xu)} \phi(t,x) dx - \int_0^a \left(\frac{u}{x}\right)^{\rho+1} \frac{\sin\left(\tau(log\left(\frac{u}{x}\right)\right)}{x\, log\left(\frac{u}{x}\right)} \phi(t,x) dx \right] \right.\right.$$

$$\left.\left. + \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sinh(t-v)r}{(t-v)} dt \left[\int_0^a \frac{a^{2p} x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau(log(xu))}{x\, log(xu)} \phi(v,u) dx - \int_0^a \left(\frac{u}{x}\right)^{\rho+1} \frac{\sin\left(\tau(log\left(\frac{u}{x}\right)\right)}{x\, log\left(\frac{u}{x}\right)} \phi(v,u) dx \right] \right| \right.$$

$$= sup \left| K_{a,b}(v)\,\lambda_{c,d}(u)\,u^{q+1}\,D_v^l\,D_u^q \left\{ \frac{-1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sinh(rt_1)}{t_1} dt_1 \left[\int_{\infty}^{d} \frac{a^{2p}x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau x_1)}{x_1} \phi(t,x) dx_1 \right.\right.\right.$$

$$- \int_{\infty}^{d} \left(\frac{u}{x}\right)^{\rho-1} \frac{\sin(\tau x_1)}{x_1} \phi(t,x) dx_1 - \int_{\infty}^{d} \frac{a^{2p}x^{-\rho+1}}{u^{\rho+1}} \frac{\sin(\tau x_1)}{x_1} \phi(v,u) dx_1$$

$$\left.\left.\left. + \int_{\infty}^{d} \left(\frac{u}{x}\right)^{\rho-1} \frac{\sin(\tau x_1)}{x_1} \phi(v,u) dx_1 \right] \right\} \right|$$

$$= sup \left| K_{a,b}(v)\,\lambda_{c,d}(u)\,u^{q+1}\,D_v^l\,D_u^q \left\{ \frac{-1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sinh(rt_1)}{t_1} dt_1 \left[\int_{\infty}^{d} \frac{a^{2p}x^{-\rho+1}}{u^{\rho+1}} [\phi(t_1+v,x) \right.\right.\right.$$

$$\left.\left.\left. - \phi(v,u)] \frac{\sin(\tau x_1)}{x_1} dx_1 - \int_{\infty}^{d} \left(\frac{u}{x}\right)^{\rho-1} \frac{\sin(\tau x_1)}{x_1} [\phi(t_1+v,x) - \phi(v,u)] dx_1 \right] \right\} \right|$$

Since ϕ is continuous, as in Zemanian[9],[10].

$\gamma_{l,q}[A(v,u) - \phi(v,u)] \to 0$

Hence proved.

4.3 Inversion Theorem

Let $AM_f\{f(t,x)\} = F(s,p)$, for $s > 0$ and $\rho_1 < p < \rho_1$. Also let $r\ and\ \tau\ be\ real\ variables\ such\ that$

$\infty < r < \infty, 0 < \tau < a$.Then in the sense of convergence in D^*,

$$f(t,x) = \lim_{\substack{r\to\infty \\ \tau\to\infty}} \frac{-1}{4\pi^2} \int_{-r}^{r} \int_{\rho-i\tau}^{\rho+i\tau} se^{st}x^{-p}F(s,p)\,ds\,dp$$

Where $p = \rho + i\omega, also\ s\ and\ p\ are\ fixed\ real\ numbers\ with\ -r < s < r\ and\ \rho_1 < p < \rho_2$.

Proof: - Let $\phi(t,x) \in D$. Choose the real numbers c and d such that $c < s < d$ and the real numbers $a\ and\ b\ such\ that \rho_1 < a < p < b < \rho_2$.

We have to show that

$$\langle f(t,x), \phi(t,x)\rangle = \lim_{\substack{r\to\infty \\ \tau\to\infty}} \left\langle \frac{-1}{4\pi^2} \int_{-r}^{r} \int_{\rho-i\tau}^{\rho+i\tau} se^{st}x^{-p}F(s,p)\,ds\,dp, \phi(t,x) \right\rangle \quad (4.3.1)$$

Now, the integral on $s\ and\ p\ is\ a\ continuous\ function\ of\ t\ and\ x\ and\ therefore\ the\ right\ hand\ side\ of$ (4.3.1) without the limit notation can be written as

$$\frac{-1}{4\pi^2} \int_{-\infty}^{\infty} \int_{0}^{a} \phi(t,x) \int_{-r}^{r} \int_{-\tau}^{\tau} se^{st}x^{-p}F(s,p)\,ds\,d\omega\,dt\,dx, \qquad where, p = \rho + i\omega, r, \tau > 0 \qquad (4.3.2)$$

Since $\phi(t,x)$ is of bounded support and the integrand is a continuous function of t, x, s, w, the order of integration may be changed and we write,

$$\frac{-1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{a}\phi(t,x)\int_{-r}^{r}\int_{-\tau}^{\tau}se^{st}x^{-p}F(s,p)dsd\omega dtdx=\frac{-1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}F(s,p)\int_{-\infty}^{\infty}\int_{0}^{a}\phi(t,x)\,se^{st}x^{-p}dtdxdsd\omega$$

$$=\frac{-1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}\left\langle f(v,u),\frac{1}{s}e^{-sv}\left(\frac{a^{2p}}{u^{p+1}}-u^{p-1}\right)\right\rangle\int_{-\infty}^{\infty}\int_{0}^{a}\phi(t,x)\,se^{st}x^{-p}dtdxdsd\omega$$

$$=\left\langle f(v,u),\frac{-1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{1}{s}e^{-sv}\left(\frac{a^{2p}}{u^{p+1}}-u^{p-1}\right)\right\rangle\int_{-\infty}^{\infty}\int_{0}^{a}\phi(t,x)\,se^{st}x^{-p}dtdxdsd\omega$$

The order of integration for the repeated integral here in may be changed because again $\phi(t,x)$ is of bounded support and the integrand is continuous function of $t, x\ s, w$ upon doing this we obtain,

$$=\left\langle f(v,u),\frac{-1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{a}\phi(t,x)\int_{-r}^{r}\int_{-\tau}^{\tau}\frac{1}{s}e^{-sv}\left(\frac{a^{2p}}{u^{p+1}}-u^{p-1}\right)se^{st}x^{-p}dtdxdsd\omega\right\rangle$$

$$=\left\langle f(v,u),\frac{-1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{a}\phi(t,x)\left[\int_{-r}^{r}e^{(t-v)s}ds\right]\left[\int_{-\tau}^{\tau}\left(\frac{a^{2p}}{u^{p+1}}-u^{p-1}\right)x^{-p}d\omega\right]dtdx\right\rangle$$

… … … (4.3.3)

$$Consider,\int_{-r}^{r}e^{(t-v)s}ds=\left[\frac{e^{(t-v)s}}{(t-v)}\right]_{-r}^{r}=\left[\frac{e^{(t-v)r}}{(t-v)}-\frac{e^{-(t-v)r}}{(t-v)}\right]$$

$$=\frac{1}{(t-v)}\left[e^{(t-v)r}-e^{-(t-v)r}\right]=\frac{1}{(t-v)}[2sinh(t-v)r]$$

$$=\frac{2sinh(t-v)r}{(t-v)}\ (4.3.4)$$

$$\int_{-\tau}^{\tau}\left(\frac{a^{2p}}{u^{p+1}}-u^{p-1}\right)x^{-p}d\omega=\int_{-\tau}^{\tau}x^{-p}\frac{a^{2p}}{u^{p+1}}d\omega-\int_{-\tau}^{\tau}x^{-p}u^{p-1}d\omega$$

$$=\int_{-\tau}^{\tau}x^{-p}u^{-p-1}a^{2p}d\omega-\int_{-\tau}^{\tau}x^{-p}u^{p-1}d\omega$$

$$=\int_{-\tau}^{\tau}x(xu)^{-p-1}a^{2p}d\omega-\int_{-\tau}^{\tau}\left(\frac{u}{x}\right)^{p-1}\frac{1}{x}d\omega$$

$$=\int_{-\tau}^{\tau}x(xu)^{-\rho-i\omega-1}a^{2p}d\omega-\int_{-\tau}^{\tau}\left(\frac{u}{x}\right)^{\rho+i\omega-1}\frac{1}{x}d\omega$$

$$=a^{2p}\int_{-\tau}^{\tau}x(xu)^{-\rho-1}(xu)^{-i\omega}d\omega-\int_{-\tau}^{\tau}\left(\frac{u}{x}\right)^{\rho-1}\left(\frac{u}{x}\right)^{i\omega}\frac{1}{x}d\omega$$

$$=a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\int_{-\tau}^{\tau}(xu)^{-i\omega}d\omega-\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\int_{-\tau}^{\tau}\left(\frac{u}{x}\right)^{i\omega}d\omega$$

$$= a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\int_{-\tau}^{\tau}(xu)^{t}(idt)-\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\int_{-\tau}^{\tau}\left(\frac{u}{x}\right)^{t}(-idt)$$

$$putting\ \ t=-i\omega\ and\ t=i\omega\ respectivity$$

$$= a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\,i\int_{-\tau}^{\tau}(xu)^{t}dt+i\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\int_{-\tau}^{\tau}\left(\frac{u}{x}\right)^{t}dt$$

$$= a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\,i\left[\frac{(xu)^{t}}{\log(xu)}\right]_{-\tau}^{\tau}+i\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\left[\frac{\left(\frac{u}{x}\right)^{t}}{\log\left(\frac{u}{x}\right)}\right]_{-\tau}^{\tau}$$

$$= a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\,i\left[\frac{(xu)^{-i\omega}}{\log(xu)}\right]_{-\tau}^{\tau}+i\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\left[\frac{\left(\frac{u}{x}\right)^{i\omega}}{\log\left(\frac{u}{x}\right)}\right]_{-\tau}^{\tau}$$

$$= a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\,i\left[\frac{(xu)^{-i\tau}}{\log(xu)}-\frac{(xu)^{i\tau}}{\log(xu)}\right]+i\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\left[\frac{\left(\frac{u}{x}\right)^{i\tau}}{\log\left(\frac{u}{x}\right)}+\frac{\left(\frac{u}{x}\right)^{-i\tau}}{\log\left(\frac{u}{x}\right)}\right]$$

$$= -i\left\{a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\left[\frac{(xu)^{i\tau}}{\log(xu)}-\frac{(xu)^{-i\tau}}{\log(xu)}\right]-\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\left[\frac{\left(\frac{u}{x}\right)^{i\tau}}{\log\left(\frac{u}{x}\right)}+\frac{\left(\frac{u}{x}\right)^{-i\tau}}{\log\left(\frac{u}{x}\right)}\right]\right\}$$

$$= -i\left\{a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\frac{1}{\log(xu)}\left[(xu)^{i\tau}-(xu)^{-i\tau}\right]-\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\frac{1}{\log\left(\frac{u}{x}\right)}\left[\left(\frac{u}{x}\right)^{i\tau}-\left(\frac{u}{x}\right)^{-i\tau}\right]\right\}$$

$$= -i\left\{a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\frac{[2isin(\tau\, log(xu))]}{\log(xu)}-\frac{1}{x}\left(\frac{u}{x}\right)^{\rho-1}\frac{\left[2isin\left(\tau\, log\left(\frac{u}{x}\right)\right)\right]}{\log\left(\frac{u}{x}\right)}\right\}$$

$$= a^{2p}\frac{(x)^{-\rho}}{(u)^{\rho+1}}\frac{sin(\tau\, log(xu))}{\log(xu)}-\left(\frac{u}{x}\right)^{\rho-1}\frac{sin\left(\tau\, log\left(\frac{u}{x}\right)\right)}{x\log\left(\frac{u}{x}\right)}$$

$$= a^{2p}\frac{(x)^{-\rho+1}}{(u)^{\rho+1}}\frac{sin(\tau\, log(xu))}{x\log(xu)}-\left(\frac{u}{x}\right)^{\rho-1}\frac{sin\left(\tau\, log\left(\frac{u}{x}\right)\right)}{x\log\left(\frac{u}{x}\right)}\quad(4.3.5)$$

From (4.3.4) and (4.3.5), (4.3.3) becomes

$$= \left\langle f(v,u),\frac{-1}{4\pi^{2}}\int_{-\infty}^{\infty}\int_{0}^{a}\frac{2sinh(t-v)r}{(t-v)}\left[a^{2p}\frac{(x)^{-\rho+1}}{(u)^{\rho+1}}\frac{sin(\tau\, log(xu))}{x\log(xu)}-\left(\frac{u}{x}\right)^{\rho-1}\frac{sin\left(\tau\, log\left(\frac{u}{x}\right)\right)}{x\log\left(\frac{u}{x}\right)}\right]\phi(t,x)dtdx\right\rangle$$

Taking $r\to\infty\ and\ \tau\to\infty\ and\ using\ Lemma\ (2), we\ get$

$$\langle f(v,u),\phi(v,u)\rangle=\lim_{\substack{r\to\infty\\ \tau\to\infty}}\left\langle\frac{-1}{4\pi^{2}}\int_{-r}^{r}\int_{\rho-i\tau}^{\rho+i\tau}se^{st}x^{-p}F(s,p)dsdp,\phi(t,x)\right\rangle$$

This completes the proof.

4.4 Uniqueness Theorem:

If $AM_f\{f(t,x)\} = F(s,p)$, for $s,p \in \Omega_f$ *and* $AM_f\{g(t,x)\} = G(s,p), for\ s,p \in \Omega_g$, $s > 0$ and

$\rho_1 < Re\ p < \rho_2$. If $\Omega_f \cap \Omega_g$ is not empty and if $F(s,p) = G(s,p)$, for $s \in \Omega_f \cap \Omega_g$ and $p \in \Omega_f \cap \Omega_g$.

Then $f = g$ in the sense if equality $D^*(I)$.

Proof :- Let $f\ and\ g$ must assign the same value to each $\phi \in$ D.

By inversion theorem and equating $F(s,p) and\ G(s,p)$ in $\langle f - g,\ \phi(t,x)\rangle$

$$\langle f - g, \phi(t,x)\rangle = \lim_{\substack{r\to\infty \\ \tau\to\infty}} \left\langle \frac{-1}{4\pi^2} \int_{-r}^{r} \int_{-\tau}^{\tau} (F - G)(s,p) s e^{st} x^{-p} ds dp, \phi(t,x) \right\rangle = 0$$

Thus $f = g$ in $D^*(I)$.

V. CONCLUSIONS

In this paper, we have rigorously established both the inversion theorem and the uniqueness theorem for generalized Aboodh-Finite Mellin Transform. The inversion theorem confirms that, under appropriate conditions on the testing function space, the original function can be fully reconstructed from its transformed counterpart. This result not only validates the operational reliability of the transform but also ensures its applicability in solving differential equations, boundary value problems and inverse problems across mathematics, physics, and engineering. Equally significant is the uniqueness theorem, which guarantees that the transform representation of a function is one-to-one within the prescribed domain. This implies that no two distinct functions share the same transform.

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