International Journal of Mathematics and Statistics Invention (IJMSI)
e-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 14 Issue 3 || May. – June. 2026 || PP: 31-37

On Existence and Uniqueness Theorem for Fuzzy Integral Equation

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ABSTRACT:*In this paper we study the existence and uniqueness of solution of Fuzzyintegral equation.*

KEYWORDS: *Fuzzy integral equation, existence of solution.*

Date of Submission: 22-05-2026 Date of Acceptance: 05-06-2026

I. INTRODUCTION

The topic of Fuzzy Differential Equations and Fuzzy Integral Equations have been developed in recent years, in both theoretical and numerical point of view. In relation to fuzzy control fuzzy integral equations is rapidly developed in recent years. Also, these are encountered in various fields of science and in numerous applications, including elasticity, plasticity, heat and mass transfer, oscillation theory, filtration theory, electrostatics, biomechanics, electrical engineering etc. We know that fuzzy integral equations require appropriate and applicable definitions of fuzzy function and fuzzy integral of fuzzy function. Fuzzy mapping function was first introduced by Zadeh [1]. Durbois and Parde[2, 3] presented on elementary fuzzy calculus based on extension principle. Kaleva [4] choose to define the integral of fuzzy function using Lebesgue type concept of integration. In [7], Andrej V. Plotnikov and Natalia V. S. proved the existence and uniqueness theorem with- out using the embedding theorem of Kaleva. Different approaches were used to solve fuzzy integral equations by several Authors [5,6, 8-13].

In present paper, we consider the Fuzzy integral equation of the type

$$X(t) = A(t)\left[X_0 + \int_0^t F\left(s, A^{-1}(s)X(s), A^{-1}(s)\int_0^s k(s,\tau)X(\tau)d\tau\right)ds\right] \quad \ldots\ldots\ (1.1).$$

where, $t \in [0,d] = J \subset \mathbb{R}_+$ is time, $X \in E^n$ is phase variable, $A(t)$ is $n \times n$ dimensional matric valued function, $F: \mathbb{R}_+ \times E^n \times E^n \to E^n$ is a Fuzzy mapping, $X_0 \in E^n$.
Our aim is to prove the existence and uniqueness theorem of fuzzy integral equation given by (1.1).

II. PRELIMINARIES AND NOTATIONS

Let conv($\mathbb{R}^n$) be a set of all nonempty (convex) compact subset from space $\mathbb{R}^n$,$h(A,B) = \min_{r\geq 0}\{A \subseteq S_r(B), B \subseteq S_r(A)\}$ be Hausdorff distance between sets A and B,$S_r(A)$ is r-neighbourhood of set A. E^n be set of all functions $u: \mathbb{R}^n \to [0,1]$ satisfying following conditions.

1. u is normal
2. u is fuzzy convex i.e. for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$
3. u is upper semicontinuous,
4. $[u]^0 = cl\{x \in \mathbb{R}^n : u(x) > \alpha\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number and E^n is fuzzy number space. Let $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. Then from above conditions a-level set $[u]^\alpha \in conv(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$. The fuzzy mapping θ is defined by,

$$\theta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Define $D: E^n \times E^n \to [0, \infty)$ by $D(u, v) = \sup_{0 \le \alpha \le 1} h([u]^\alpha, [v]^\alpha)$. Then D is a metric on E^n. Further

1. (E^n, D) is compact metric space.
2. $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$.
3. $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v \in E^n, \lambda \in \mathbb{R}$.

Definition 1. A mapping $F: [0, T] \to E^n$ is measurable if for all $\alpha \in [0,1]$ the map $F_\alpha: [0, T] \to conv(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

Definition 2. A mapping $F: [0, T] \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $||x(t)|| \le h(t)$ for every $x(t) \in F_\alpha(t)$.

Definition 3. The integral of a fuzzy mapping $F: [0, T] \to E^n$ is defined levelwise by

$$\left[\int_0^T F(t) dt\right]^\alpha = \int_0^T F_\alpha(t) dt$$

$$\int_0^T F_\alpha(t) dt = \left\{\int_0^T f(t) dt : f: [0, T] \to \mathbb{R}^n \text{ is measurable selection of } F_\alpha: [0, T] \to conv(\mathbb{R}^n)\right\}$$

for all $\alpha \in [0,1]$

Definition 4. Measurably and integrable bounded mapping $F: [0, T] \to E^n$ is said to be integrable over $[0, T]$ if $\int_0^T F(t) dt \in E^n$.

Note that if $F: [0, T] \to E^n$ is measurable and integrably bounded, then F is integrable. Also, if $F: [0, T] \to E^n$ is continuous then it is integrable. Proposition 2.1. Let $F, G: [0, T] \to E^n$ be integrable and $X \in \mathbb{R}$, then

1. $\int_0^T \big(F(t) + G(t)\big) dt = \int_0^T F(t) dt + \int_0^T G(t) dt$;
2. $\int_0^T \lambda F(t) dt = \lambda \int_0^T F(t) dt$;
3. $D\big(F(t), G(t)\big)$ is integrable;
4. $D\left(\int_0^T F(t) dt, \int_0^T G(t) dt\right) \le \int_0^T D\left(F(t), G(t)\right) dt$.

Definition 5. Fuzzy mapping $X: J \to E^n$ is called solution of integral equation (1.1) if it is continuous and satisfies integral equation (1.1) on interval J.

We list here the following hypotheses which are used in our further discussion.

$(\boldsymbol{H1})$: For any fixed t, the fuzzy mapping $F(.,X,.)$ is continuous.

$(\boldsymbol{H2})$: There exist a positive constant L such that

D(F(t, X', Y'), F(t, X",Y")) ≤L[D(X', X") + D(Y',Y")]for all (t, X', Y'), (t, X",Y") ∈ Q.

$(\boldsymbol{H3})$: There exist a positive constant K such that

$D\big(F(t, X, Y), \hat{\theta}\big) \le K\left(1 + D\big(X, \hat{\theta}\big) + D\big(Y, \hat{\theta}\big)\right)$ for all $(t, X, Y) \in Q$.

$(\boldsymbol{H4})$: The matrix valued function $A(t), A^{-1}(t)$ are continuous.

$(\boldsymbol{H5})$: There exists a positive constant α_1, α_2 such that

$||A(t)|| \le \alpha_1, ||A^{-1}(t)|| \le \alpha_2$, for all $t \in J$.

Proposition 2.1: Let $F, G: [0, T] \to E^n$ be integrableand$\lambda \in$ R,then

(i) $\int_0^T \big(F(t) + G(t)\big)dt = \int_0^T F(t)dt + \int_0^T G(t)dt$;

(ii) $\int_0^T \lambda F(t)dt = \lambda \int_0^T F(t)dt$;

(iii) $D(F(t), G(t))$ is integrable;

(iv) $D(\int_0^T F(t)dt, \int_0^T G(t)dt) \leq \int_0^T D\big(F(t), G(t)\big)dt$.

III. MAIN RESULTS

In this section we state and prove results related to existence and uniqueness of solution of Fuzzy integral equation.

Theorem3.1:- If the hypotheses (H1) – (H5) are satisfied in the domain $J \times E^n \times E^n$, then equation (1.1) has a unique solution on the interval J.

Proof. I: Firstly, we show the existence of solution Consider, the successive approximations of solution

$$X^0(t) = A(t)X_0,$$

$$X^{k+1}(t) = A(t)\left[X_0 + \int_0^t F\left(s, A^{-1}(s)X^k(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^k(\tau)d\tau\right)ds\right]$$

for $0 \leq t \leq d$.

By conditions (H1),(H2) and (H4), $X^k(t)$ is continuous on J for all $k \in \mathbb{N}$. First we prove that sequence $\{X^k(t)\}_{k=0}^{\infty}$ is uniformly bounded. Let,

$$D(X^0(t), X_0) = D(A(t)X_0, X_0)$$

$$D(X^0(t), X_0) \leq D(A(t)X_0, \theta) + D\big(X_0, \hat{\theta}\big)$$

$$\leq \alpha_1 D\big(X_0, \hat{\theta}\big) + D\big(X_0, \hat{\theta}\big)$$

$$\leq (\alpha_1 + 1)D\big(X_0, \hat{\theta}\big),$$

$$D\big(X^1(t), X^0(t)\big) = D\left(A(t)\left[X_0 + \int_0^t F\left(s, A^{-1}(s)X^0(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^0(\tau)d\tau\right)ds\right], X^0(t)\right)$$

$$= D\left(A(t)\left[X_0 + \int_0^t F\left(s, A^{-1}(s)X_0(s), A^{-1}(s)\int_0^s k\,(s,\tau)X_0(\tau)d\tau\right)ds\right], A(t)X_0(t)\right)$$

$$\leq \alpha_1 D\left(X_0 + \int_0^t F\left(s, X_0, X_0\int_0^s k\,(s,\tau)d\tau\right)ds, X_0\right)$$

$$\leq \alpha_1 D\left(\int_0^t F\,(s, X_0, Y_0)ds, \hat{\theta}\right)$$

$$\leq \alpha_1 D\left(\int_0^t F\,(s, X_0, Y_0), \hat{\theta}\right)ds$$

$$\leq \alpha_1 K\left(1 + D\big(X_0, \hat{\theta}\big) + D\big(Y_0, \hat{\theta}\big)\right)\int_0^t ds$$

$\leq \alpha_1 K\left(1 + D\big(X_0, \hat{\theta}\big) + D\big(Y_0, \hat{\theta}\big)\right)t.$

$$D\big(X^2(t),X^1(t)\big)$$

$$= D\left(\begin{matrix} A(t)\left[+\int_0^t F\left(s, A^{-1}(s)X^1(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^1(\tau)d\tau\right)ds\right], \\ X_0 \\ A(t)\left[+\int_0^t F\left(s, A^{-1}(s)X^0(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^0(\tau)d\tau\right)ds\right]\end{matrix}\right)$$

$$\leq \alpha_1 D\left[\int_0^t F\left(s, A^{-1}(s)X^1(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^1(\tau)d\tau\right)ds, \int_0^t F\left(s, A^{-1}(s)X^0(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^0(\tau)d\tau\right)ds\right]$$

$$\leq \alpha_1 \int_0^t D\left(F\left(s, A^{-1}(s)X^1(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^1(\tau)d\tau\right), F\left(s, A^{-1}(s)X^0(s), A^{-1}(s)\int_0^s k\,(s,\tau)X^0(\tau)d\tau\right)\right)ds$$

$$\leq \alpha_1 \int_0^t L\left[D\big(A^{-1}(s)X^1(s), A^{-1}(s)X^0(s)\big) + D\left(A^{-1}(s)\int_0^s k\,(s,\tau)X^1(\tau)d\tau, A^{-1}(s)\int_0^s k\,(s,\tau)X^0(\tau)d\tau\right)\right]ds$$

$$\leq \alpha_1\alpha_2 L\int_0^t \left[D\big(X^1(s),X^0(s)\big) + D\left(\int_0^s k\,(s,\tau)X^1(\tau)d\tau, \int_0^s k\,(s,\tau)X^0(\tau)d\tau\right)\right]ds$$

$$\leq \alpha_1\alpha_2 L\int_0^t \left[D\big(X^1(s),X^0(s)\big) + K_T\int_0^s D\,\big(X^1(\tau),X^0(\tau)\big)d\tau\right]ds$$

$$\leq \alpha_1\alpha_2 L\int_0^t D\,\big(X^1(s),X^0(s)\big)(1+K_T s)ds$$

$$\leq \alpha_1\alpha_2 L(1+K_T d)\int_0^t D\,\big(X^1(s),X^0(s)\big)ds$$

Let $M_1 = \max_{t\in[0,d]} D\big(X^1(t),X^0(t)\big)$. Then

$$M_1 \leq \alpha_1\alpha_2 L(1+K_T d)\int_0^d M_1\,ds = \alpha_1\alpha_2 L(1+K_T d)dM_1$$

$$D\big(X^2(t),X^1(t)\big) \leq \alpha_1\alpha_2 L(1+K_T d)\int_0^t D\,\big(X^1(s),X^0(s)\big)ds$$

$$\leq \alpha_1\alpha_2 L(1+K_T d)\int_0^t \Big(\alpha_1 Kd\big(1+D(X_0,\theta)\big)\Big)\,ds$$

$$\leq (\alpha_1\alpha_2 L)^2 Kd^2\big(1+D(X_0,\theta)\big)(1+K_T d)$$

$$D\big(X^3(t),X^2(t)\big) \leq \alpha_1\alpha_2 L\int_0^t D\,\big(X^2(s),X^1(s)\big)(1+K_T s)ds$$

$$\leq \alpha_1\alpha_2 L(1+K_T d)\int_0^t D\,\big(X^2(s),X^1(s)\big)ds$$

$$\leq \alpha_1\alpha_2 L(1+K_T d)\int_0^t \Big((\alpha_1\alpha_2 L)^2 Kd^2\big(1+D(X_0,\theta)\big)(1+K_T d)\Big)\,ds$$

$$\leq (\alpha_1\alpha_2 L)^3 Kd^3\big(1+D(X_0,\theta)\big)(1+K_T d)^2$$

In general,

$$D\big(X^{n+1}(t), X^n(t)\big) \le (\alpha_1\alpha_2 L)^n K d^n\big(1 + D(X_0, \theta)\big)(1 + K_T d)^{n-1}\frac{t^n}{n!}$$

$$\le (\alpha_1\alpha_2 L)^n K d^n\big(1 + D(X_0, \theta)\big)(1 + K_T d)^{n-1}\frac{d^n}{n!}$$

Thus,

$$\max_{t\in[0,d]} D(X^{n+1}, X^n) \le (\alpha_1\alpha_2 L)^n K\frac{d^{2n}}{n!}\big(1 + D(X_0, \theta)\big)(1 + K_T d)^{n-1}$$

Let $b = K\big(1 + D(X_0, \theta)\big)(1 + K_T d)\sum_{i=1}^{\infty}\frac{\left(\alpha_1\alpha_2 L(1+K_T d)\right)^i}{i!}$.

Then $\max_{t\in[0,d]} D(X^{n+1}(t), X_0) \le b$.

Therefore, sequence $\{X^k(t)\}_{k=0}^{\infty}$ is uniformly bounded.

Now, let us show that sequence of fuzzy mappings $\{X^k(t)\}_{k=0}^{\infty}$ is a cauchy sequence for $n, p \in \mathbb{N}$.

$$D\big(X^{n+p}(t), X^p(t)\big) \le \sum_{k=p}^{n+p-1} D\left(X^{k+1}(t), X^k(t)\right)$$

$$\le \sum_{k=p}^{n+p-1}(\alpha_1\alpha_2 L)^k K\frac{d^{2k}}{k!}\big(1 + D(X_0, \theta)\big)(1 + K_T d)^{k-1}$$

$$\le (\alpha_1\alpha_2 L)^p K d^p\big(1 + D(X_0, \theta)\big)(1 + K_T d)^{p-1}\sum_{k=0}^{n-1}\frac{(\alpha_1\alpha_2 L d^2)^k}{k!}$$

$$\le (\alpha_1\alpha_2 L)^p K\frac{d^{2p}}{p!}\big(1 + D(X_0, \theta)\big)(1 + K_T d)^{p-1}$$

$$\le b\big(\alpha_1\alpha_2 L(1 + K_T d)\big)^p\frac{d^p}{p!}$$

$$\le b\{\alpha_1\alpha_2 L(1 + K_T d)\}^p\frac{d^p}{p!}$$

Hence sequence $\{X^k(t)\}$ is cauchy sequence. Its limit is continuous fuzzy mapping that we will denote by $X(t)$. By using the conditions of Theorem3.1 fuzzy mapping $X(t)$ satisfies equation (1.1) i.e $X(t)$ is solution of (1.1) on the interval $[0, d]$.

II: Secondly, we prove the uniqueness of solution Suppose that there exist at least two different solutions $X(t)$ and $Z(t)$ of (1.1) on the interval $[0, d]$. Let $\rho = \max_{t\in[0,d]} D\big(X(t), Z(t)\big) > 0$.

$$X(t) = A(t)\left[X_0 + \int_0^t F\left(s, A^{-1}(s)X(s), A^{-1}(s)\int_0^s k(s,\tau)X(\tau)d\tau\right)ds\right]$$

$$Z(t) = A(t)\left[Z_0 + \int_0^t F\left(s, A^{-1}(s)Z(s), A^{-1}(s)\int_0^s k(s,\tau)Z(\tau)d\tau\right)ds\right]$$

$$D\big(X(t), Z(t)\big) = D\begin{pmatrix} A(t)\left[X_0 + \int_0^t F\left(s, A^{-1}(s)X(s), A^{-1}(s)\int_0^s k(s,\tau)X(\tau)d\tau\right)ds\right], \\ A(t)\left[Z_0 + \int_0^t F\left(s, A^{-1}(s)Z(s), A^{-1}(s)\int_0^s k(s,\tau)Z(\tau)d\tau\right)ds\right] \end{pmatrix}$$

$$\leq \alpha_1 D\left[\begin{matrix} X_0 + \int_0^t F\left(s, A^{-1}(s)X(s), A^{-1}(s)\int_0^s k\,(s,\tau)X(\tau)d\tau\right)ds, \\ Z_0 + \int_0^t F\left(s, A^{-1}(s)Z(s), A^{-1}(s)\int_0^s k\,(s,\tau)Z(\tau)d\tau\right)ds \end{matrix}\right]$$

$$\leq \alpha_1 D(X_0, Z_0)$$
$$+ \alpha_1 \int_0^t D\left(F\left(s, A^{-1}(s)X(s), A^{-1}(s)\int_0^s k\,(s,\tau)X(\tau)d\tau\right), F\left(s, A^{-1}(s)Z(s), A^{-1}(s)\int_0^s k\,(s,\tau)Z(\tau)d\tau\right)\right)ds$$

$$\leq \alpha_1 D(X_0, Z_0) + \alpha_1 L \int_0^t \Big[D\big(A^{-1}(s)X(s), A^{-1}(s)Z(s)\big) + D\left(A^{-1}(s)\int_0^s k\,(s,\tau)X(\tau)d\tau, A^{-1}(s)\int_0^s k\,(s,\tau)Z(\tau)d\tau\right)\Big]ds$$

$$\leq \alpha_1 D(X_0, Z_0) + \alpha_1\alpha_2 L \int_0^t \left[D\big(X(s), Z(s)\big) + D\left(\int_0^s k\,(s,\tau)X(\tau)d\tau, \int_0^s k\,(s,\tau)Z(\tau)d\tau\right)\right]ds$$

$$\leq \alpha_1 D(X_0, Z_0) + \alpha_1\alpha_2 L \int_0^t \left[D\big(X(s), Z(s)\big) + K_T \int_0^s D\,\big(X(\tau), Z(\tau)\big)d\tau\right]ds$$

$$\leq \alpha_1 D(X_0, Z_0) + \alpha_1\alpha_2 L \int_0^t D\,\big(X(s), Z(s)\big)(1 + K_T s)ds$$

$$\leq \alpha_1 D(X_0, Z_0) + \alpha_1\alpha_2 L(1 + K_T d)\int_0^t D\,\big(X(s), Z(s)\big)ds$$

Let $\rho(t) = D\big(X(t), Z(t)\big)$. Then

$$\rho(t) \leq \alpha_1 D(X_0, Z_0) + \alpha_1\alpha_2 L(1 + K_T d)\int_0^t \rho\,(s)ds$$

By Gronwall's inequality,

$$\rho(t) \leq \alpha_1 D(X_0, Z_0)e^{\alpha_1\alpha_2 L(1+K_T d)t}$$

Since $X(t)$ and $Z(t)$ are solutions, $X_0 = Z_0$. So $D(X_0, Z_0) = 0$. Therefore, $\rho(t) = 0$ for all $t \in [0, d]$. This implies $X(t) = Z(t)$. Hence equation (1.1) has a unique solution on the interval J.

IV. ACKNOWLEDGEMENT

The authors would like to express their gratitude to the anonymous referees for their valuable comments which significantly improved the original manuscript.

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