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Minimum Size of Generating Set for Transitive $p-$Group G of Degree p^3.

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ABSTRACT: *The exact number of transitive $p-$groups of degree p^3 is unknown up to date and in this article we determine this, up to isomorphism, and we also realise that the elements of this set are largely influenced by the prime ideal generated by p. The letter p denotes an arbitrary but fixed prime number.*

KEYWORDS: *size, transitive, generating, set.*

I. INTRODUCTION

Let G be a group acting transitively on a non – empty set Ω containing more than one element. The exact number of transitive p – groups of degree p^3 is unknown to date and we
We achieve an improvement on a result by Audu in [5] on the minimum size of generating set for transitive p – groups of degree p^3.

II. RESULTS

1.2 MINIMUM SIZE OF GENERATING SET FOR G

In this section we determine the actual minimum size of generating set for transitive p – groups of degree p^3, for any prime number p.

1.2.1 Lemma
Let p be a fixed prime and G be a transitive p – group of degree p^3, then the largest order of G is $p^{p(p+1)+1}$.

Proof
If G is a transitive p – group of degree p^3, then $|G| = p^n$, where n = 3, 4, … , r, where

r $\in$N is such that $p^r | p^3!$. But

$$p^3! = \prod_{k=1}^{p^3-1} p^3(p^3-k) = p^3\prod_{k=1}^{p^3-1}(p^3-k) \qquad (1.1)$$

Now $\prod_{k=1}^{p^3-1}(p^3-k)$ is a product of (p^3 -1) terms out of which (p^2 – 1) contain p as a factor. More precisely, it consists of (p^2 – 1) – (p – 1) = (p^2 – p) terms containing p as a factor while the remaining (p^2 – 1) - (p^2- p) = (p – 1) contain p^2 as a factor.

Thus equation (1.1) contains $p^3.p^{p^2-p}.p^{2(p-1)} = p^{p(p+1)+1}$ as the highest power of p dividing p^3!
Consequently, r = p (p + 1) + 1 as required.

1.2.2 Remark
We notice that an important aspect of Lemma 1.2.1 is that it specifies those transitive p – groups of degree p^3 that can exist. In other words, transitive p – groups of degree p^3 of orders greater than $p^{p(p+1)+1}$ or of orders less than p^i (i = 1, 2) do not exist.

Moreover, in [1] we found that a non – abelian transitive p – group of degree p^3 and exponent p^3 can neither be generated by two different generators of orders p^3 and p^2 nor be generated by two different generators each of order p^2 simultaneously, but that such a group can only be generated by exactly one generator of order p^3 with the remaining generators each being of order p.

Also, in [1] we saw that a non – abelian transitive p – group of degree p^3 and exponent p^2 cannot be generated by two different generators each of orders p^2 and that such a group can only be generated by exactly one generator of order p^2 with other different generators each being of order p.

To start with, we notice that transitive p – groups of degree p^3 must be of exponent p^3 or p^2 or p, otherwise transitivity fails.

These observations will help us to find the exact values of *r (G)*, which denotes the minimum size of generating set for G, as in the following:

1.2.3 Lemma
Let p be a prime number and let G be a non – abelian transitive p – group of degree p^3, of exponent p^3 and of order p^n, then
(i) if G is generated by a generator of order p^3, then $r(G) = n - 2$ for $n \geq 4$

(ii) if G is not generated by a generator of order p^3, then $r(G) = n - 3$ for $n \geq 7$

Proof:
For each n = 3, 4, . . . , p (p + 1) +1, we may write $|G| = p^{3i+2s+(n-3i-2s)}$, i = 1, 2, …, ω_n, $n = 3\omega_n + s$, $s \in A = \{0, 1, 2\}$, $\omega_n \in \mathbb{N}$, $3i \leq 3i + 2s < n$. So that G contains

$r(G) = (n - 2i - s)$ generators out of which i are of order p^3, s are of order p^2 and the remaining $(n - 3i - 2s)$ are of order p.

(i) if G is generated by a generator of order p^3, then i = 1, s = 0 (by the result in [1]), substituting these in the above, we see that $r(G) = n - 2$, for $n \geq 4$.

(ii) if G is not generated by a generator of order p^3 then i = 0, s = 3. Thus G must be generated by 3 different generators each of order p^2 and the subgroup of G generated by these 3 generators must be abelian of order p^6 (in [1]). As G is non – abelian, the order of G must be greater than 6. Substituting these in the above, we get

$r(G) = n - 3$ for $n \geq 7$.

1.2.4 Lemma
Let p be a prime number and let G be a non – abelian transitive p – group of degree p^3, of exponent p^2 and of order p^n, then
(i) if G is generated by a generator of order p^2, then $r(G) = n - 1$ for $n \geq 3$

(ii) if G is not generated by a generator of order p^2, then $r(G) = n$ for n = 6, 7.

Proof:
For each n = 3, …, p (p + 1) +1, we have $|G| = p^{2i+(n-2i)}$, i =1, 2, 3, … ω_n, $n = 2\omega_n + s$,

$s \in B = \{0, 1\}$, $\omega_n \in \mathbb{N}$, $2i < n$, and in this case G is generated by $r(G) = (n - i)$ generators out of which i are of order p^2 and the remaining $(n - 2i)$ are of order p.

(i) if G is generated by a generator of order p^2, then i = 1(in [1]. Substituting this in the above, we see that $r(G) = (n - 1)$, for $n \geq 3$,

(ii)) if G is not generated by a generator of order p^2, then i = 0. In this case, all the different generators of G are each of order p. But as G is non – abelian of exponent p^2, G must be of rank 6 (by our main result in [1]) or rank 7 (by the proof of Lemma 1.2.5.). Hence

$r(G) = n$, for n = 6, 7.

1.2.5 Lemma

Let p be a prime number and let G be a non – abelian transitive p – group of degree p^3, of exponent p, then $r(G) = 3, 4$ or 5.

Proof:
Since G is of exponent p, then all the generators of G are of order p, $|G| = p^s$, where s is an integer with $s \geq 3$ and again by our main result in [1], we have s = 3, 4 or 5. Hence $r(G) = 3, 4$ or 5.
Since we know the transitive p – groups of degree p^3 that can exist, we have, irrespective order, the following:

1.2.6 Proposition
Let p be a prime number and let G be a transitive p – group of degree p^3. If $r(G)$ denotes the minimum size of generating set for G, then

$$1 \leq r(G) \leq p(p + 1).$$

Proof:
By Lemma 1.2.1, such a group exists when $|G| = p^n$, with n = 3, 4, . . . , $p(p + 1) + 1$, substituting these in various values of $r(G)$ in Lemmas 1.2.3., 1.2.4., 1.2.5 and using the fact that $r(G)$ is a positive integer, we see that the maximal value of $r(G)$ is $p(p + 1)$ and its least value is 1./.
With the aid of Lemma 1.2.3, we may generalize our main result in [1] as follows:

1.2.7 Theorem
Let p be a prime and G a non – abelian transitive p – group of degree p^3 and exponent p such that $r(G) \neq 3$. Then every transitive p – group of degree p^3 containing G as a normal subgroup is of exponent p^2.

Proof:
The proof is essentially the same as in our main result in [1]. Here, by Lemma 1.2.5,
$r(G) = 4$ or 5 with $|G| = p^5$ or p^6 and since G is of exponent p, we must have

$r(G') = r(G) + 1$ and $|G'| = p^6$ or p^7 where G' is a p – group of degree p^3 containing G as a normal subgroup.

1.2.8 Remark
If we set m = 3 in Audu (1986), we then obtain for any transitive p – group G of degree p^3, p being any prime, the minimal size of generating set for G,

$r(G)$, is such that $r(G) \leq 1 + \sum_{i=0}^{1} p^i = 2 + p$.

But we observe that there are many transitive p – groups G' of degree p^3, p being any prime, such that $2 + p \leq r(G')$. This is the case when, for instance, G' is a transitive p – group of degree p^3, exponent p^2 and order p^{4+p} by Lemma 1.2.4 (i).

Consequently our result in Lemma 1.2.5 is an improvement of Audu (1986) result when m = 3.

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