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A Short Note On The Order Of a Meromorphic Matrix Valued Function.

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ABSTRCT: *In this paper we have compared the orders of two meromorphic matrix valued functions A(z) and B(z) whose elements satisfy a similar condition as in Nevanlinna-Polya theorem on a complex domain D.*

KEYWORDS*: Nevanlinna theory, Matrix valued meromorphic functions.*

Preliminaries: We define a meromorphic matrix valued function as in [2].

By a matrix valued meromorphic function A(z) we mean a matrix all of whose entries are meromorphic on the whole (finite) complex plane.

A complex number z is called a pole of A(z) if it is a pole of one of the entries of A(z), and z is called a zero of A(z) if it is a pole of $[A(z)]^{-1}$.

For a meromorphic $m \times m$ matrix valued function A(z),

$$\text{let } m(r, A) = \frac{1}{2\pi}\int_0^{2\pi} \log \left\| A\left(re^{i\theta}\right)\right\| d\theta \tag{1}$$

where A has no poles on the circle $|z| = r$.

$$\text{Here, } \|A(z)\| = \max_{\substack{\|x\| = 1 \\ x \in C^n}} \|A(z)x\|$$

$$\text{Set } N(r, A) = \int_0^r \frac{n(t, A)}{t}\, dt \tag{2}$$

where n(t, A) denotes the number of poles of A in the disk $\{z : |z| \le t\}$, counting multiplicities.

Let T(r, A) = m(r, A) + N(r, A)

The order of A is defined by $\rho = \lim_{r \to \infty} \sup \dfrac{\log T(r, A)}{\log r}$

We wish to prove the following result

Theorem: Let $A = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$ and $B = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix}$ be two meromorphic matrix valued functions. If f_k and g_k (k=1, 2) satisfy

$$|f_1(z)|^2 + |f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2 \tag{1}$$

on a complex domain D, the n $\rho_A = \rho_B$, where ρ_A and ρ_B are the orders of A and B respectively.

We use the following lemmas to prove our result.

Lemma 1[3] [The Nevanlinna –Polya theorem]

Let n be an arbitrary fixed positive integer and for each k (k = 1, 2, … n) let f_k and g_k be analytic functions of a complex variable z on a non empty domain D.

If f_k and g_k (k = 1,2,…n) satisfy

$$\sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k(z)|^2$$

on D and if $f_1, f_2, \ldots f_n$ are linearly independent on D, then there exists an $n \times n$ unitary matrix C, where each of the entries of C is a complex constant such that

$$\begin{bmatrix} g_1(z) \\ g_2(z) \\ \\ \\ \\ g_n(z) \end{bmatrix} = C \begin{bmatrix} f_1(z) \\ f_2(z) \\ \\ . \\ \\ f_n(z) \end{bmatrix}$$

holds on D.

Lemma 2 Let f_k and g_k be as defined in our theorem (k = 1,2). Then there exists a 2×2 unitary matrix C where each of the entries of C is a complex constant such that $B = CA$ (2)

where A and B are as defined in the theorem.

Proof of Lemma 2

We consider the following two cases.

Case A: If f_1, f_2 are linearly independent on D, then the proof follows from the Nevanlinna – Polya theorem.

Case B: If f_1 and f_2 are linearly dependent on D, then there exists two complex constants c_1, c_2, not both zero such that

$$c_1 f_1(z) + c_2 f_2(z) = 0 \tag{3}$$

We discuss two subcases

Case B_1 : If $c_2 \neq 0$, then by (3) we get $f_2(z) = \dfrac{-c_1}{c_2} f_1(z)$ (4)

holds on D.

If we set $b = \dfrac{-c_1}{c_2}$, then by (4) we have $f_2(z) = b\, f_1(z)$ on D. (5)

Hence (1) takes the form

$$(1 + |b|^2)\, |f_1(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2 . \tag{6}$$

We may assure that $f_1 \not\equiv 0$ on D. Otherwise the proof is trivial.

Hence by (6), we get

$$\left|\frac{g_1(z)}{f_1(z)}\right|^2 + \left|\frac{g_2(z)}{f_1(z)}\right|^2 = 1 + |b|^2 \tag{7}$$

Taking the Laplacians $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ of both sides of (7) with respect to z = x + iy (x, y real), we get

$$\left| \left(\frac{g_1(z)}{f_1(z)} \right)' \right|^2 + \left| \left(\frac{g_2(z)}{f_1(z)} \right)' \right|^2 = 0 \tag{8}$$

Since $\Delta \left| P(z) \right|^2 = 4 \left| P'(z) \right|^2$, [14] where P is an analytic function of z, by (8), we get

$$\left(\frac{g_1(z)}{f_1(z)} \right)' = 0, \qquad \left(\frac{g_2(z)}{f_1(z)} \right)' = 0$$

Hence, $g_1(z) = c\, f_1(z)$ and $g_2(z) = d\, f_1(z)$ (9)

where c, d are complex constants.

Substituting (9) in (7), we get $\left| c \right|^2 + \left| d \right|^2 = 1 + \left| b \right|^2$ (10)

Let us define

$$U := \begin{bmatrix} \frac{1}{\sqrt{1 + \left| b \right|^2}} & -\frac{\bar{b}}{\sqrt{1 + \left| b \right|^2}} \\ \frac{b}{\sqrt{1 + \left| b \right|^2}} & \frac{1}{\sqrt{1 + \left| b \right|^2}} \end{bmatrix} \tag{11}$$

and $$V := \begin{bmatrix} \frac{c}{\sqrt{1 + \left| b \right|^2}} & -\frac{\bar{d}}{\sqrt{1 + \left| b \right|^2}} \\ \frac{d}{\sqrt{1 + \left| b \right|^2}} & \frac{\bar{c}}{\sqrt{1 + \left| b \right|^2}} \end{bmatrix} \tag{12}$$

Then, it is easy to prove that

$$U \begin{bmatrix} \sqrt{1 + \left| b \right|^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix} \tag{13}$$

$$\text{and } V \begin{bmatrix} \sqrt{1 + \left| b \right|^2} \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{14}$$

Set $C = V\, U^{-1}$ (15)

Since all 2×2 unitarly matrices form a group under the standard multiplication of matrices, by (15), C is a 2×2 unitary matrix.

Now, by (13), we have $U^{-1}\begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix}$ (16)

Thus, we have on D,

$$C\begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} = f_1(z)\ C\begin{pmatrix} 1 \\ b \end{pmatrix} \quad , \text{ by (5)}$$

$$= f_1(z)\ V\left(U^{-1}\begin{pmatrix} 1 \\ b \end{pmatrix}\right), \text{ by (15)}$$

$$= f_1(z)\ V\begin{pmatrix} \sqrt{1+|b|^2} \\ 0 \end{pmatrix}, \text{ by (16)}$$

$$= f_1(z)\begin{pmatrix} c \\ d \end{pmatrix}, \text{ by (14)}$$

$$= \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix}, \quad \text{by (9)}$$

Hence, the result.

Case B_2. Let $c_2 = 0$ and $c_1 \neq 0$.

Then by (3), we get $f_1 = 0$.

Hence, by (1), $|f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2$ (17)

holds on D.

By (17) and by a similar argument as in case B_1, we get the result.

Proof of the theorem

By Lemma 2, we have B = CA where A and B are as defined in the theorem.

Therefore, T(r, B) = T(r, CA)

using the basics of Nevanlinna theory[2], we can show that,

T(r, B) ≤ T(r, A) as T(r,C)=o{T(r,f}

On further simpler simplifications, we get

$$\rho_B \leq \rho_A . \tag{18}$$

By inter changing f_k and g_k (k=1, 2) in Lemma 2,

we get A =CB , which implies

T(r, A) ≤ T(r, B) and hence $\rho_A \leq \rho_B$ (19)

By (18) and (19), we have $\rho_A = \rho_B$

Hence the result.

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