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Transient Thermal Stresses Due to Instantaneous Internal Heat Generation in a Thin Hollow Cylinder

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ABSTRACT: *The present paper deals with the determination of temperature change, displacement and thermal stresses in a thin hollow cylinder defined by $a \le r \le b$ with internal heat generation. Heat dissipates by convection from inner circular surface $(r = a)$ and outer circular surface $(r = b)$. Also, initially the thin hollow cylinder is at arbitrary temperature $f(r)$. Here we modify Kulkarni et al. (2008) for homogeneous heat convection along radial direction. The governing heat conduction equation has been solved by the method of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for temperature change, displacement and stresses have been computed numerically and illustrated graphically.*

KEYWORDS: *Transient thermal stresses, instantaneous internal heat generation, thin hollow cylinder.*

I. INTRODUCTION

During the last century the theory of elasticity has found of considerable applications in the solution of engineering problems. Thermoelasticity contains the generalized theory of heat conductions, thermal stresses. A considerable progress in the field of air-craft and machine structures, mainly with gas and steam turbines and the emergence of new topics in chemical engineering have given rise to numerous problems in which thermal stresses play an important role and frequently even a primary role. Deshmukh (2002) discussed generalized transient heat conduction problem in a thin hollow cylinder. Durge and Khobragade (2009) studied an inverse problem of an elastic vibration in thin hollow cylinder. Deshmukh *et al.* (2011) discussed the thermal stresses in a hollow circular cylinder subjected to arbitrary initial temperature and time dependent heat flux is applied at the outer circular boundary whereas inner circular boundary is at zero heat flux. Roy Choudhary (1972) discussed the quasi-static thermal stresses in a thin circular disk subjected to transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and fixed circular edge thermally insulated.

Most recently Bhongade and Durge (2013) discussed study of transient thermal stresses in finite hollow cylinder with internal heat generation.

The present paper deals with the determination of displacement and thermal stresses in a thin hollow cylinder defined by $a \le r \le b$ with internal heat generation. Heat dissipates by convection from inner circular surface $(r = a)$ and outer circular surface $(r = b)$. Also initially the thin hollow cylinder is at arbitrary temperature $f(r)$. Here we modify Kulkarni *et al.* (2008) for homogeneous heat convection along radial direction. The governing heat conduction equation has been solved by the method of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for temperature change, displacement and stresses have been computed numerically and illustrated graphically. A mathematical model has been constructed of a thin hollow cylinder with the help of numerical illustration by considering copper (pure) thin hollow cylinder. No one previously studied such type of problem. This is new contribution to the field.

The direct problem is very important in view of its relevance to various industrial mechanics subjected to heating such as the main shaft of lathe, turbines and the role of rolling mill, base of furnace of boiler of a thermal power plant, gas power plant and the measurement of aerodynamic heating.

II. FORMULATION OF THE PROBLEM

Consider a thin hollow cylinder occupying space D defined by $a \le r \le b$. Initially the thin hollow cylinder is at arbitrary temperature $f(r)$. Heat dissipates by convection from the inner circular surface $(r = a)$ and outer circular surface $(r = b)$. For time $t > 0$, heat is generated within the thin hollow cylinder at the rate $q(r, t)$. Under these prescribed conditions, the displacement and thermal stresses in a thin hollow cylinder due to internal heat generation are required to be determined.

Following Roy Choudhuri (1972), we assume that a thin hollow cylinder of small thickness h is in a plane state of stress. In fact, "the smaller the thickness of the thin hollow cylinder compared to its diameter, the nearer to a plane state of stress is the actual state". The displacement equations of thermoelasticity have the form

$$U_{i,kk} + \left(\frac{1+v}{1-v}\right) e_{,i} = 2\left(\frac{1+v}{1-v}\right) a_t T_{,i} \tag{1}$$

$$e = U_{k,k}\ ; k, i = 1,2 \tag{2}$$

where

U_i - displacement component

e - dilatation

T – temperature

And v and a_t are respectively, the Poisson's ratio and the linear coefficient of thermal expansion of the thin hollow cylinder material.

Introducing

$$U_i = \phi_{,i} \quad i = 1, 2$$

we have

$$\nabla_1{}^2 \phi = (1+v) a_t T \tag{3}$$

$$\nabla_1{}^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}$$

$$\sigma_{ij} = 2\mu\left(\phi_{,ij} - \delta_{ij}\,\phi_{,kk}\right) i, j, k = 1, 2 \tag{4}$$

where μ is the Lame constant and δ_{ij} is the Kronecker symbol.

In the axially-symmetric case

$$\phi = \phi(r,t), T = T(r,t)$$

and the differential equation governing the displacement potential function is $\phi(r,t)$ given in Noda (2003) as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = (1+v) a_t T \tag{5}$$

$$\frac{\partial \phi}{\partial r} + h_{s_1}\phi = 0 \text{ at } r = a \tag{6}$$

$$\frac{\partial \phi}{\partial r} + h_{s_2}\phi = 0 \text{ at } r = b \tag{7}$$

The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial \phi}{\partial r} \tag{8}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \phi}{\partial r^2} \tag{9}$$

In the plane state of stress within the thin hollow cylinder

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \tag{10}$$

The temperature of the hollow cylinder satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{q(r,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{11}$$

with the boundary conditions,

$$h_{s_1} T + \frac{\partial T}{\partial r} = 0 \; at \;\; r = a, t > 0 \tag{12}$$

$$h_{s_2} T + \frac{\partial T}{\partial r} = 0 \;\; at \;\; r = b, \; t > 0 \tag{13}$$

and the initial condition

$$T = f(r) \text{ at } t = 0,\ a \le r \le b \tag{14}$$

where k is the thermal conductivity of the material of the thin hollow cylinder, α is the thermal diffusivity of the material of the thin hollow cylinder, $q(r, t)$ is internal heat generation and h_{s_1} and h_{s_2} are the relative heat transfer coefficients on the inner and outer surface of the thin hollow cylinder.
Equations (1) to (14) constitute the mathematical formulation of the problem.

III. SOLUTION OF THE HEAT CONDUCTION EQUATION

To obtain the expression for temperature $T(r, t)$, we introduce the finite Hankel transform over the variable r and its inverse transform defined in Ozisik (1968) are

$$\bar{T}(\beta_m, t) = \int_{r'=a}^{b} r' K_0(\beta_m, r')\, T(r', t)\, dr' \tag{15}$$

$$T(r, t) = \sum_{m=1}^{\infty} K_0(\beta_m, r)\, \bar{T}(\beta_m, t) \tag{16}$$

where

$$K_0(\beta_m, r) = \frac{\mathrm{R}_0(\beta_m, r)}{\sqrt{N}} \tag{17}$$

and

$$\mathrm{R}_0(\beta_m, r) = \left[\frac{J_0(\beta_m r)}{\beta_m J_0'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{\beta_m Y_0'(\beta_m b) + h_{s_2} Y_0(\beta_m b)}\right], \tag{18}$$

The normality constant

$$N = \frac{b^2}{2}\left(1 + \frac{{h_{s_2}}^2}{{\beta_m}^2}\right) {R_0}^2(\beta_m b) - \frac{a^2}{2}\left(1 + \frac{{h_{s_1}}^2}{{\beta_m}^2}\right) {R_0}^2(\beta_m a) \tag{19}$$

and $\beta_1, \beta_2, \beta_3$ …… are the positive roots of the transcendental equation

$$\frac{\beta_m J_0'(\beta_m a) + h_{s_1} J_0(\beta_m a)}{\beta_m J_0'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{\beta_m Y_0'(\beta_m a) + h_{s_1} Y_0(\beta_m a)}{\beta_m Y_0'(\beta_m b) + h_{s_2} Y_0(\beta_m b)} = 0 \tag{20}$$

where $J_n(x)$ is Bessel function of the first kind of order n and $Y_n(x)$ is Bessel function of the second kind of order n.
On applying the finite Hankel transform defined in the Eq. (15) and its inverse transform defined in Eq. (16), one obtains the expression for temperature as

$$T(r, t) = \sum_{m=1}^{\infty} e^{-\alpha {\beta_m}^2 t} K_0(\beta_m, r) \times \left\{ \begin{array}{c} \int_{r'=a}^{b} r' K_0(\beta_m, r') f(r')\, dr' \\ + \frac{\alpha}{k} \int_{t'=0}^{t} e^{\alpha {\beta_m}^2 t'} \left[\int_{r'=a}^{b} r' K_0(\beta_m, r') q(r', t') dr'\right] dt' \end{array} \right\} \tag{21}$$

Displacement Potential and Thermal Stresses
Using Eq. (21) in Eq. (5), one obtains

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = (1 + v) a_t \sum_{m=1}^{\infty} e^{-\alpha {\beta_m}^2 t} K_0(\beta_m, r) \times \left\{ \begin{array}{c} \int_{r'=a}^{b} r' K_0(\beta_m, r') f(r')\, dr' \\ + \frac{\alpha}{k} \int_{t'=0}^{t} e^{\alpha {\beta_m}^2 t'} \left[\int_{r'=a}^{b} r' K_0(\beta_m, r') q(r', t') dr'\right] dt' \end{array} \right\} \tag{22}$$

Solving Eq. (22), one obtains

$$\phi(r, t) = -(1 + v) a_t \sum_{m=1}^{\infty} \frac{1}{{\beta_m}^2} e^{-\alpha {\beta_m}^2 t} K_0(\beta_m, r) \times \left\{ \begin{array}{c} \int_{r'=a}^{b} r' K_0(\beta_m, r') f(r')\, dr' \\ + \frac{\alpha}{k} \int_{t'=0}^{t} e^{\alpha {\beta_m}^2 t'} \left[\int_{r'=a}^{b} r' K_0(\beta_m, r') q(r', t') dr'\right] dt' \end{array} \right\} \tag{23}$$

Using Eq. (23) in Eqs. (8) and (9), one obtains expression for thermal stresses as

$$\sigma_{rr} = 2\mu(1 + \vartheta) a_t \sum_{m=1}^{\infty} \frac{1}{r \beta_m} \frac{1}{\sqrt{N}} e^{-\alpha {\beta_m}^2 t} \times \left[\frac{J_0'(\beta_m r)}{\beta_m J_0'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{Y_0'(\beta_m r)}{\beta_m Y_0'(\beta_m b) + h_{s_2} Y_0(\beta_m b)}\right]$$

$$\times \begin{Bmatrix} \int_{r'=a}^{b} r' K_0(\beta_m, r') f(r')\, dr' \\ + \frac{\alpha}{k} \int_{t'=0}^{t} e^{\alpha\, \beta_m{}^2 t'} \left[\int_{r'=a}^{b} r' K_0(\beta_m, r') q(r', t') dr' \right] dt' \end{Bmatrix} \quad (24)$$

$$\sigma_{\theta\theta} = -2\mu(1+\vartheta) a_t \sum_{m=1}^{\infty} \frac{1}{\sqrt{N}} e^{-\alpha\, \beta_m{}^2 t}$$
$$\times \left[\frac{J_1{}'(\beta_m r)}{\beta_m J_0{}'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{Y_1{}'(\beta_m r)}{\beta_m Y_0{}'(\beta_m b) + h_{s_2} Y_0(\beta_m b)} \right]$$
$$\times \begin{Bmatrix} \int_{r'=a}^{b} r' K_0(\beta_m, r') f(r')\, dr' \\ + \frac{\alpha}{k} \int_{t'=0}^{t} e^{\alpha\, \beta_m{}^2 t'} \left[\int_{r'=a}^{b} r' K_0(\beta_m, r') q(r', t') dr' \right] dt' \end{Bmatrix} \quad (25)$$

IV. SPECIAL CASE AND NUMERICAL CALCULATIONS

Setting

(1) $f(r) = r^2$

$$F(\beta_m) = \frac{1}{\sqrt{N}} \left[\frac{b^3 J_1(\beta_m b) - a^3 J_1(\beta_m a) - 2\left(b^2 J_2(\beta_m b) - a^2 J_2(\beta_m a)\right)}{\beta_m J_0'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{b^3 Y_1(\beta_m b) - a^3 Y_1(\beta_m a) - 2\left(b^2 Y_2(\beta_m b) - a^2 Y_2(\beta_m a)\right)}{\beta_m Y_0{}'(\beta_m b) + h_{s_2} Y_0(\beta_m b)} \right]$$

(2) $q = q_i\, \delta(r - r_0)\, \delta(t - \tau)$

where *r* is the radius measured in meter, $\delta(r)$ is well known DIract delta function of argument *r*.

The internal heat source $q(r, t)$ is an instantaneous line heat source of strength q_i situated at the centre of the thin hollow cylinder along the radial direction and releases its heat instantaneously at the time $t = \tau = 10$sec.

$$\bar{q}\,(\beta_m, t) = \frac{r_0\, q_i\, \delta(t-\tau)}{\sqrt{N}}$$
$$\times \left[\frac{J_0(\beta_m r_0)}{\beta_m J_0'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{Y_0(\beta_m r_0)}{\beta_m Y_0{}'(\beta_m b) + h_{s_2} Y_0(\beta_m b)} \right]$$

$a = 1m,\ b = 2m,\ h_{s_1} = 13 \ and\ \ h_{s_2} = 17.$

$r_0 = 1.5m, q_i = 90W/mK$ and $\tau = 10$sec.

Material Properties

The numerical calculation has been carried out for a copper (pure) hollow cylinder with the material properties defined as,

Thermal diffusivity $\alpha = 112.34 \times 10^{-6}\ m^2 s^{-1}$,
Specific heat $c_\rho = 383 J/Kg$,
Thermal conductivity $k = 386$ W/m K
Shear modulus $G = 48\ G\ pa$,
Youngs modulus $E = 130\ Gpa$,
Poisson ratio $\vartheta = 0.35$,
Coefficient of linear thermal expansion $a_t = 16.5 \times 10^{-6} m^2 s^{-1}$,
Lame constant$\mu = 26.67$

Roots of Transcendental Equation

The $\beta_1 = 3.8214$, $\beta_2 = 7.0232$, $\beta_3 = 10.1672$, $\beta_4 = 13.3292$, $\beta_5 = 16.4657$ are the roots of transcendental equation $\frac{\beta_m J_0{}'(\beta_m a) + h_{s_1} J_0(\beta_m a)}{\beta_m J_0{}'(\beta_m b) + h_{s_2} J_0(\beta_m b)} - \frac{\beta_m Y_0'(\beta_m a) + h_{s_1} Y_0(\beta_m a)}{\beta_m Y_0{}'(\beta_m b) + h_{s_2} Y_0(\beta_m b)} = 0.$

The numerical calculation and the graph has been carried out with the help of mathematical software Matlab.

V. DISCUSSION

In this paper a thin hollow cylinder is considered and determined the expressions for temperature, displacement and stresses due to internal heat generation within it. As a special case mathematical model is constructed for considering copper (pure) thin hollow cylinder with the material properties specified above.

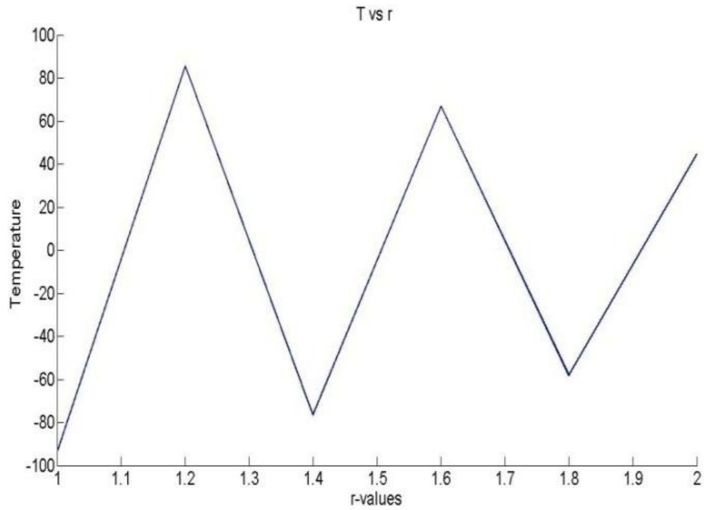

Fig. 1 Temperature function T.

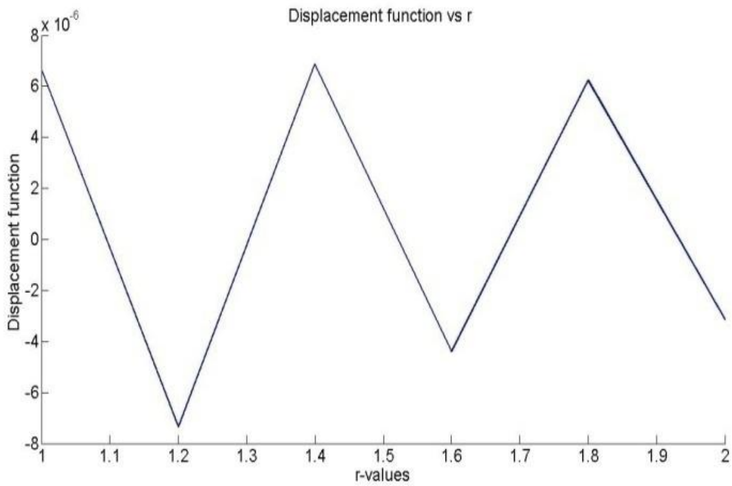

Fig.2 Displacement function ϕ.

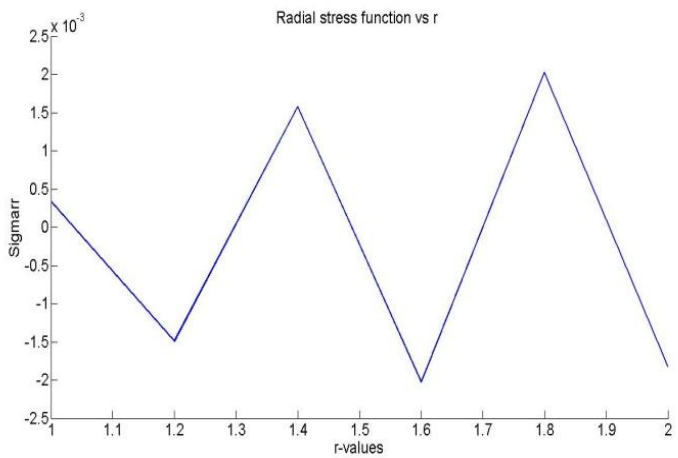

Fig. 3 Radial stress function σ_{rr}.

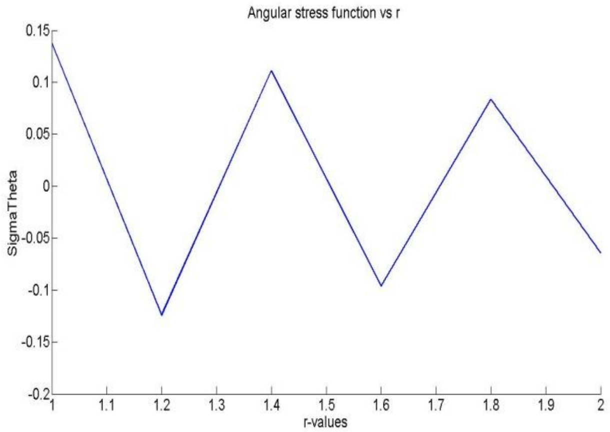

Fig. 4 Angular stress function $\sigma_{\theta\theta}$.

From figure 1, it is observed that temperature increases for $1 \leq r \leq 1.2$, $1.4 \leq r \leq 1.6$ and $1.8 \leq r \leq 2$ along radial direction. Temperature decreases for $1.2 \leq r \leq 1.4$ and $1.6 \leq r \leq 1.8$. Overall behavior of temperature along radial direction is decreasing from the inner circular surface to outer circular surface of a thin hollow cylinder. From figure 2, it is observed that the displacement function ϕ decreases for $1 \leq r \leq 1.2$, $1.4 \leq r \leq 1.6$ and $1.8 \leq r \leq 2$ along radial direction. It increases for $1.2 \leq r \leq 1.4$ and $1.6 \leq r \leq 1.8$ along radial direction. Overall behavior of displacement function along radial direction is decreasing from the inner circular surface to outer circular surface of a thin hollow cylinder.

From figure 3, it is observed that the radial stress σ_{rr} decreases from for $1 \leq r \leq 1.2$, $1.4 \leq r \leq 1.6$ and $1.8 \leq r \leq 2$ along radial direction. It increases for $1.2 \leq r \leq 1.4$ and $1.6 \leq r \leq 1.8$ along radial direction. Overall behavior of radial stress σ_{rr} is tensile towards outer circular surface along radial direction.

From figure 4, it is observed that the angular stress function $\sigma_{\theta\theta}$ for $1 \leq r \leq 1.2$, $1.4 \leq r \leq 1.6$ and $1.8 \leq r \leq 2$ along radial direction. It increases for $1.2 \leq r \leq 1.4$ and $1.6 \leq r \leq 1.8$ along radial direction. Overall behavior of angular stress function $\sigma_{\theta\theta}$ is compressive towards outer circular surface along radial direction.

VI. CONCLUSION

We can conclude that due to instantaneous internal heat generation, temperature and displacement decreases from the inner circular surface to a outer circular surface along radial direction of thin hollow cylinder, whereas the radial stress σ_{rr} is tensile in nature and the angular stress function $\sigma_{\theta\theta}$ is compressive in nature along radial direction of thin hollow cylinder. The results obtained here are useful in engineering problems particularly in the determination of state of stress in a, base of furnace of boiler of a thermal power plant, gas power plant and the measurement of aerodynamic heating.

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