International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 5 || May. 2014 || PP-20-29

On Path-Following Dynamical Systems Trajectories for Linear Optimization Problems in Standard Form

Ukwu Chukwunenye
Department of Mathematics University of Jos P.M.B 2084 Jos, Nigeria

ABSTRACT : *This article examined path-following methods for solving linear programming problems. Then it collected key relevant results in Ukwu [1], formulated analogous continuous dynamical systems, and established a sequence of propositions which were invoked to prove that the system's trajectories converge to the optimal solution of a linear programming problem in standard form, under appropriate conditions. These results were made possible by the exploitation of norm properties, their derivatives and the theory of ordinary differential equations, paving the way for the pursuits of satisficing solutions of some linear optimization problems that may not require exact optimal solutions but particular solutions at specified tolerance levels, within feasible guidelines or constraints.*

KEYWORDS: *Convergent, Dynamics, Karmarkar, Optimal, Trajectories.*

I. INTRODUCTION AND MOTIVATION

The application of linear programming to business, management, engineering and structured decision processes has been quite remarkable. Since the development of the simplex method in 1947 by G.B. Dantzing, there has been a flurry of research activities in the designing of solution methods for linear programming, mostly aimed at realizing more effective and efficient algorithmic computer implementations and computing complexity reductions. In the Fall of 1984, Karmarkar [2] of AT & Bell Laboratories proposed a new polynomial-time algorithm for solving linear optimization problems. The new algorithm not only possesses better complexity than the Simplex method in the worst-case scenario, but also shows the potential to rival the Simplex algorithm for large-scale, real-world applications. This development quickly captured the attention of Operations Research community. Radically different from the Simplex method, Karmarkar's original algorithm considers a linear programming problem over a simplex structure and moves through the interior of the polytope of feasible domain by transforming the space at each step to place the current solution at the center of the polytope in the transformed space. Then the solution is moved in the direction of projected steepest descent far enough to avoid the boundary of the feasible region in order to remain interior. Next, the inverse transformation of the improved solution is taken to map it back to the original space to obtain a new interior solution. The process is repeated until an optimum is obtained with a desired level of accuracy.
Karmarkar's standard form for linear programming can be described as follows:

$$(LP_1)\begin{cases}\min\ c^T x \\ s.t.\ Ax = 0 \\ \quad e^T x = 1 \\ \quad x \ge 0\end{cases} \tag{1}$$

where A is an $m \times n$ matrix of full row rank, $e = (1,1,...,1)^T$ is the column vector of n ones, c is an *n*-dimensional column vector and T denotes transpose.
The basic assumptions of Karmarkar's algorithm include:

$$Ae = 0 \tag{2}$$

the optimal objective value of (1) is zero. (3)

Notice that if we define $x^0 = \frac{e}{n}$, then assumption (2) implies that x^0 is a feasible solution of (1) and each component of x^0 has the positive value $\frac{1}{n}$. Any feasible solution *x* of (1) is called an interior feasible solution if each component of *x* is positive. This implies that *x* is not on the boundary of the feasible region; needless to say that the constraint $e^T x = 1$ in (1) implies that $\sum_{j=1}^{n} x_j = 1$. Therefore $e = (1,1,...,1)^T$, leading to the conclusion that a consistent problem in Karmarkar's standard form has a finite optimum. In Fang and Puthenpura [3], it is shown that any linear programming problem in standard form can be expressed in Karmarkar's standard form. Karmarkar's algorithm and its specifics are well-exposed in [3]. Karmarkar's algorithm is an interior-point iterative scheme for solving linear programming problems.

Interior-point methods approach the optimal solution of the linear program from the interior of the feasible region by generating a sequence of parameterized interior solutions. The specifics of these methods have already been discussed in [1]. The primary focus of this article will be on path-following methods. The basic idea of path-following is to incorporate a barrier function into the linear objective. By parameterizing the barrier function, corresponding minimizers form a path that leads to an optimal solution of the linear program.

The main motivation for this work comes from the work of Shen and Fang [4], in which the "generalized barrier functions" for linear programming were defined to create an ideal interior trajectory for path-following. The key components such as the moving direction and the criterion of closeness required for a path-following algorithm were introduced for designing a generic path-following algorithm with convergence and polynomiality proofs under certain conditions.

This work is aimed at exploiting the convergence results in [4] to a parameterized continuous dynamical system. This would lead to the construction of appropriate energy and Lyapunov functions which would be utilized to show that the trajectories of the dynamical system converge to the optimal solution of the linear program under appropriate assumptions.

One is not aware of any interior-point dynamic solver reported in the literature. Most dynamic solvers have been used for the neural network approach. Such investigations can be referred to in Bertsekas [5], Cohen and Grossberg [6], Hopfield and Tanks [7], Wang [8, 9, 10, 11],and Zah [12]. In section 4, we formulate our supporting propositions and main results. Section 5 presents our conclusions and direction of a follow-up research.

II. INTERIOR-POINT METHODS

Interior-point methods such as Affine Scaling Methods, Potential Reduction Methods and Path-Following Methods are well exposed in [1]. Our main result hinges on Path-following methods.

2.1 Path-following Methods

Consider a linear program:

$$(P)\begin{cases} \min\ c^T x \\ s.t.\ Ax = b \\ \quad x \geq 0 \end{cases} \tag{4}$$

Let $W = \{x \in \mathbf{R}^n : Ax = b,\, x \geq 0\}$ and $W_0 = (x \in \mathbf{R}^n : Ax = b,\, x > 0)$ be the interior feasible domain.

Let the following assumptions hold:

A has full row rank, (5)

$W \equiv \overline{W_0} \neq \Phi$, where $\overline{W_0}$ is the closure of W_0, and Φ is the empty set (6)

W is compact. (7)

2.2 Definition

A function $\phi : W \to \overline{\mathbf{R}}$ is called a generalized barrier function for linear programming (GBLP), if

$(P1)$ $\phi : W \to \overline{\mathbf{R}}$ is proper, strictly convex and differentiable, where $\overline{\mathbf{R}}$ is the extended real line.

Remark

The properness property of ϕ is equivalent to the requirement that $\phi(x)$ be strictly bounded below by $-\infty$ for all $x \in W$ and be strictly bounded above by ∞ for some $x \in W$.

(P2) if the sequence $\{x^k\} \subset W_0$ converges to x with the i^{th} component, $x_i = 0$ then $\lim_{k\to\infty} \left(\nabla\phi\left(x^k\right)\right)_i = -\infty$

(P3) the effective domain of ϕ contains W_0. Equivalently, $W_0 \subset \{\ x \in W\ s.t\ \phi(x) \text{ is finite }\ \}$.

Let $\mu > 0$ and define an augmented primal problem (P_u) associated with a *GBLP* function follows:

$$(P_u)\begin{cases}\min\ c^T x+\mu\phi(x)\\ s.t.\ Ax=b\\ \quad x\ge 0\end{cases}\tag{8}$$

Then we have the following results from [4]:

[1] (P_u) has a unique optimal solution, denoted by $x(\mu)$, in W_0,

[2] $\{c^T z(\mu)\}$ is a monotone decreasing function in μ,

[3] The set $\{x(\mu):\mu>0\}$ characterizes an interior, continuous curve in W_0.

[4] Given a decreasing positive sequence μ_k such that $\lim_{k\to\infty}\mu_k=0$, if $x^*=\lim_{k\to\infty}x(\mu_k)$ then x^* is the optimum of (*P*).

[5] Suppose that $\bar{x}$ is a given interior feasible solution to (*P*) and Δx solves the problem:

$$(P^1_\mu)\begin{cases}\min\left[c+\mu\nabla\phi(\bar{x})^T\Delta x\right]\\ s.t.\ \ A\Delta x=0\\ \|\tilde{X}^{-1}\Delta x\|^2\le\beta^2<1,\end{cases}\tag{9}$$

where $\tilde{X}$ is any positive definite symmetric matrix and $0<\beta<1$. Then, Δx defines a moving direction at $\bar{x}$ below:

$$\Delta x=-\tilde{X}\left[I-\tilde{X}A^T\left(A\tilde{X}^2A^T\right)^{-1}A\tilde{X}\right]\tilde{X}\left(c+\mu\Delta\phi(\bar{x})\right).$$

Under appropriate condition on $\tilde{X}$, *c* and ϕ, it is proved in [4], that any convergent feasible sequence of solutions to (P_u) must converge to the optimal solution to (*P*) as $\mu\to 0^+$.

The next section collects a sequence of lemmas in [1] needed in the proof of the asymptotic behavior of the system's trajectories.

III. CONTINUOUS DYNAMICAL SYSTEMS RESULTS

In this section, we formulate an analogous continuous dynamical system and prove that the system's trajectories converge to the optimal solution of (*P*) under some appropriate conditions.

Let *A* be an m x n matrix of full row rank. Let $\mathbf{R}^n_+$ denote the set $\{x\in\mathbf{R}^n : x\ge 0\}$. For $x\in W$, let $z=x+r$ for some $r\in R^n$. Then $Ax=b\Leftrightarrow Az-Ar=b$. Taking $r=-A^T\left(AA^T\right)^{-1}b$, we see that $Az=0$. Therefore, the system $Az=0$ is consistent if and only if the system $Ax=b$ is consistent. In the sequel we let $r\in\mathbf{R}^n$ be such that $Ar=-b$ and $z=x+r$.

Let β be a constant such that $0<\beta<1$. Let $\tilde{X}$ be any positive definite symmetric matrix of order *n* and let *p* and *q* be norm conjugates of each other such that $1\le p,q\le\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.

Let a_j denote the j^{th} row of *A* for $j\in\{1,2,...,m\}$. For fixed *A*, define the map $\|A(\cdot)\|:\mathbf{R}^n\to\mathbf{R}^1_+$ by

$$\|Az\|_p=\left(\sum_{j=1}^{m}|a_jz|^p\right)^{\frac{1}{p}}.$$

<u>Remark</u>

$\|Az\|_p$ is called the *p*-norm of the function Az. Unless explicitly stated we use the 2-norm in this article.

Define $\|A\|_2=\max_{z\ne0}\left\{\frac{\|Az\|_2}{\|z\|_2}\right\}$. For all $x\in W$ and for any given $r\in\mathbf{R}^n$ such that $z=x+r$ consider the function:

$$E_{p,p,r,\mu,\lambda}(z) \equiv E(z)$$

$$= c^T(z-r) + p\|Az\|_p + r\sum_{j=1}^{n}(z-r)_j^- + \mu\phi(z-r) + \lambda\left(\|\tilde{X}^{-1}z\|_2^2 - \beta^2\right) \qquad (10)$$

where

$$w^- = \max\{-w, 0\} = \begin{cases} -w \text{ if } w < 0 \\ 0 \text{ if } w \geq 0 \end{cases} \text{ and } w^+ = \max\{w, 0\} = \begin{cases} w \text{ if } w > 0 \\ 0 \text{ if } w \leq 0 \end{cases},$$

for $w \in \mathbf{R}$; $\mu > 0$, $r \geq 0$, $p > 0$, $\lambda \geq 0$.

Note the following:

[1] Property (P1) of definition 2.2 implies that, $\phi(z-r) > -\infty$ for all z in the null-space of A and $\phi(z-r) < \infty$ for at least one z in the null-space of A.

[2] Properties (P1) and (P2) of definition 2.2 imply that $\|\phi(z-r)\|_2 < \infty$, for.

[3] Property (P2) of definition 2.2 implies that if $\{z^k : k = 1, 2, ...\}$ is any positive convergent sequence in the null-space of A such that $\lim_{k\to\infty}(z^k - r) = \hat{z} - r$, with $(\hat{z}-r)_j = 0$, for some $j \in \{1, 2, ..., n\}$, then $\lim_{k\to\infty}\left(\nabla_z\phi(z^k - r)\right)_j = -\infty$. The latter ensures that the minimization of $E(z)$ is never achieved at the boundary of the set $\{z : z - r \geq 0\}$, using a gradient projection method in the minimization program.

[4] $p\|Az\|_p$ is penalty for the violation of $z \in N(A)$.

[5] $r\sum_{j=1}^{n}(z-r)_j^-$ penalizes violations of $z - r \geq 0$.

[6] $\lambda\left(\|\tilde{X}^{-1}z\|_2^2 - \beta^2\right)$ is the Lagrangian term associated with the constraint $\|\tilde{X}^{-1}z\|_2^2 - \beta^2 \leq 0$.

The stage is now set for our dynamical system formulation and the proof of our main result with the aid of the results in [1].

Let $0 < \beta < 1$ and

$S_1 = \{z \in \mathbf{R}^n : Az - 0\} \equiv N(A)$, $S_2 = \{z \in \mathbf{R}^n : \|\tilde{X}^{-1}z\|_2^2 - \beta^2 \leq 0\}$, and $T = \{z \in \mathbf{R}^n : z - r \geq 0\}$,

where r is given and defined as on the previous page. Let t and t_0 be any pair of time variables such that $t \geq t_0 \geq 0$, and let z_0 be an n-dimensional column vector.

For a differentiable function $D : \mathbf{R}^n \to \mathbf{R}^1$, let $\nabla_z D(z)$ denote the gradient of $D(z)$ with respect to z. Observe that $\nabla_z D(z) \in \mathbf{R}^n$ for each $z \in \mathbf{R}^n$.

Consider the following dynamical system:

$$\dot{z}(t) = -\nabla_z E_{p,p,r,\mu,\lambda}(z(t));\ t \geq t_0 \geq 0 \qquad (11)$$

$$z(t_0) = z_0 \qquad (12)$$

$$z_0 \in S_2 \cap \text{int}(T). \qquad (13)$$

Then $\sum_{j=0}^{n}(z_0 - r)_j^- = 0$, using the definition, $w^- = \max\{-w, 0\}$ and the fact that $z_0 - r > 0$, by virtue of z_0 being in int(T).

System (11) can be treated as a control system of the form:

$$\dot{z} = -c + A^T\upsilon^{(1)} + \upsilon^{(3} \qquad (14)$$

where $\upsilon^{(1)} \equiv \upsilon^{(1)}(z) = -p\nabla_{Az}\|Az\|_p$, $\upsilon^{(2)} = \upsilon^{(2)}(z) = -\mu\nabla\phi(z-r)$ and

$\upsilon^{(3)} \equiv \upsilon^{(3)}(z) = -\lambda\nabla_z\left(\|\tilde{X}^{-1}z\|_2^2 - \beta^2\right)$.

$\upsilon^{(1)}, \upsilon^{(2)}$, and $\upsilon^{(3)}$ can be regarded as controls. These controls will be implemented such that the trajectories of (11), (12) and (13) will be forced into the feasible region $S_1 \cap S_2 \cap \text{int}(t)$ and maintained there while moving in a direction that decreases $c^T(z-r)$. The following sequence of lemmas will be found useful in the sequel.

Let I_n be the identity matrix of order *n* and let $P = I_n - A^T(AA^T)^{-1}A$ be the projection matrix onto the null-space of *A*.

The following results were established in [1]

3.1 Lemma

The dynamics of system (11) when restricted to $S_1 \cap S_2 \cap \text{int}(t)$ are described by:

$$\dot{z} = -P\nabla_z E(z) \tag{15}$$

$$Az = 0 \tag{16}$$

3.2 Lemma

If $0 \neq z \in S_1 \cap S_2$, then $P\nabla_z\left(\left\|\tilde{X}^{-1}z\right\|_2^2 - \beta^2\right) \neq 0$

3.3 Lemma

For any $p \geq 1$

and $\forall z \notin S_1$,

$$\nabla_z\left(\|Az\|_p\right) = \frac{A^T}{\|Az\|_p^{p-1}}\left(|a_1 z|^{p-1}\operatorname{sgn}(a_1 z), |a_2 z|^{p-1}\operatorname{sgn}(a_2 z), \cdots, |a_m z|^{p-1}\operatorname{sgn}(a_m z)\right)^T \tag{17}$$

3.4 Lemma (Noble [13], p429)

For any square matrix *M*,

$$\|M\|_2 = \left\{\text{maximum eigenvalue of } M^T M\right\}^{1/2}$$

3.5 Definition

The core of the target S_1 for the dynamical system:

$$\dot{z} = -\nabla_z E(z) \tag{18}$$

is the set $\{z_0 \in \mathbf{R}^n\}$ of all initial points that can be driven to the target S_1 in finite time and maintained there, thereafter by an appropriate implementation of some feasible control procedure.

Denote this set by $\text{core}(S_1)$. The following lemma demonstrates that under certain conditions $\text{core}(S_1)$ is nonempty.

3.6 Lemma

If:

$$z^T A^T (AA^T)^{-1} A\nabla_z E(z) \geq k\|Az\|_p \tag{19}$$

for some $p \geq 1$ and $\forall z \notin S_1$, with some $k > 0$, then the trajectories of:

$$\dot{z} = -\nabla_z E(z) \tag{20}$$

$$z(0) = z_0 \tag{21}$$

hit S_1 in finite time and remain there thereafter.

Our task is to construct appropriate energy and Lyapunov functions and impose appropriate conditions under which the trajectories of:

$$\dot{z} = -\nabla_z E(z) \quad (22)$$

$$z(0) = z_0 \quad (23)$$

converge to $z_\mu - r$, where $z_\mu - r$ is an optimal solution of (5) for each $\mu > 0$. Then we can appeal to Theorem 2.2 of [4] to assert the convergence to the optimal solution of (P), noting that a positive decreasing sequence of parameters $\mu^{\bar{k}}$ with limit 0 may be used in place of μ.
This above approach holds a lot of promise for an extension of our result to neural networks where dynamical systems, energy and Lyapunov functions are used extensively.

IV. MAIN RESULT

The following tool is needed for the proof of the main result:
Let d_1 be a nonnegative constant such that:

$$\left\| w^T A^T \left(AA^T\right)^{-1} \right\|_q \le d_1 \left\| w^T A^T \left(AA^T\right)^{-1} \right\|_2 \quad (24)$$

$\forall\ w \in \mathbf{R}^n$ and $\forall q \ge 1$. See Stoer and Bulirsch [14], p. 185.

Let $d_2 = \left[\max\{u_1, u_2, ..., u_n\}\right]^{\frac{1}{2}}$, where u_i is an eigenvalue of $\tilde{X}^{-2}$ for $i \in \{1, 2, ..., n\}$. See [13], p. 429.

<u>Remark</u>
The positive definiteness of $\tilde{X}$ guarantees the existence and the positivity of d_2. Also the existence of d_1 is assured by the equivalence property of *p*-norms. Therefore d_1 and d_2 are well-defined. Furthermore, $\left\|\tilde{X}^{-1}\right\|_2 = d_2$ and $\left\|\tilde{X}^{-1}\right\|_2^2 = d_2^2$, using ([13], p. 429) and the fact that α is an eigenvalue of M if and only if α^j is an eigenvalue of M^j for any positive integer *j*.

4.1 Theorem
Let $1 \le p \le \infty$ and let the following condition hold:

$$p - k \ge \left(d_1 \|c\|_2 + \mu \left\|\nabla_z \phi(z-r)\right\|_2 + 2d_1 d_2 \lambda\right) \left\| A^T \left(AA^T\right)^{-1} \right\|_2 \quad (25)$$

where k, μ and λ are sufficiently small, *p* is sufficiently large and $z \in S_2 \cap \text{int}(T)$. If $z \in S_1^c \cap S_2 \cap \text{int}(T)$, then condition (19) of lemma (3.4) holds and the equilibrium of the system (3.2) is asymptotically stable. Furthermore, if μ in (3.2) is replaced by a positive decreasing sequence of parameters $\left\{\mu^{\bar{k}}, \bar{k} = 1, 2, ...\right\}$ with limit being zero, then the solution of (3.2) in x-space converges to the optimal solution of the linear programming problem (*P*).

Proof
The strategy for the proof will be as follows:
[1] We show that the condition (19) of lemma (3.4) holds, which will assure the feasibility of the trajectories of (3.2) at some time $\bar{t}$ and thereafter.
[2] We show that the function $E_{p,p,\lambda,\mu,r}(z)$ is a nonincreasing time function along the trajectories of (3.4) and that the time derivative vanishes only at an equilibrium solution of (3.4).
[3] We show that the equilibria of (3.4) satisfy the Karush-Kuhn-Tucker Optimality Condition. This would guarantee the uniqueness of the equilibrium.
[4] Finally we construct a Lyapunov function which is negative definite along the trajectories of the system (3.4) and then invoke the Lyapunov theory to conclude that the trajectories converge to the unique equilibrium of (3.4). Consequently, the results of Chapter two and, in particular, Theorem 2.2 of [15] can be used to conclude that the trajectories converge to the optimal solution of problem *P*.

Suppose the trajectories of (3.2) are restricted to S_1. By lemma (3.1), the dynamics of (3.2) are given by:

$$\dot{z} = -P\nabla_z E(z) \quad (26)$$

If $z \in S_1^c \cap \text{int}(T)$, then Lemma (3.4) implies that the trajectories of (3.2) reach S_1 in finite time and could be forced to remain there thereafter, if (19) holds, now:

$$\dot{z} = -c - p\nabla_z \left(\|Az\|_p \right) - \mu \nabla_z \phi(z-r) - \lambda \nabla_z \left(\left(\|\tilde{X}^{-1} z\|_2^2 - \beta^2 \right) \right) \quad (27)$$

Next, we expand $z^T A^T \left(AA^T \right)^{-1} A\nabla_z E(z)$ as follows:

$$z^T A^T \left(AA^T \right)^{-1} A\nabla_z E(z)$$

$$= z^T A^T \left(AA^T \right)^{-1} Ac + z^T A^T \left(AA^T \right)^{-1} Ap\nabla_z \left(\|Az\|_p \right)$$

$$+ \mu z^T A^T \left(AA^T \right)^{-1} A\nabla_z \phi(z-r) + 2\lambda z^T A^T \left(AA^T \right)^{-1} A\tilde{X}^{-2} z \triangleq T_1 + T_2 + T_3 + T_4 \quad (28)$$

From Lemma (3.3), we get:

$$T_2 = \frac{p z^T A^T \left(AA^T \right)^{-1} AA^T}{\|Az\|_p^{p-1}} \left[|a_1 z|^{p-1} \operatorname{sgn}(a_1 z), |a_2 z|^{p-1} \operatorname{sgn}(a_2 z), \cdots, |a_m z|^{p-1} \operatorname{sgn}(a_m z) \right]^t$$

$$= \frac{p (Az)^T}{\|Az\|_p^{p-1}} \left[|a_1 z|^{p-1} \operatorname{sgn}(a_1 z), |a_2 z|^{p-1} \operatorname{sgn}(a_2 z), \cdots, |a_m z|^{p-1} \operatorname{sgn}(a_m z) \right]^t$$

$$= \frac{p}{\|az\|_p^{p-1}} \left(a_1 z, a_2 z, \ldots a_m z \right) \left[|a_1 z|^{p-1} \operatorname{sgn}(a_1 z), |a_2 z|^{p-1} \operatorname{sgn}(a_2 z), \cdots, |a_m z|^{p-1} \operatorname{sgn}(a_m z) \right]^t \quad (29)$$

Hence:

$$T_2 = \frac{p}{\|Az\|_p^{p-1}} \sum_{i=1}^{m} \left(|a_i z|^{p-1} (a_i z) \operatorname{sgn}(a_i z) \right) = \frac{p}{\|Az\|_p^{p-1}} \sum_{i=1}^{m} |a_i z|^p \ \left(\text{using } \upsilon \operatorname{sgn}(\upsilon) = |\upsilon| \right)$$

$$= \frac{p}{\|Az\|_p^{p-1}} \|Az\|_p^p = p\|Az\|_p \quad (30)$$

$$|T_1| = \left| z^T A^T \left(AA^T \right)^{-1} Ac \right| = \left| c^T A^T \left(AA^T \right)^{-1} Az \right| \le \left\| c^T A^T \left(AA^T \right)^{-1} \right\|_q \|Az\|_p$$

$$\le d_1 \left\| c^T A^T \left(AA^T \right)^{-1} \right\|_2 \|Az\|_p \le d_1 \left\| c^T \right\|_2 \left\| A^T \left(AA^T \right)^{-1} \right\|_2 \|Az\|_p \quad (31)$$

(by Hölders inequality and [12], p. 482)

where p is the conjugate exponent of q; that is $1 \le p, q \le \infty, \frac{1}{p} + \frac{1}{q} = 1$.

$$|T_4| = 2\lambda \left| z^T A^T \left(AA^T \right)^{-1} A\tilde{X}^{-1} z \right| \le 2\lambda \left\| z^T \tilde{X}^{-2} A^T \left(AA^T \right)^{-1} \right\|_q \|Az\|_p$$

$$\le 2d_1\lambda \left\| z^T \tilde{X}^{-2} A^T \left(AA^T \right)^{-1} \right\|_2 \|Az\|_p \le 2d_1\lambda \left\| z^T \tilde{X}^{-1} \right\|_2 \left\| \tilde{X}^{-1} \right\|_2 \left\| A^T \left(AA^T \right)^{-1} \right\|_2 \|Az\|_p$$

$$\le 2d_1\lambda d_2 \left\| A^T \left(AA^T \right)^{-1} \right\|_2 \|Az\|_p$$

since $z \in S_2$, $0 < \beta < 1$ and in view of the definition of d_2 following (24) by a similar reasoning,

$$|T_3| \le d_1 \mu \left\| \nabla_z \phi(z-r) \right\|_2 \left\| A^T \left(AA^T \right)^{-1} \right\|_2 \|Az\|_p$$

Let: $$S = T_1 + T_2 + T_3 + T_4 \quad (32)$$

Then: $$S \ge k\|Az\|_p \quad (33)$$

$\forall z \notin S_1$ with some *k*, if:

$$-|T_1| + T_2 - |T_3| - |T_4| \ge k\|Az\|_p \quad (34)$$

since $z^T A^T \left(AA^T\right)^{-1} A\nabla_z E(z) \geq -|T_1| + |T_2| - |T_3| - |T_4|$. Consequently (19) is satisfied if (34) is satisfied. Moreover (34) is satisfied if:

$$p - k - d_1\left(\|c\|_2 + \mu\|\nabla_z\phi(z-r)\|_2 + 2d_2\lambda\right)\left\|A^T\left(AA^T\right)^{-1}\right\|_2 \geq 0 \qquad (35)$$

So that:

$$p - k \geq d_1\left(\|c\|_2 + \mu\|\nabla_z\phi(z-r)\|_2 + 2d_2\|\lambda\|_2\right)\left\|A^T\left(AA^T\right)^{-1}\right\|_2 \qquad (36)$$

(36) is satisfied $\forall z \in S_1^c \cap S_2 \cap \operatorname{int}(T)$ if μ, k and λ are sufficiently small, p is sufficiently large and for any sequence $\left\{z^{\bar{k}}\right\}_1^x$, such that $\left(z^{\bar{k}} - r\right)_j \nrightarrow 0$ as $\bar{k} \to \infty$ in view of (P2) of Definition 2.2. Now, (36) $\Rightarrow$ (35) $\Rightarrow$ (34) $\Rightarrow$(19). Hence, the null-space feasibility condition, $z(t) \in S_1$ can be restored only in the interior of *T*, if (36) holds.

To complete the proof, we need the following sequence of results which shows that starting from an interior feasible point, the trajectories of (3.4) transformed to the *x*-space will converge to the optimal solution of the corresponding linear program (*P*).

4.2 Proposition

The function $E_{p,p,\lambda,\mu,r}(z)$ defined in (10) is a nonincreasing function of time on the trajectories of (3.4).

Proof

$$\frac{d}{dt}E(z) = \nabla_z^T E(z)\dot{z} = \nabla_z^T E(z)\left[-P\nabla_z E(z)\right] = -\nabla_z^T E(z) P^T P\nabla_z E(z)\left(\text{using } P = P^T = P^2\right)$$

$$= -\|P\nabla_z E(z)\|_2^2 \leq 0 \qquad (37)$$

as desired.

4.2 Proposition

The time derivative of $E(z)$ vanishes only at any equilibrium of (26).

Proof

An equilibrium of (26) satisfies the relation $P\nabla_z E(z) = 0$.

$$\frac{d}{dt}E(z) = 0 \qquad (38)$$

if $-\|P\nabla_z E(z)\|_2^2 = 0$, by (37). Therefore:

$$P\nabla_z E(z) = 0 \qquad (39)$$

showing that any solution to (38) satisfies (39) and consequently is an equilibrium point (26).

4.3 Proposition

An equilibrium of (26) satisfies the Karush-Kuhn-Tucker Optimality Condition.

Proof

By the definition of equilibria:

$$P\nabla_z E(z) = 0 \qquad (40)$$

(40) implies that $\nabla_z E(z)$ is in null-space of *P*. $AP = 0$ and so:

$$P^T A^T = PA^T = 0 \qquad (41)$$

(40) and (41) show that $\nabla_z E(z)$ is in the range space of A^T, thus there exists $a\ w \in \mathbf{R}^m$ such that:

$$\nabla_z E(z) = A^T w \qquad (42)$$

We deduce from (42) that:

$$c + \mu\nabla_z\phi(z-r) + 2\lambda\tilde{X}^{-2}z = A^T w \qquad (43)$$

Let $s = -\mu\nabla_z\phi(z-r) - 2\lambda\tilde{X}^{-2}z$. Then:

$$A^T w + s - c = 0 \quad (44)$$
$$Az = 0$$
$$\mu \nabla_z \phi(z - r) + 2\lambda \tilde{X}^{-2} z = -s$$
$$z - r > 0.$$

The condition (44) coincides with the Karush-Kuhn-Tucker Optimality Condition (46) of $\nabla_z E(z)$ in [4] if and only if:

$$\lambda \tilde{X}^{-2} z = 0 \quad (45)$$

in which case the positive definiteness of $\tilde{X}$ assures that:

$$\lambda z = 0 \quad (46)$$

Note also that the convexity of the set $S_1 \cap S_2 \cap \text{int}(T)$ assures the uniqueness of solution to the Karush-Kuhn-Tuker Optimality Condition and consequently the uniqueness of the equilibrium of (26). From (37) we deduce that $\frac{d}{dt} E(z) < 0$ for $z \neq z_\mu$, where z_μ is the unique equilibrium of (26). This establishes that the equilibrium of the system (26) is asymptotically stable. Now let $\mu \to 0$ to deduce that $z_\mu - r \to x^*$, the optimal solution of (P). (See [4], theorem 2.2).

The condition $\lambda z = 0$ in (46) is a complementary-slackness-like condition: If $z_j \neq 0$ for some $j \in \{1,2,\ldots,n\}$, then $\lambda = 0$; and $\lambda > 0$

For potential Neural Network applications we also use the Lyapunov theory to show that the trajectories of (26) converge to the unique equilibrium of (26). To this end, we need to construct an appropriate Lyapunov function.

4.5 Definition

A function $V : D \subset \mathbf{R}^n \to \mathbf{R}$ is called a Lyapunov function if V is continuously differentiable and positive definite on D. if V is negative definite along the trajectories of the dynamical system $\frac{d}{dt} x(t) = f(x(t))$, where $f(0) = 0$, and $x \in D$, then the equilibrium $x = 0$ is asymptotically stable. (See [13], pp. 203-205)

It is not clear that $z = 0$ is the unique equilibrium of (26); therefore we must perform a linear translation of the equilibrium to the origin.

Let z_μ be the equilibrium solution of (4.2). Let $y = z + z_\mu$. Then:

$$\dot{z} = -P\nabla_z E(z + z_\mu) + P\nabla_z E(z_\mu) = -P\nabla_z E(z + z_\mu) \quad (47)$$

since $P\nabla_z E(z_\mu) = 0$.

Clearly $z = 0$ is equilibrium of the transformed system and is unique. Let:

$$V(z) = \left\| P\nabla_z E(z + z_\mu) \right\|_2^2 \quad (48)$$

Then $V(0) = 0$ and $V(z) > 0$ if $z \neq 0$. Also $V(z)$ is continuous in *z*. therefore $V(z)$ is a Lyapunov function. Note that:

$$\frac{d}{dt} V(z(t)) = 2\left[P\nabla_z E(z + z_\mu)\right]^T \dot{z} = -2\left\| P\nabla_z E(z + z_\mu) \right\|_2^2 \quad (49)$$

which vanishes if and only if $z = 0$. Therefore $V(z)$ the derivative of is negative definite along the solution (on the trajectories) of (47). We now appeal to Lyapunov theory to assert that the origin is an asymptotically stable solution of (47). Consequently the equilibrium z_μ of the system (26) is asymptotically stable. Now let $\mu \to 0$. Then $z_\mu - r \to x^*$, the optimal solution of (*P*). This comes from theorem 2.2 of [4].

VI. CONCLUSION

This work was motivated partly by interior-point concepts and largely by the path-finding methods in [4] for solving linear programming problems. Many real-life problems which could be formulated as linear programming problems are dynamic in nature; for example, the inventory level of some item at a given time and changes in demand levels of some consumer goods due to price fluctuations and seasonal variation. Also on-line optimization may be required in many application areas, such as satellite guidance, robotics and oil outputs from oil wells and refinery operations. Some of these problems may not require exact optimal solutions but particular solution at specified tolerance levels, within feasible guidelines or constraints. In particular solutions at positive levels may be desired for all decision variables, implying that interior solutions are desired. These and many other problems of the continuous variety could be more realistically modelled by continuous dynamical systems. Unfortunately, research in this direction has been based mainly on neural network approach, none of which is interior-point oriented. In section 3, we formulated an interior-point based dynamical system for solving linear programming problems in standard form. The key ideas for this formulation came from the examination of [4]. Then, we stated that under certain conditions, the solutions of our dynamical system would converge to the solution of a corresponding linear programming problem in standard form. We proceeded cite some relevant results in [1] and establish a sequence of propositions which would be needed to prove that our solutions would have the right convergence property-the convergence of the trajectories of the dynamical system to the optimal solution of the linear program under appropriate assumptions. This approach holds a lot of promise for an extension of our result to neural networks where dynamical systems, energy and Lyapunov functions are used extensively.

REFERENCES

[1] Ukwu (2014o). *"On interior-point methods, related dynamical systems results and cores of targets for linear programming". International Journal of Mathematics and Statistics Invention (IJMSI), Vol. 2, Issue 5, April 2014.*

[2] Karmarkar, N.K.,(1984).*"A new polynomial-time algorithm for linear programming," Combinatorica, Vol.4, pp. 373-395.*

[3] Fang, S.C., and Puthenpura, S.(1984).*"Linear optimization and extensions," Prentice Hall, Englewood Cliffs, N.J.*

[4] Shen, R.L. and Fang, S.C.(1994).*"On the generalized path-following methods for linear programming," Optimization, Vol. 30, pp. 235-249.*

[5] Bertsekas, D.P.(1975).*"Necessary and sufficient conditions for a penalty methods to be exact," Mathematical programming, Vol. 9, pp. 87-99.*

[6] Cohen, A., and Grossberg, S.(1983).*"Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," IEEE Transactions on Systems, Man, and Cybernetics, Sept./Oct., Vol. SMC-13, No. 5, pp. 815-825.*

[7] Hopfield, J. and Tanks, D.(1983).*"Simple neural optimizations networks: An A/D converter signal decision circuit, and, and a linear programming circuit," IEEE Trans., CAS-33, (5), pp.533-541.*

[8] Wang, J. (1991).*"A time-varying recurrent neural system for convex programming," Proc. Int. Joint Conf. on Neural Networks, Seattle, Vol. 1, pp. 147-152.*

[9] Wang, J.(1992).*"Analogue neural network for solving the assignment problem," Electronics Letters, May , Vol. 28, No.11, pp. 1047-1050.*

[10] Wang, J.(1993).*"Analysis and design of recurrent neural network for linear programming," IEEE Trans. On Circuits and Systems- I, Sept., Vol. 40, No. 9, pp. 613-618.*

[11] Wang, J.(1991).*"On the asymptotic properties of recurrent neural networks for optimization," Int. J. Pattern Recognition and Artificial Intelligence, Vol. 5, No. 4, pp. 581-601.*

[12] Zah, H. (1995). "*Solving linear programming problems with neural networks: A comparative study," IEEE Transactions on NeuralNetworks, Jan. Vol. 6, No.1. pp. 94-105.*

[13] Noble, B..(1969). *"Applied linear algebra," Prentice-Hall, Inc. N.J.*

[14] Stoer, J., and Bulirsch, R.(1991).*"Introduction to numerical analysis," 2nd Ed., Springer-Verlg, N.Y.*