Some Results on $\varepsilon$-Trans-Sasakian Manifolds

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ABSTRACT: In this paper, we have studied 3-dimensional $\varepsilon$-trans-Sasakian manifold. Some basic results regarding 3-dimensional trans-Sasakian manifolds have been obtained. Locally $\varphi$-recurrent, locally $\varphi$-symmetric and $\varphi$-quasi conformally symmetric 3-dimensional $\varepsilon$-trans-Sasakian manifolds are also studied. Further some results on generalized Ricci-recurrent $\varepsilon$-trans-Sasakian manifold were given.

KEYWORDS: $\varepsilon$-trans-Sasakian manifold, locally $\varphi$-symmetric, locally $\varphi$-recurrent, quasi conformal curvature tensor, generalized Ricci-recurrent manifold.

I. INTRODUCTION

In the Gray-Hervella classification of almost Hermitian manifolds [10], there appears a class $W_\varepsilon$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $M$ is called a trans-Sasakian structure [17] if the product manifold $M \times \mathbb{R}$ belongs to the class $W_\varepsilon$. The class $C_\varepsilon \oplus C_\varepsilon$ coincides with the class of trans-Sasakian structures of type $(\alpha, \beta)$. In [15], local nature of the two subclasses, namely $C_1$ and $C_2$ structure of trans-Sasakian structures are characterized completely. Further trans-Sasakian structures of type $(0, 0), (0, \beta)$ and $(\alpha, 0)$ are cosymplectic [2], $\beta$-Kenmotsu [11] and $\alpha$-Sasakian [11] respectively. In 2003, U. C. De and M. M. Tripathi [7] obtained the explicit formulae for Ricci operator, Ricci tensor and curvature tensor in a 3-dimensional trans-Sasakian manifold. In 2007, C. S. Bagewadi and Venkatesha [1] studied some curvature tensors on a trans-Sasakian manifold. And in 2010, S. S. Shukla and D. D. Singh [19] studied $\varepsilon$-trans-Sasakian manifold. In their paper they have obtained fundamental results on $\varepsilon$-trans-Sasakian manifold. A Riemannian manifold is called locally symmetric due to Cartan if its Riemannian curvature tensor $R$ satisfies the relation $\nabla R = 0$, where $\nabla$ denotes the operator of covariant differentiation [13]. Similarly the Riemannian manifold is said to be locally $\varphi$-symmetric if $\varphi^2(\nabla g(Y, Z)) = 0$, for all vector fields $X, Y$ and $W$ orthogonal to $\xi$. This notion was introduced by T. Takahashi [20] for Sasakian manifolds. As a proper generalization of locally $\varphi$-symmetric manifolds, $\varphi$-recurrent manifolds were introduced by U. C. De and et al. [8]. Further locally $\varphi$-Quasi conformally symmetric manifolds were introduced and studied in [5]. In 2002, J. S. Kim and et al. [12] studied generalized $\varepsilon$-trans-Sasakian manifolds. A non-flat Riemannian manifold $M$ is called a generalized Ricci-recurrent manifold [6], if its Ricci tensor $\mathcal{R}$ satisfies the condition,

\[(\nabla g)(\mathcal{R})(Y, Z) = A(\mathcal{R})(Y, Z) + B(\mathcal{R})(Y, Z),\]

where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$ and $A, B$ are 1-forms on $M$. In particular, if the 1-form $B$ vanishes identically, then $M$ reduces to Ricci - recurrent manifold [18] introduced by E. M. Patterson. The paper is organized as follows: In section 2, preliminaries about the paper are provided. In section 3, the expressions for scalar curvature and Ricci tensor are obtained for three-dimensional $\varepsilon$-trans-Sasakian manifolds. In section 4, three-dimensional locally $\varphi$-recurrent $\varepsilon$-trans-Sasakian manifold are studied. Here we proved that 3-dimensional $\varepsilon$-trans-Sasakian manifold with $\alpha$ and $\beta$ constant is locally $\varphi$-recurrent if and only if the scalar curvature is constant. Further in section 5, three-dimensional $\varepsilon$-Quasi conformally symmetric $\varepsilon$-trans-Sasakian manifold are studied and proved that a 3-dimensional $\varepsilon$-trans-Sasakian manifold with $\alpha$ and $\beta$ constant is locally $\varphi$-Quasi conformally symmetric if and only if the scalar curvature is constant. Finally in section 6, some results on generalized Ricci-recurrent $\varepsilon$-trans-Sasakian manifolds were given.

II. PRELIMINARIES

Let $M$ be an $\varepsilon$- almost contact metric manifold [9] with an almost contact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ that is, $\varphi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is an indefinite metric such that

\[(2.1) \quad \varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0.\]

\[(2.2) \quad g(\xi, \xi) = \varepsilon, \quad g(X, \xi) = \varepsilon g(X, \xi),\]

\[(2.3) \quad g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y),\]

for any vector fields $X, Y$ on $M$, where $\varepsilon$ is $1$ or $-1$ according as $\xi$ is space like (or) time like.

An $\varepsilon$-almost contact metric manifold is called an $\varepsilon$-trans-Sasakian manifold [19], if

\[(2.4) \quad (\nabla_X \varphi)(Y) = a(g(X, Y)\xi - \eta(Y)\xi) + b g(\varphi(X), Y)\xi - \eta(Y)\varphi(X).\]
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Further in an ε-trans-Sasakian manifold, the following holds true:

\[
R(X, Y)\xi = (\alpha^2 - \beta^2)\eta(X)\eta(Y) + 2\alpha\beta\eta(Y)\phi X - \eta(X)\phi Y + e(Y)\phi X - (X\alpha)\phi Y + Y\beta\phi^2 X - (X\beta)\phi^2 Y,
\]

where \(\nabla\) is the Levi-Civita connection with respect to \(g\).

Definition 2.1. A non-flat Riemannian manifold \(M\) is called a generalized Ricci-recurrent manifold \([12]\), if its Ricci tensor \(S\) satisfies the condition

\[
(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),
\]

where \(\nabla\) denotes Levi-Civita connection of the Riemannian metric \(g\) and \(A\) and \(B\) are 1-forms on \(M\).

Definition 2.2. An ε-trans-Sasakian manifold is said to be locally \(\phi\)-symmetric manifold \([4]\), if

\[
(\nabla_X \phi)(Y, Z) = 0.
\]

Definition 2.3. An ε-trans-Sasakian manifold is said to be a \(\phi\)-recurrent manifold \([3]\) if there exist a non zero 1-form \(\alpha\) such that

\[
\phi^2(\nabla_X \phi)(Y, Z) = A(W)g(Y, Z),
\]

for any arbitrary vector field \(X, Y, Z\) and \(W\). If \(X, Y, Z\) and \(W\) are orthogonal to \(\xi\), then the manifold is called locally \(\phi\)-recurrent manifold. If the 1-form \(\alpha\) vanishes, then the manifold reduces to a \(\phi\)-symmetric manifold.

III. THREE DIMENSIONAL ε-TRANS-SASAKIAN MANIFOLD

Since conformal curvature tensor vanishes in a three dimensional Riemannian manifold, we get

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Y)Y),
\]

where \(r\) is the scalar curvature.

Theorem 3.1. In a three dimensional ε-trans-Sasakian manifold, the Ricci operator is given by

\[
QX = \left[\frac{1}{\alpha} - e(\alpha^2 - \beta^2) - e(\xi B)\right]X - \left[\frac{1}{\alpha} + \xi B - 3e(\alpha^2 - \beta^2)\right]\eta(X)\xi
\]

\[
+ e(\phi(\text{grad}\alpha) - \text{grad}\beta)\eta(X) - (\phi\xi)B(X)\xi - (X\beta)\xi.
\]

Proof: Substitute \(Z\) by \(\xi\) in (3.1), we get

\[
R(X, Y)\xi = g(Y, \xi)QX - g(X, \xi)QY + S(Y, \xi)X - S(X, \xi)Y - \frac{r}{2}\eta(Y)X - \eta(X)Y.
\]

Putting \(Y = \xi\) in (3.3), we get

\[
eQX = R(X, \xi)\xi + g(X, \xi)Q\xi - S(\xi, \xi)X + S(\xi, \xi)\xi + \frac{2}{\alpha}(X - \eta(X))\xi.
\]

Using (2.2), (2.7) and (2.11) in (3.4), we get (3.2).

Theorem 3.2. In a three dimensional ε-trans-Sasakian manifold, the Ricci tensor and curvature tensor are given by

\[
S(X, Y) = \left[\frac{1}{\alpha} - e(\alpha^2 - \beta^2) + \xi B\right]g(X, Y)
\]

\[
- \left[\frac{1}{\alpha} + \xi B - 3e(\alpha^2 - \beta^2)\right]\eta(X)\eta(Y).
\]
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Theorem 4.3. A three dimensional $\varepsilon$-trans-current $\varepsilon$-Trans-Sasakian manifold with $\alpha$ and $\beta$ constants is locally $\phi$-recurrent if and only if the scalar curvature is constant.

Proof: Equation (3.5) follows from (3.2). Using (3.5) and (3.2) in (3.1), we get (3.6).

Proof: Taking the covariant differentiation of the equation (3.6), we have

\[ (\nabla_W R)(X,Y)Z = \frac{\text{d}(W)}{2} - 4\varepsilon (\text{d}a(W) - \text{d}b(W) + 2(\nabla_W (\xi))^2) \big( g(Y,Z)X - g(X,Z)Y \big) \]

Putting $W = l_i e_j$ in (4.1), where $\{e_i\}, i = 1, 2, 3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

\[ R(X,Y)Z = \lambda (g(Y,Z)X - g(X,Z)Y) \]

where $\lambda = -\frac{\text{d}(w)}{2a}\phi_i$ is a scalar.

Corollary 4.1. A three-dimensional $\varepsilon$-trans-Sasakian manifold with $\alpha$ and $\beta$ constants is locally $\phi$-symmetric if and only if the scalar curvature is constant.

Theorem 4.4. A three dimensional $\varepsilon$-trans-Sasakian manifold with $\alpha$ and $\beta$ constants is locally $\phi$-Ricci symmetric if and only if the scalar curvature is constant.

Proof: Now differentiating (3.2) covariantly along $W$ we obtain

\[ (\nabla_W Q)(X) = \frac{\text{d}(W)}{2} - \frac{\text{d}(W)}{2} \eta(X) \xi \]

Applying $\phi$ on both side of (4.3) and using (2.1) we have,

\[ \phi^2 (\nabla_W Q)(X) = \frac{\text{d}(W)}{2} (-X + \eta(X) \xi) \]
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If $X$ is orthogonal to $\xi$, we get

$$\phi^2 (\nabla_W Q)(X) = -\frac{dr(W)}{2} X. \quad (4.5)$$

THREE DIMENSIONAL LOCALLY $\varphi$-QUASICONFORMALLY SYMMETRIC $\varepsilon$-TRANS-SASAKIAN MANIFOLD

The quasiconformal curvature tensor on a Riemannian manifold is given by [5]

$$C^\ast(X, Y)Z = a\mathcal{R}(X, Y)Z + b[\mathcal{S}(Y, Z)X - \mathcal{S}(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{3}(\xi) + 2b\left[ g(Y, Z)X - g(X, Z)Y \right]. \quad (5.1)$$

where $a$ and $b$ are constants and $r$ is the scalar curvature of the manifold.

**Theorem 5.5.** A three dimensional $\varepsilon$-trans-Sasakian manifold with $\alpha$ and $\beta$ constants is locally $\varphi$-quasiconformally symmetric if and only if the scalar curvature is constant.

**Proof:** Using (3.2), (3.5) and (3.6) in (5.1) we have,

$$C^\ast(X, Y)Z = (a + b)\left( [\xi] - 2\varepsilon(a^2 - \beta^2) + 2\xi] \right) \left( g(Y, Z)X - g(X, Z)Y \right) - \frac{r}{3}(\xi) + 2b\left[ g(Y, Z)X - g(X, Z)Y \right]. \quad (5.2)$$

Taking the covariant differentiation of the above equation and assuming $\alpha$ and $\beta$ as constants we have,

$$\nabla_W C^\ast(X, Y)Z = (a + b)\left( \left[ \frac{dr(W)}{3} \right] g(Y, Z)X - g(X, Z)Y \right) - \frac{r}{3}(\xi) + 2b\left[ g(Y, Z)X - g(X, Z)Y \right]. \quad (5.3)$$

Now assume that $X$, $Y$ and $Z$ are horizontal vector fields. Using (2.1) in (5.3), we get

$$\phi^2 (\nabla_W C^\ast)(X, Y)Z = (a + b)\left[ \frac{dr(W)}{3} \right] \left( g(Y, Z)X - g(X, Z)Y \right) \quad (5.4)$$

Suppose $\phi^2 (\nabla_W C^\ast)(X, Y)Z = 0$ then either $a + b = 0$ or $dr(W) = 0$. If $a + b = 0$ then substituting $a = -b$ in (5.1) we find

$$\phi^2 (\nabla_W C^\ast)(X, Y)Z = aC(X, Y)Z. \quad (5.5)$$

Where $C$ is the Weyl conformal curvature tensor. But in a 3-dimensional Riemannian manifold $C = 0$ which implies $C^\ast = 0$ and so $a + b = 0$. Therefore $dr(W) = 0$.

Using Corollary 4.1 and Theorem 5.5, we state the following Corollary:

**Corollary 5.2.** A three-dimensional $\varepsilon$-trans-Sasakian manifold is locally $\varphi$-quasiconformally symmetric if and only if it is locally $\varphi$-symmetric.

IV. GENERALIZED RICCI-RECURRENT $\varepsilon$-TRANS-SASAKIAN MANIFOLD

**Theorem 6.6.** The 1-forms $A$ and $B$ of a generalized Ricci-recurrent $(2n+1)$ dimensional $\varepsilon$-trans-Sasakian manifold are related by
In particular, we get
\begin{align*}
B(\xi) &= 2\alpha \left[ \alpha^2 - \beta^2 - \epsilon(\xi \beta) \right] - \left( \alpha^2 - \beta^2 - \epsilon(\xi \beta) \right) A(\xi).
\end{align*}

**Proof:** We have
\begin{align*}
\langle \nabla_X S(Y, Z) \rangle &= S(X, Y) - S(Y, X) - S(Y, V_X Z).
\end{align*}

Using (2.14) in (6.3), we get
\begin{align*}
\langle V_X S(Y, Z) \rangle + \langle V_Y S(Z, X) \rangle + \langle V_Z S(X, Y) \rangle &= 0.
\end{align*}

Putting $Y = Z = \xi$ in (6.4), we obtain
\begin{align*}
\langle V_X S(\xi, \xi) \rangle + \epsilon B(\xi) &= XS(\xi, \xi) - 2\epsilon S(\xi, \xi).
\end{align*}

which in view of (2.5), (2.11) and (2.13) reduces to (6.1). The equation (6.2) is obvious from (6.1).

A Riemannian manifold is said to admit cyclic Ricci tensor if
\begin{align*}
\langle V_X S(Y, Z) \rangle + \langle V_Y S(Z, X) \rangle + \langle V_Z S(X, Y) \rangle &= 0.
\end{align*}

**Theorem 6.7.** In a $(2n+1)$-dimensional generalized Ricci-recurrent $\epsilon$-trans-Sasakian manifold with cyclic Ricci tensor satisfies
\begin{align*}
\langle V_X S(Y, Y) \rangle + \langle V_Y S(Z, X) \rangle + \langle V_Z S(X, Y) \rangle &= 0.
\end{align*}

**Proof:** From the definition of generalized Ricci-recurrent manifold and (6.6), we get
\begin{align*}
\langle V_X S(Y, Y) \rangle + \langle V_Y S(Z, X) \rangle + \langle V_Z S(X, Y) \rangle &= 0.
\end{align*}

Putting $Y = Z = \xi$ in the above equation we get,
\begin{align*}
\langle V_X S(\xi, \xi) \rangle + \epsilon B(\xi) &= XS(\xi, \xi) - 2\epsilon S(\xi, \xi).
\end{align*}

which in view of (2.11) and (6.2) gives (6.7).

**REFERENCES**