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Bhaskaracharya's Method for Diophantine Equation

Dr. Manisha M.Acharya
M.D.College, Parel, Mumbai.

ABSTRACT: *In the book written by Bhaskaracharya 'Siddanth Shiromani' there are four parts viz, Lilavati, Bijaganita, Ganitadhaya and Goladhya. Of these Lilavati and Bijaganita are two books related to Mathematics and Algebra respectively. In both the books Bhaskaracharya has given examples and methods to solve problems numerically and geometrically. Some problems and the methods to solve are so simple that even a high school student can easily grasp it and understand. But some of them are so complex that even a scholar in the subject finds it difficult to analyse and see the solution for it. In this paper the method is discussed to solve a linear Diophantine equation,that is, an equation of the type* $ax \pm b = cy$ *where x and y are variables while a, b, c are constants. This method is called as Kuttak Method. In his book Lilavati, in shloka numbered 234A Bhaskaracharya has given method to solve particularly the equation* $100x \pm 90 = 63y$*. In this paper the author has discussed the Kuttak Method to solve this particular equation and also application of Kuttak method.*

KEYWORDS: *Bhaskaracharya, Siddanth Shiromani, Lilavati, Bijaganita, Diophantine Equation ,Kuttak Method.Equation* $ax \pm b = cy$.

I. INTRODUCTION

Bhaskaracharya was born in 1114 and he completed his granth siddhantashiromani in 1150, that is, he completed his granth when he was just 36.In ancient days, scientific theories, literature used to be expressed in poetry form that is shlokas. Usually formulae and methods were used and mentioned without proofs.Bhaskaracharya while executing different formulae in mathematics makes us aware of various spiritual, ecological and practical repercussions associated with maths which is really astonishing.Most of the ancient mathematics and astronomy is covered in his script. In this paper, we discuss Bhaskaracharya's method for solving Diophantine Equation. It is customary to apply the term Diophantine Equation to any equation in one or more unknowns that is to be solved in the integers. Bhaskaracharya has called this equation as kuttak equation and the method to solve this equation as kuttak method. He has discussed this equation and its solution method in his book Lilavati which is one of the sections of siddhant –siromani.

Kuttak is an equation of the type $ax \pm b = cy$ where a,b,c are constant and x,y are unknowns. The equation is solved to get integral values of x and y.

As per Bhaskaracharya dividend 'a' is known quantity which is being multiplied by an unknown multiplier(x) is called as भाज्य (Bhajya) and a given augment 'b' which is added to a product or subtracted from it is called as क्षेपक (Shepak) and the divisor 'c' is called as हार (Har). An attempt is made, in this paper to mention those shlokas related to kuttak, the meanings of shloka and various kuttak examples.

II. BHASKARACHARYA'S ALGORITHM TO SOLVE KUTTAK

श्लोक (Shloka) –

भाज्यो हारः क्षेपकश्चापवर्त्यः। केनाप्यादौ संभवे कुट्टकार्थम् ।।
येन च्छिन्नो भाज्यहारौ न तेन । **#esHeM®ewleodogäcegefÎäcesJe**।।235।।
परस्परं भाजितयोर्ययोर्यः । शेषस्तयोः स्यादपवर्तनं सः ।।
तेनापवर्तेन विभाजितौ यौ । तौ भाज्यहारौ दृढसंज्ञकौ स्तः ।।236।।
मिथो भजेत्तौ दृढभाज्यहारौ। यावत् विभाज्ये भवतीह रूपं ।।
फलान्यधोधस्तदधो निवेश्य । क्षेपस्ततः शून्यमुपांतिमेन ।।237।।
स्वोर्ध्वे हतेऽन्त्येन युते तदन्त्यम । त्यजेन्मुहुः स्यादिती राशियुग्मम् ।।
ऊर्ध्वो विभाज्येन दृढेन तष्टः । फलं गुणो स्यादधरो हरेण ।।238।।

एवं तदैवात्र याद समास्ताः । स्युर्लब्धयश्चेद्विषमास्तदानीम् ।।
यदागतौ लब्धिगुणौ विशोध्यौ । स्वतक्षणाच्छेषमितौ तु तौ स्तः ।।239।।

श्लोक 243

क्षेपजे तक्षणाच्छुद्वे गुणाप्ती स्तो वियोगजे ।।243।।

श्लोक 248

क्षेप विशुद्धिं परिकल्प्य रूपं ।
पृथक् पृथग्ये गुणकारलब्धी ।।
अभीप्सित-क्षेपविशुद्धनिघ्ने ।
स्वहारतष्टे भवतस्तयोस्ते ।।248।।

Meaning of Shlokas : Consider Kuttak $ax \pm b = cy$ where a is भाज्य (Bhajya), that is, dividend. 'b' is क्षेपक (Shepak), that is, augment and 'c' is हार (Har), that is divisor. Equation is solvable if and only if GCD of 'a' and 'c' divides 'b'. If a,b and c have any common factor then divide the whole equation by that common factor. Thus, we get an equation in its lowest proportional. Then divide the reduced dividend and divisor reciprocally till the remainder in the dividend is an unit.
Place the quotients under each other in succession, below these an augment and below augment put zero. This arrangement is called as वल्ली (Valli).

Further the method is described in three steps :-

Step I – In *Valli* multiply the last but one figure by the figure immediately above it, add the last figure to the product. Write the value obtained in next column in the front of last quotient.

Step II – Multiply this value by the quotient which is above the last quotient in the *valli*. To this product add the value obtained in the setp I. Write this product in the second column above the value obtained in step I. Continue this process till the first quotient written in *Valli*.

Step III – Divide the upper most quantity obtained in the second column by the reduced dividend and denote the remainder by 'm'. Further, divide the next or the second quantity in the column by the reduced divisor and denote the remainder by 'n'.

If the number of quotients in *Valli* is even and augment is positive or if the number of quotients is odd and augment is negative then 'm' defines the value for y. Which is called as लाब्धि (Labhadi) and 'n' defines the value of x called as गुण (Gun).

But if the number of quotients is even and augment is negative or if the number of quotients is odd and augment is positive, then "the Reduced dividend – m", is *Labhadi*, that is, value of y while "the Reduced Divisor – n" is *Gun*, that is, value of x.

In the last shloka it is said that Kuttak can be modified by replacing *shepak* by ± 1. It means, the given kuttak is solved by replacing *Shepak* by $\pm$ 1and find the values of *Gun* and *Labhadi.* Then multiply these values by the *Shepak* given in the original Kuttak and obtain values of *Gun* and *Labhadi* for original Kuttak.

Examples: We give following examples to explain Bhaskaracharya's Algorithm.

$100x + 90 = 63y$.
All the three values a, b, c, that is, 100,90,63 are in their lowest proportional and *Shepak* is 90. Hence we consider a kuttak replacing 90 by 1, that is, consider
$100x' + 1 = 63y'$.

Here *Bhajya* is 100 and *Har* is 63. Divide 100 and 63 reciprocally till the remainder is one.

$100 = 63 \times 1 + 37$

$63 = 37 \times 1 + 26$
$37 = 26 \times 1 + 11$
$26 = 11 \times 2 + 4$
$11 = 4 \times 2 + 3$
$4 = 3 \times 1 + 1$

वल्ली (Valli) is formed by writing quotients one after the other in succession, below these quotients 1 is written and then lastly O is written.

Quotients	2nd Column
1	27
1	17
1	10
2	7
2	3
1	1
1	
0	

Dividing the upper most quantity 27 by *Bhajya* 100, the remainder is 27 and lower quantity 17 by 63 the remainder is 17. Here *Shepak* 90 is positive and number of quotients in *Valli* is even. Hence y'=27 and x'=17.
Further to obtain values of y and x multiply both 'y' and 'x' by 90 that is, 27 x 90 =2430 and 17 x 90 = 1530. Divide 2430 by 100 and 1530 by 63. Then remainders are 30 and 18 respectively. Hence *Labhadi* that is, value of y is 30 and *Gun*, that is value of x is 18.

Further for t€Z the *Gun* series is 18+63t = 18, 81,144 ……… It gives all values for x and *Labhadi* series is : 30+100t = 30, 130, 230 ………. gives all values for y.

2.3.2] $100x - 90=63y$

Here *Shepak* is negative hence instead of x=18 and y=30, the values are
x= 63-18 = 45 and y= 100-30=70.

Hence *Gun* series is 45 + 63t = 45,108,171 ……….. and *Labhadi* series is 70+100t=70, 170, 270 ………………….

2.3.3] $420x + 21 = 91y$
Here 7 is a common factor. Hence dividing the equation throughout by 7 we write $60x + 3 = 13y$.

Further consider a kuttak replacing 3 by 1; that is, $60x'+1 =13y'$
Since both 60, 13 are in their lowest proportional now. Divide 60 by 13 reciprocally till the remainder is one.

$60 = 13 \times 4 +8$
$13 = 8 \times 1 + 5$
$8 = 5 \times 1 + 3$
$5 = 3 \times 1 + 2$
$3 = 2 \times 1 + 1$

Hence *Valli* is.

Quotients	**2nd** Coloumn
4	23
1	5
1	3
1	2
1	1
1	
0	

Dividing 23 by 60 and 5 by 13. We get remainders 23 and 5 respectively. Further, the *Shepak* is positive and number of quotients in Valli is odd. Hence
y'=60-23=37 and x' = 13-5=8.

Further to get values of x and y.
Multiply both y' and x' by 3. Divide the products by 60 and 13 respectively. Then the remainders are 51 and 11.

Hence y=51 and x=11.
Further *Labhadi* series is y = 51 + 60t = 51, 111 ………………. And *Gun* series is
x = 11+13t = 11, 24 ………………….

III. APPLICATIONS

In this section we discuss few applications of Bhaskaracharya's algorithm to solve kuttak equations.

If 'd' is the GCD of two integers a & b then there exist integers 'm' and 'n' such that d=ma+nb.

To find integers m and n means to solve kuttak ma-d=nb. We consider particular example : Express GCD 5 of 1705 and 625 in the form 1705m+625n.

Hence we solve Kuttak 1705m-5=625n. Since 5 is a common factor, dividing the equation throughout by 5 we get.

341m-1=125n

Divide 341 by 125 reciprocally till the remainder is one.

$341=125\times2+91$
$125= 91\times1+34$
$91 = 34\times2+23$
$34 = 23\times1+1$
$23= 11\times2+1$
Hence *Valli* is

Quotients	2nd Column
2	30
1	11
2	8
1	3
2	2
1	
0	

Dividing 30 by 341 and 11 by 125 remainders are 30 and 11 respectively.

Here *Shepak* is negative and number of quotients in *Valli* is odd. Hence m=11 and n=30.
Hence $1705\times11-5=625\times30$.
That is $1705\times11+625\times(-30)=5$
Further for t€z, m=11+125t and n=30+341t

A person went to a bank. He withdrew some cash which he did not count. After that he purchased toys worth Rs. 19. But in the bank the amount in paise was replaced by rupees and that of rupees by paise by the cashier.

Hence even after spending Rs. 19, he had doubled amount of the cheque remaining with him. So for what amount cheque was withdrawn?

This types of problem can also be solved using Kuttak method.

Here let the cheque be of amount Rs. X and Y Paise.
Hence 100y + x – 1900 = 2(100x+y)
100y + x -1900 = 200x +2y
98y = 199x + 1900
Hence we get a Kuttak
199x +1900 = 98y.
It can be solved by Bhaskaracharya's Algorithm and obtain values of x and y.

A band of 25 pirates stole a sack of gold coins. When they tried to divide equally among them 18 coins remain. In the ensuing fight 16 of the pirates died. Again it was divided equally to find 8 coins were left. Again a fight resulted in killing 5 more pirate. This time they decided not to fight but to donate the remaining coins if left any after dividing equally among left out 4 pirates. Hence they had to donate 3. What was the least number of gold coins they could have stolen?

Let x denote the number of coins stolen.

Then we get congruences
$x \equiv 18 \pmod{25}$
$x \equiv 8 \pmod{9}$
$x \equiv 3 \pmod{4}$

Nowadays these linear congruences are solved using Chinese Remainder Theorem. But we can also get solution by Kuttak method. We get following Kuttak equations.

$x = 25k + 18$
$x = 9z + 8$
$x = 4y + 3$

$\therefore$ $4y+3=9z+8$ and $4y + 3 = 25k + 18$
$\therefore$ $4y = 9z + 5 = 25k + 15$
Hence $9z = 25k + 10$.

We solve the Kuttak $25k + 10 = 9z$,

Firstly consider a Kuttak replacing 10 by 1,

That is $25k' + 1 = 9z'$

Divide 25 by 9 reciprocally till remainder is one.

$25 = 9 \times 2 + 7$
$9 = 7 \times 1 + 2$
$7 = 2 \times 3 + 1$.
Hence *Valli* is

Quotients	2nd Column
2	11
1	4
3	3
1	
0	

Dividing 11 by 25 and 4 by 9 we get remainders 11 and 4 respectively.

Further *Shepak* is positive and number of quotients in *Valli* is odd. Hence $z' = 25-11 = 14$ and $k' = 9-4=5$. To get values of z and k multiply both z' and k' by l0. Divide the product $11 \times 10 = 110$ and $5 \times 10 = 50$ respectively by 25 and 9. Then remainders are 15 and 5. Hence $z = 15$ and $k = 5$.

Hence putting value z in $x = 9z + 8$ or value of k in $x = 25k + 18$ we get answer as $x = 143$.

There are many other many other applications of kuttak such as for calculating *Ahagran and Grahabhraman* etc. which has been given in his *Bijaganita and Ganitadhyaya*

IV. CONCLUSION

Nowadays Euclidean Algorithm is the most commonly used method for solving kuttak. This method is found in Euclid's Elements . Bhaskaracharya deduced the method which complemented the Euclidean method using his Intelligence. Before winding up we would like to end up with an anecdote. In an interview an IIM student was asked whether he would like to ask ten easy questions or one difficult question . Being a very creative and intelligent ,the student chose a second option of difficult question. Interviewer asked " Tell me day first or night ?" Without waiting for a second the student replied it is the day first.The interviewer frowned and asked why not night? Immediately student replied you were to ask me one difficult question.Thus the great Bhaskaracharya was very intelligent and creative mathematician who always thought out of the existing frame work.

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