

## The Self Purification Model for Water Pollution

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**ABSTRACT :** Mathematical model have been developed for self purification of river water. Since the ecology of the river depends largely on the quantity of dissolved oxygen in its water, this dissolved oxygen (DO) seems to be a convenient criterion for measuring the degree of pollution of a river as far as organic pollution is concerned. However, even the term organic pollution embodies a great number of different materials and the question assessing this load of pollutants is raised. Considering that the effect of all kinds of organic matter will be consumption of dissolved oxygen, it is usual to measure the load of organic pollution by the quantity of oxygen necessary to completely oxidize this load by bacteriological breakdown, i.e. by its biological oxygen demand (BOD). The present analysis deals with the polluted water in a river. Mathematical models is formulated which simulates the variations of the parameters  $D = DO$  (Dissolved oxygen and  $B = BOD$  (Biological oxygen demand) over time at each point of a river (or reach of a river). The validity of the prospect use of the model depends heavily on the validity of the equations which have been used and this depends on knowledge of accurate hydrolic parameters advection, diffusion and reaeration. These parameters are fairly well known by a theoretical approach when compared to biodegradation and other phenomena. A details comprehensive field measurement survey is necessary to determine empirically the bulk biodegradation coefficients to be introduce into the model. Under these conditions a complete understanding of the mechanism of self purification can be obtained. This mathematical models is very helpful for the study of oxygen in rivers.

**KEYWORDS:** Dissolved Oxygen, Molecular Diffusion, Surface Reaeration, Water Quality Modeling

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### I. INTRODUCTION

Choosing a criterion to characterize the degree of pollution of river is not easy because several quite different kinds of pollution (organic matter, radioactivity, heat, poisonous chemicals, pathogenic germs) are found in a river. The effect also are quit different, for example turbidity, death of the fauna, or eutrophication. In fact, pollution of rivers by organic matter (most municipal and industrial effluents) is certainly the most significant in terms of quantity and by effect on the river. In river water organic matter is naturally eliminated by a biochemical degradation performed by bacteria. As long as dissolved oxygen is available, the biochemical breakdown can be considered equivalent to an oxidization reaction which will lower the level of dissolved oxygen and therefore deteriorate the ecological balance in the river. When there is no oxygen left in the water, the breakdown of organic matter becomes anaerobic. Fortunately, atmospheric oxygen enters the water through the surface, and thus a river is capable of eliminating a definite amount of organic matter by itself, this ability of the river to regenerate is called 'Self purification'.

Since the ecology of the river depends largely on the quantity of dissolved oxygen in its water, this dissolved oxygen (DO) seems to be a convenient criterion for measuring the degree of pollution of a river as far as organic pollution is concerned. However, even the term organic pollution embodies a great number of different materials and the question assessing this load of pollutants is raised. Considering that the effect of all kinds of organic matter will be consumption of dissolved oxygen, it is usual to measure the load of organic pollution by the quantity of oxygen necessary to completely oxidize this load by bacteriological breakdown, i.e. by its biological oxygen demand (BOD) Chevereau formulated Mathematical Models for oxygen balance in rivers. Streeter and Phelps obtained pollution equation by making important assumptions for the investigation of biochemical breakdown camp. Elder has proposed value of the semiempirical coefficients. A for a very wide open channel. He has taken  $A = 0.067, 0.23$  and  $10$  for the mean vertical diffusion coefficient over depth, mean transverse diffusion coefficient and for the mean longitudinal diffusion coefficient respectively. O Conner and Dobbins developed a formula for surface reaeration coefficient based on a theoretical analysis which yields good result. Gannon founds  $k_1$  coefficients ranging from  $0.01$  to  $1.0 \text{ day}^{-1}$ . He also advocated two step explicit method.

In natural stream, molecular diffusion is very weak compared with turbulent diffusion and therefore only turbulent diffusion is considered. The study of diffusion goes back to Prandtl's first experiment turbulence studies. The basis of prandtl's turbulence theory are clearly shown in Bakhmelff's book. The turbulent diffusion in a very wide rectangle channel assuming two dimensional flow was investigated by Elder. The present analysis deals with the polluted water in a river. Mathematical models is formulated which simulates the variations of the parameters  $D = DO$  (Dissolved oxygen and  $B = BOD$  (Biological oxygen demand) over time at each point of a river (or reach of a river).

#### Symbols

A	=	An empirical coefficient
B	=	(BOD) Biological oxygen demand.
COD	=	Chemical oxygen demand.
C	=	Concentration of dissolved oxygen of the particle.
CA	=	The rate of input/output of DO
CS	=	The Saturation Concentration of oxygen
D	=	Oxygen deficit
DO	=	The value of the D at time = o
DC	=	Critical value of D
D1	=	The longitudinal diffusion coefficient
DO	=	Dissolved oxygen
g	=	The weight acceleration
h	=	The Height of the fall = the depth
k	=	A coefficient depending on the shape of weir and on quality of the water.
$K_1, K_2$	=	Biological coefficients
L	=	Polluted water in a river with a concentration of organic pollution
LA	=	The rate of input / output of BOD
LO	=	Value of time $t = O$
S	=	Slope of the energy line
t	=	Time
T	=	The water temperature
U	=	The mean velocity
$u^*$	=	the wall velocity
UDO	=	Ultimate oxygen demand
X	=	The abscissa along the river.

## II. EQUATIONS AND MODELING

**Pollution Equations:** Consider a particle of polluted water in a river with a concentration of organic pollution  $B(=BOD)$  denoted by  $L$ . This particle undergoes several processes which will change its concentration  $L$ . Among these, the main ones are advection, diffusion, biodegradation of organic matter, and other inputs of organic matter. Let  $C$  be the concentration of dissolved oxygen of particle. The processes which will change  $C$  are mainly advection, diffusion oxidization of organic matter, reaeration through the surface, and other inputs or outputs of organic matter.

A mathematical model for organic pollution concentration  $L$  describing the variations over time is

$$\frac{\partial L}{\partial t} = D_L \frac{\partial^2 L}{\partial X^2} - U \frac{\partial L}{\partial X} - K_1 L + L_A \quad (1)$$

And the equation describing the oxygen concentration  $C$  is

$$\frac{\partial C}{\partial t} = D_L \frac{\partial^2 C}{\partial X^2} - U \frac{\partial C}{\partial X} - K_1 L + K_2 (C_s - C) - C_A \quad (2)$$

Equation (1) describes organic pollution where (i)  $C$  and  $L$  are assumed to be uniformed in the cross section and (ii) is constant over time and the flow assumed steady. In the Streeter and Phelps equation the biochemical breakdown term is written

This shows that the pollution load measured by its total BOD (BOD) is degraded according to an exponential law (Fig-1) at a constant rate,  $K_1$  called biodegraded coefficient

Replacing e by 10,

The equation (5) becomes

$$L = LO 10^{-K_1 t} \quad (6)$$

provided that  $K_1 = 2.3 KO$ , Streeter and Phelps estimated, based on laboratory experiments that KO was approximately equal to 0.1 day<sup>-1</sup>. Under these conditions, there is a relation between the initial pollution load LO measured by  $B = [BOD]$  and the load Lt after a time t of incubation; for instance, if t = 5 days,

Then

$$BOD = 1.45 BOD_5$$

Therefore, the determination of  $B = BOD$  which theoretically needs an infinite time (which is impossible), can be obtained by determining the BOD ( $B_5$ ) by way of laboratory BOD ( $=B_5$ ) tested and applying equation (7)

**Surface Reaeration:** In the Streeter Phelps equation, this term is written as

$$K_2 (C_s - C) \quad (8)$$

Where  $K_2$  is a coefficient which depends on local hydraulic conditions.

The molecular diffusion of dissolved oxygen in water is very small (2.10-5 m<sup>2</sup>/s). But it is generally acknowledged that except close to the surface and to the river bed, turbulent diffusion is sufficient to mix the dissolved oxygen in uniform manner over a vertical section. Field measurement support this statement, so

$$K_2 = \frac{u^m}{H^n} \quad (9)$$

In which m would be approximately 1 and n approximately is 1.5. Most author consider C as a Constant coefficient, a coefficient depending on the diffusion coefficient, or on Froude's number. With m = 1, n = 1.5 and C being a constant coefficient a "mean formula" is obtained which is satisfactory in practical cases. It is generally admitted that the coefficient K<sub>2</sub> is dependent on water temperature according to the following equation.

$$K_2(T) = K_2(20^\circ C) \theta^{(T-20)}$$

In which  $\theta$  is approximately 1.025. But this value of  $\theta$  is not the most accurate value; but pollution models do not need such accuracy.

For unsteady flow, equation (1) can be written as

$$\frac{\partial(SL)}{\partial t} = \frac{\partial(SD_L \frac{\partial L}{\partial X})}{\partial X} - \frac{\partial(USL)}{\partial X} - SK_1 L + SL_A \quad (10)$$

Where S is slope of energy line.

$$\text{Or } \frac{\partial S}{\partial t} + S \frac{\partial L}{\partial t} = \frac{\partial(SD_L \frac{\partial L}{\partial X})}{\partial X} - \frac{\partial(USL)}{\partial X} - \frac{US \partial L}{\partial X} - L \frac{\partial(US)}{\partial X} - SK_1 L + S L_A \quad (11)$$

If U and S satisfy the mass equation then

$$\frac{\partial S}{\partial t} = - \frac{\partial(US)}{\partial X}$$

Using (12) in (11), we get

$$S \frac{\partial L}{\partial t} = \frac{\partial (SD_L \frac{\partial L}{\partial X})}{\partial X} - \frac{\partial (U A \partial L)}{\partial X} - S K_1 L + S L_A \quad (13)$$

**Analytical Solution:** Analytical solution can help in understanding the mechanism of self purification in river and the influence of the main parameters. Consider the case in which the variations concentration in BOD (=B) and oxygen are due only to biodegradation and reaeration through the surface. Assume that the flow is steady and that diffusion is ignored. If  $K_1$  and  $K_2$  are constants, the concentration in oxygen C of a Particle moving at velocity U is given by

$$\frac{dC}{dt} = -K_1 L + K_2 (C_s - C) \quad (14)$$

Assume,  $D = C_s - C$ ,  
Where D is the oxygen deficit  
So (14) becomes

$$\frac{dD}{dt} = -\frac{dC}{dt} = -[K_1 L + K_2 D]$$

$$\text{Or } \frac{dD}{dt} = K_1 L - K_2 D$$

$$\text{Or } \frac{dD}{dt} + K_2 D = K_1 L \quad (15)$$

$$\text{I.F.} = e^{\int K_2 dt} = e^{K_2 t}$$

The solution of (15) is given by

$$\begin{aligned} D e^{K_2 t} &= \int K_1 L e^{K_2 t} dt \\ &= \int K_1 (L_0 e^{-K_1 t}) e^{K_2 t} dt \end{aligned} \quad (16)$$

inserting L from (5)

$$\begin{aligned} D e^{K_2 t} &= K_1 L_0 \int e^{(K_2 - K_1)t} dt \\ \text{or } D e^{K_2 t} &= \frac{K_1 L_0 e^{(K_2 - K_1)t}}{(K_2 - K_1)} + E \end{aligned} \quad (17)$$

Where E is constant of integration initially,  $t = 0$ ,  $D = D_0$ . Therefore,

from (17), we have

$$E = D_0 - \frac{K_1 L_0}{K_2 - K_1}$$

Putting E in (17), we get

$$\begin{aligned}
 D e^{K_2 t} &= \frac{K_1 L_0}{(K_2 - K_1)} e^{K_2 t} e^{-K_1 t} + D_0 - \frac{K_1 L_0}{(K_2 - K_1)} \\
 \text{or } D e^{K_2 t} &= \frac{K_1 L_0}{(K_2 - K_1)} e^{K_2 t} \cdot e^{-K_1 t} - \frac{K_1 L_0}{(K_2 - K_1)} + D_0 \\
 \text{or } D &= \frac{K_1 L_0}{(K_2 - K_1)} e^{-K_1 t} - \frac{K_1 L_0}{(K_2 - K_1) e^{K_2 t}} + \frac{D_0}{e^{-K_2 t}} \\
 \text{or } D &= \frac{K_1 L_0}{(K_2 - K_1)} (e^{-K_1 t} - e^{-K_2 t}) + D_0 e^{-K_2 t} \quad (18)
 \end{aligned}$$

Here  $L_0$  ;  $D_0$  ; denote the value of  $L$  and  $D$  at time  $t = 0$

Using  $k_1$  and  $k_2$  in place of  $K_1$ ,  $K_2$  and replacing  $e$  by  $10$ , we get

$$D = \frac{k_1 L_0}{(k_2 - k_1)} (10^{-k_1 t} - 10^{-k_2 t}) + D_0 10^{-k_2 t} \quad (19)$$

The graphical representation of this equation is shown as a sag curve in Fig 2. Worst condition in a river are observed at a point where the dissolved oxygen concentration is a minimum, therefore the point of the centre when  $D$  is equal to a maximum  $DC$  and called the critical deficit, appears at the critical time  $t_c$ , so for critical deficit  $DC$  and critical time  $t_c$ .

$$\frac{dD}{dt} = 0$$

Now, differentiation (19) w.r. to  $t$ , we get

$$\begin{aligned}
 \frac{dD}{dt} &= \frac{k_1 L_0}{(k_2 - k_1)} [10^{-k_1 t} (-k_1 \log_e 10) - 10^{-k_2 t} (-k_2 \log_e 10)] + D_0 10^{-k_2 t} (-k_2 \log_e 10) \\
 \text{or } \frac{dD}{dt} &= \frac{k_1 L_0}{(k_2 - k_1)} [-k_1 10^{-k_1 t} \log_e 10 + k_2 10^{-k_2 t} \log_e 10] - D_0 k_2 10^{-k_2 t} \log_e 10 \quad (20)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dD}{dt} &= \left[ \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ -k_1 10^{-k_1 t} + k_2 10^{-k_2 t} \right\} - D_0 k_2 10^{-k_2 t} \right] \log_e 10 \right] \\
 \frac{dD}{dt} &= \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ -k_1 10^{-k_1 t} + k_2 10^{-k_2 t} \right\} - D_0 k_2 10^{-k_2 t} \right] \times \log_e 10 \quad (21)
 \end{aligned}$$

Again differentiating, we have

$$\frac{d^2 D}{dt^2} = \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ -k_1 10^{-k_1 t} (-\log_e 10) + k_2 10^{-k_2 t} (-k_2 \log_e 10) \right\} - D_0 k_2 10^{-k_2 t} (-k_2 \log_e 10) \right] \times \log_e 10$$

Or

$$\frac{d^2D}{dt^2} = \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ k_1^2 10^{-k_1 t} - k_2^2 10^{-k_2 t} \right\} + D_0 k_2^2 10^{-k_2 t} \right] (\log_e 10)^2 \quad (22)$$

When  $\frac{dD}{dt} = 0$ , from equation (21), we have

$$\text{Or } 0 = \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ -k_1 10^{-k_1 t} + k_2 10^{-k_2 t} \right\} - D_0 k_2 10^{-k_2 t} \right] \times \log_e 10$$

$$\text{Or } 0 = \frac{k_1 L_0}{(k_2 - k_1)} \left\{ -k_1 10^{-k_1 t} + k_2 10^{-k_2 t} \right\} - D_0 k_2 10^{-k_2 t}$$

$$D_0 k_2 10^{-k_2 t} = \frac{k_1 L_0}{k_2 - k_1} \left\{ -k_1 10^{-k_1 t} + k_2 10^{-k_2 t} \right\}$$

$$\text{Or } D_0 = \frac{k_1 L_0}{(k_2 - k_1)} \left[ \frac{-k_1 10^{-k_1 t} + k_2 10^{-k_2 t}}{k_2 10^{-k_2 t}} \right] \quad (23)$$

$$\text{Or } D_0 = \frac{k_1 L_0}{k_2 - k_1} \left[ -\frac{k_1}{k_2} 10^{-k_1 t + k_2 t} + 1 \right]$$

$$\text{Or } D_0 = \frac{k_1 L_0}{k_2 - k_1} \left[ -\frac{k_1}{k_2} 10^{(k_2 - k_1)t} + 1 \right]$$

$$\text{Or } D_0 = \frac{k_1 L_0}{k_2 - k_1} \left[ 1 - \frac{k_1}{k_2} 10^{(k_2 - k_1)t} \right] \quad (24)$$

$$\text{Or } \frac{(k_2 - k_1) D_0}{k_1 L_0} = 1 - \frac{k_1}{k_2} 10^{(k_2 - k_1)t}$$

$$\text{Or } \frac{k_2}{k_1} \left[ 1 - \frac{(k_2 - k_1) D_0}{L_0 k_1} \right] = 10^{(k_2 - k_1)t} \quad (25)$$

Taking log both sides,

$$\log_{10} \left[ \frac{k_2}{k_1} \left\{ 1 - \frac{(k_2 - k_1) D_0}{L_0 k_1} \right\} \right] = \log_{10} 10^{(k_2 - k_1)t}$$

$$= (k_2 - k_1) t \cdot \log_{10} 10$$

$$\text{Or } \log_{10} \left[ \frac{k_2}{k_1} \left\{ 1 - \frac{(k_2 - k_1) D_0}{L_0 k_1} \right\} \right] = (k_2 - k_1) t$$

$$t = \frac{1}{(k_2 - k_1)} \log_{10} \frac{k_2}{k_1} \left\{ 1 - \frac{(k_2 - k_1) D_0}{L_0 k_1} \right\} = t_c (\text{say}) \quad (26)$$

Therefore Taking  $k_1 = 0.1$  in (26)

$$t_c = \frac{1}{(k_2 - 0.1)} \log_{10} \frac{k_2}{0.1} \left\{ 1 - \frac{(k_2 - 0.1)}{0.1} \times \frac{D_0}{L_0} \right\}, \text{ as } D_0 = D_c \quad (27)$$

Special Cases : For different values of  $\frac{D_0}{L_0}$ , the values of  $t_c$  from (27) are as follows:

$$t_c = \frac{1}{k_2 - 0.1} \log_{10} \left[ \frac{k_2}{0.1} \right], \text{ for } \frac{D_0}{L_0} = 0 \quad (28)$$

$$t_c = \frac{1}{k_2 - 0.1} \log_{10} \left[ \frac{k_2}{0.1} \left\{ 1 - (k_2 - 0.1) \right\} \right], \text{ for } \frac{D_0}{L_0} = 0.1 \quad (29)$$

$$t_c = \frac{1}{k_2 - 0.1} \log_{10} \left[ \frac{k_2}{0.1} \left\{ 1 - (k_2 - 0.1) 5 \right\} \right], \text{ for } \frac{D_0}{L_0} = 0.5 \quad (30)$$

$$t_c = \frac{1}{k_2 - 0.1} \log_{10} \left[ \frac{k_2}{0.1} \left\{ 1 - (k_2 - 0.1) \times 10 \right\} \right], \text{ for } \frac{D_0}{L_0} = 1.0 \quad (31)$$

$$(32) \quad t_c = \frac{1}{k_2 - 0.1} \log_{10} \left[ \frac{k_2}{0.1} \left\{ 1 - (k_2 - 0.1) \times 10 \right\} \right], \text{ for } \frac{D_0}{L_0} = 1.0$$

$$t_c = \frac{1}{k_2 - 0.1} \log_{10} \left[ \frac{k_2}{0.1} \left\{ 1 - (k_2 - 0.1) 50 \right\} \right], \text{ for } \frac{D_0}{L_0} = 5.0 \quad (33)$$

Now, from the equation (19)

$$\begin{aligned} D &= \frac{k_1 L_0}{(k_2 - k_1)} (10^{-k_1 t} - 10^{-k_2 t}) + D_0 10^{-k_2 t} \\ &= \frac{k_1 L_0}{k_2 - k_1} \left[ \frac{1}{10^{k_1 t}} - 10^{-k_2 t} \right] + D_0 10^{-k_2 t} \\ &= \frac{k_1 L_0}{k_2 - k_1} \left[ \frac{1 - 10^{-k_2 t} \times 10^{k_1 t}}{10^{k_1 t}} \right] + D_0 10^{-k_2 t} \\ &= \frac{k_1 L_0}{k_2 - k_1} \left\{ \frac{1 - 10^{-(k_2 - k_1)t}}{10^{k_1 t}} \right\} + D_0 10^{-k_2 t} \\ &= \frac{k_1 L_0}{k_2 - k_1} \left\{ 1 - \frac{1}{10^{(k_2 - k_1)t}} \right\} + \frac{D_0}{10^{(k_2 - k_1)t}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\frac{k_1 L_0}{k_2 - k_1} \left[ 1 - \frac{1}{\frac{k_2}{k_1} \left\{ 1 - \frac{D_0(k_2 - k_1)}{k_1 L_0} \right\}} \right] + \frac{D_0}{\frac{k_2}{k_1} \left\{ 1 - \frac{D_0(k_2 - k_1)}{k_1 L_0} \right\}}}{10^{k_1 t}} \\
 D &= \frac{\frac{k_1 L_0}{k_2 - k_1} \left[ \frac{k_2}{k_1} \frac{\{k_1 L_0 - D_0(k_2 - k_1)\} - k_1 L_0}{k_1 L_0} \right] + D_0}{\frac{k_2}{k_1} \left\{ 1 - \frac{D_0(k_2 - k_1)}{k_1 L_0} \right\} 10^{k_1 t}} \\
 &= \frac{\frac{k_1 L_0}{k_2 - k_1} \left[ \frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} - k_1 L_0 \right] + D_0 k_1 L_0}{k_1 L_0} \\
 &= \frac{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} 10^{k_1 t}}{k_1 L_0} \\
 &= \frac{\frac{k_1 L_0}{k_2 - k_1} \left[ \frac{k_2}{k_1} k_1 L_0 - \frac{D_0 k_2 (k_2 - k_1)}{k_1} - k_1 L_0 \right] + D_0 k_1 L_0}{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} 10^{k_1 t}} \\
 &= \frac{\frac{k_1 L_0}{k_2 - k_1} \left[ L_0 (k_2 - k_1) - \frac{D_0 k_2 (k_2 - k_1)}{k_1} \right] + D_0 k_1 L_0}{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} 10^{k_1 t}} \\
 &= \frac{L_0 \left\{ \frac{k_1}{k_2 - k_1} \right\} \left[ L_0 (k_2 - k_1) - D_0 \frac{k_2}{k_1} (k_2 - k_1) \right] + D_0 k_1}{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} 10^{k_1 t}} \\
 &= \frac{L_0 \left\{ \frac{k_1}{k_2 - k_1} X^{L_0(k_2 - k_1)} - \frac{D_0 k_2 (k_2 - k_1)}{k_1} \right\} \times \frac{k_1}{k_2 - k_1} + D_0 k_1}{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} 10^{k_1 t}} \\
 &= \frac{L_0 \left\{ k_1 L_0 - D_0 k_2 + D_0 k_1 \right\}}{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0(k_2 - k_1) \right\} 10^{k_1 t}}
 \end{aligned}$$



$$D = \frac{L_0 \left\{ k_1 L_0 - D_0 (k_2 - k_1) \right\}}{\frac{k_2}{k_1} \left\{ k_1 L_0 - D_0 (k_2 - k_1) \right\} 10^{k_1 t}} = \frac{L_0}{\frac{k_2}{k_1} 10^{k_1 t}}$$

$$D = \frac{k_1}{k_2} L_0 10^{-k_1 t} \quad (34)$$

Writing  $D_c$  for  $D$  and  $t_c$  for  $t$ , we have

$$D_c = \frac{k_1}{k_2} L_0 10^{-k_1 t_c} \quad (35)$$

$$\text{Or } \frac{D_c}{L_0} = \frac{k_1}{k_2} 10^{-k_1 t_c} \quad (36)$$

Taking  $k_1 = 0.1$

$$\frac{D_c}{L_0} = \frac{0.1}{k_2} 10^{-0.1 \times t_c} \quad (37)$$

Now, equation (22) is

$$\begin{aligned} \frac{d^2 D}{dt^2} &= \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ k_1^2 10^{-k_1 t} - k_2^2 10^{-k_2 t} \right\} + D_0 k_2^2 10^{-k_2 t} \right] (\log_e 10)^2 \\ \frac{d^2 D}{dt^2} &= \left[ \frac{k_1 L_0}{(k_2 - k_1)} \left\{ \frac{k_1^2}{10^{k_1 t}} k_2^2 10^{-k_2 t} \right\} + D_0 k_2^2 10^{-k_2 t} \right] (\log_e 10)^2 \\ &= \frac{\left[ \frac{k_1 L_0}{k_2 - k_1} \left\{ k_1^2 - k_2^2 10^{-(k_2 - k_1)t} \right\} + D_0 k_2^2 10^{-(k_2 - k_1)t} \right] \times (\log_e 10)^2}{10^{k_1 t}} \\ &= \frac{\frac{k_1 L_0}{k_2 - k_1} \left[ k_1^2 - \frac{k_2^2}{\frac{k_2}{k_1} \left\{ 1 - \frac{D_0 (k_2 - k_1)}{L_0 k_1} \right\}} \right] + \frac{D_0 k_2^2}{\frac{k_2}{k_1} \left\{ \frac{(k_2 - k_1) D_0}{L_0 k_1} \right\}}}{10^{k_1 t}} \times (\log_e 10)^2 \\ &\quad \text{(use of 25)} \\ &= \frac{\left[ \frac{k_1 L_0}{k_2 - k_1} \left\{ k_1^2 - \frac{k_1 k_2}{1 - \frac{(k_2 - k_1) D_0}{L_0 k_1}} \right\} + \frac{D_0 k_1 k_2}{1 - \frac{(k_2 - k_1) D_0}{L_0 k_1}} \right]}{10^{k_1 t}} \times (\log_e 10)^2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\left[ \frac{k_1 L_0}{k_2 - k_1} \left\{ k_1 \left( 1 - \frac{(k_2 - k_1) D_0}{L_0 k_1} \right) - k_2 \right\} + D_0 k_1 k_2 \right]}{1 - \frac{(k_2 - k_1) D_0}{L_0 k_1} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &= \frac{\frac{k_1^2 L_0}{k_2 - k_1} [k_1 \{L_0 k_1 - D_0 (k_2 - k_1)\} - L_0 k_1 k_2] + D_0 k_1^2 k_2 L_0}{\frac{L_0 k_1}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}}} \times (\log_e 10)^2 \\
 &= \frac{k_1^2 \left\{ \frac{L_0}{k_2 - k_1} [L_0 k_1^2 - D_0 k_1 (k_2 - k_1) - L_0 k_1 k_2] + D_0 L_0 k_2 \right\}}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &= \frac{k_1^2 \left\{ \frac{L_0}{k_2 - k_1} [L_0 k_1 (k_1 - k_2) - D_0 k_1 (k_2 - k_1)] + D_0 L_0 k_2 \right\}}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &= \frac{k_1^2 \left\{ \frac{L_0}{(k_2 - k_1)} x^{-L_0 k_1 (k_2 - k_1) - D_0 k_1 (k_2 - k_1) x \frac{L_0}{k_2 - k_1} + D_0 L_0 k_2} \right\}}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &= \frac{k_1^2 \{-L_0^2 k_1 - D_0 k_1 L_0 + D_0 L_0 k_2\}}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &= \frac{d^2 D}{dt^2} = \frac{-k_1^2 L_0 \{L_0 k_1 + D_0 k_1 - D_0 k_2\}}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &= \frac{-k_1^2 L_0 \{L_0 k_1 - D_0 (k_2 - k_1)\}}{\{L_0 k_1 - D_0 (k_2 - k_1)\} 10^{k_1 t}} \times (\log_e 10)^2 \\
 &\frac{d^2 D}{dt^2} = \frac{-k_1^2 L_0}{10^{k_1 t}} (\log_e 10)^2
 \end{aligned}$$

Or  $\frac{d^2 D}{dt^2} = -k_1^2 L_0 10^{-k_1 t} \cdot L_0 (\log_e 10)^2$  (38)

$$\frac{d^2 D}{dt^2} = -L_0 k_1^2 10^{-k_1 t} \left\{ \frac{1}{k_2 - k_1} \log \left[ \frac{k_2}{k_1} \left\{ 1 - \frac{D_0 (k_2 - k_1)}{L_0 k_1} \right\} \right] \right\} \times (\log_e 10)^2$$

(Use of 26)

$$\begin{aligned}
&= -k_1^2 L_0 10^{\frac{-k_1}{k_2-k_1} \log \left[ \frac{k_2}{k_1} \left\{ 1 - \frac{D_0(k_2-k_1)}{L_0 k_1} \right\} \right]} x (\log_e 10)^2 \\
&= -k_1^2 L_0 10^{\log_{10} \left[ \frac{k_2}{k_1} \left\{ 1 - \frac{D_0(k_2-k_1)}{L_0 k_1} \right\} \right]} x (\log_e 10)^2 \\
\frac{d^2 D}{dt^2} &= -k_1^2 L_0 \left[ \frac{k_2}{k_1} \left\{ 1 - \frac{D_0(k_2-k_1)}{L_0 k_1} \right\} \right]^{\frac{-k_1}{k_2-k_1}} x (\log_e 10)^2 < 0 \\
\frac{d^2 D}{dt^2} &< 0
\end{aligned}$$

Hence the value of D is maximum for the value of t given in (26). Thus D<sub>c</sub> is obtained with the aid of t.

### III. RESULTS AND CONCLUSION

Expression (19) for oxygen deficit D has been obtained in terms of biological coefficients  $k_1$  and  $k_2$ . The expressions for critical D<sub>c</sub> (35) and critical time t<sub>c</sub> (27) are also derived.

Streeter and Phelps estimated that  $k_1 = 0.1$  day<sup>-1</sup>. On this basis calculations for L/L<sub>0</sub> and D<sub>c</sub> / L<sub>0</sub> have been made. Result of calculation are entered in tables (1) and (2) and graphically shown by curve in Fig 3 and (4). Also  $k_2 t_c$  has been numerically computed for different values when  $k_1 = 0.1$ . Results are entered in tables (3) and illustrated graphically in figure (5)

From analysis of the numerical computations we observe that

- D<sub>c</sub> / L<sub>0</sub> increases as D<sub>c</sub> / L<sub>0</sub> increases (see Fig 4)
- As D<sub>c</sub> / L<sub>0</sub> increases  $k_2 t_c$  decreases. (see Fig 5)

The validity of the prospection use of the model depends heavily on the validity of the equations which have been used and this depends on knowledge of accurate hydrolic parameters advection, diffusion and reaeration. These parameters are fairly well known by a theoretical approach when compared to biodegradation and other phenomena. A details comprehensive field measurement survey is necessary to determine empirically the bulk biodegradation coefficients to be introduce into the model. Under these conditions a complete understanding of the mechanism of self purification can be obtained. This mathematical models is very helpful for the study of oxygen in rivers.

Table-1

Mathematical Model for Self Purification in River  
Variance of L/L<sub>0</sub> against t.

t	$= 10^{-0.23t}$
1	0.5588
2	0.3467
3	0.2041
4	0.1202
5	0.0707
6	0.04168
7	0.0245

Table-2  
Mathematical Model for Self Purification in River  
Variation of  $D_c / L_0$  against  $k_1/k_2$  for different value of  $D_c / L_0$ , using equation (27) & (36)

	$D_c/L_0$				
$D_c/L_0$ $k_1/k_2$	0.0	0.5	1.0	2.0	5.0
0.01	0.0098				
0.04	0.0377				
0.08	0.0727				
0.1	0.0894				
0.2	0.1679	0.149			
0.4	0.3067	0.3491			
0.6	0.4301	0.5602	0.061		
0.8	0.5429	0.6846	0.894		
1.0	1	1	1	1	
2.0	1.0953	1.3296	1.5577	3.2515	
4.0	1.7923	2.1553	2.4783	3.6211	
6.0	2.358	2.8304	3.2357	5.5551	
8.0	2.85	3.4125	3.9326	6.567	
10.0	3.2919	3.9384	4.487	7.494	

Table – 3  
Mathematical model for self purification in river  
Value of  $k_2 t_c$  for different value of  $D_0 / L_0$ , using equation (27)

$D_0/L_0$ $k_1/k_2$	0.0	0.5	1.0	2.0	5.0
0.01	2.0202				
0.02	1.7336				
0.04	1.4561				
0.08	1.1922				
0.1	1.1111				
0.2	0.8737	1.4701			
0.4	0.6632	0.3401			
0.6	0.5546	0.1845			
0.8	0.4845	0.1143	0.1401	-1.0205	
2.0	0.301	0.2041	0.1249	0	-0.243
4.0	0.2006	0.1545		0.068	0.0248
6.0	0.1561	0.1252	0.1029	0.0704	0.013
8.0	0.129	0.1064	0.0887	0.0662	0.0246
10.0	0.1111	0.0931	0.0801	0.0614	0.0288

Mathematical model for self purification in river

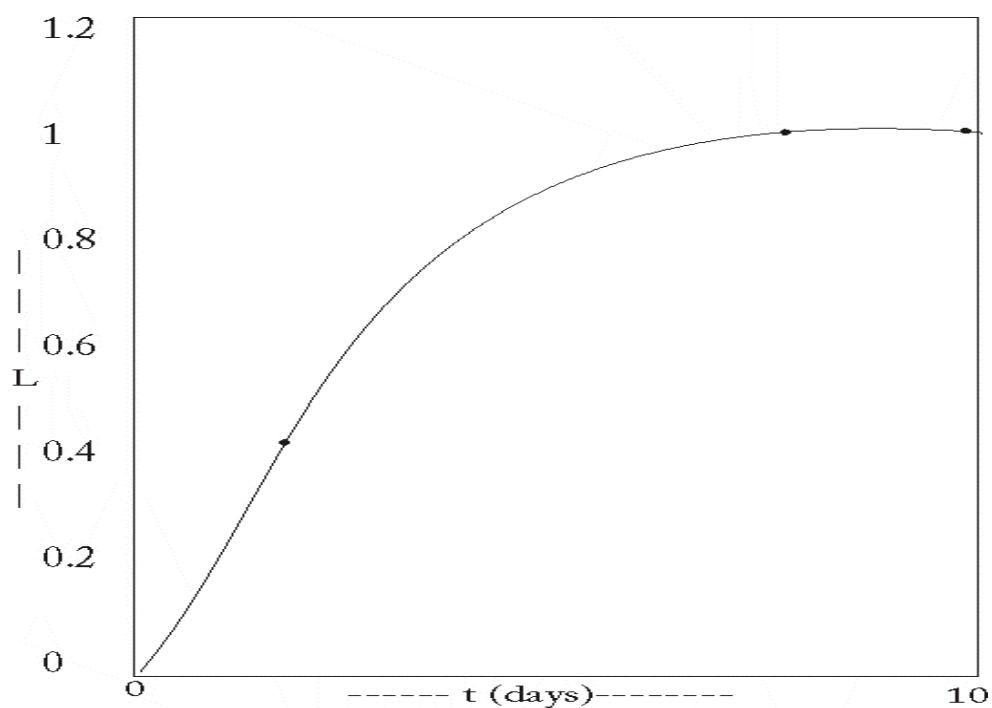


Figure- 1  
Biodegradation against  $t$

Fig. 1 Biodegradation against  $t$

Mathematical model for self purification in river

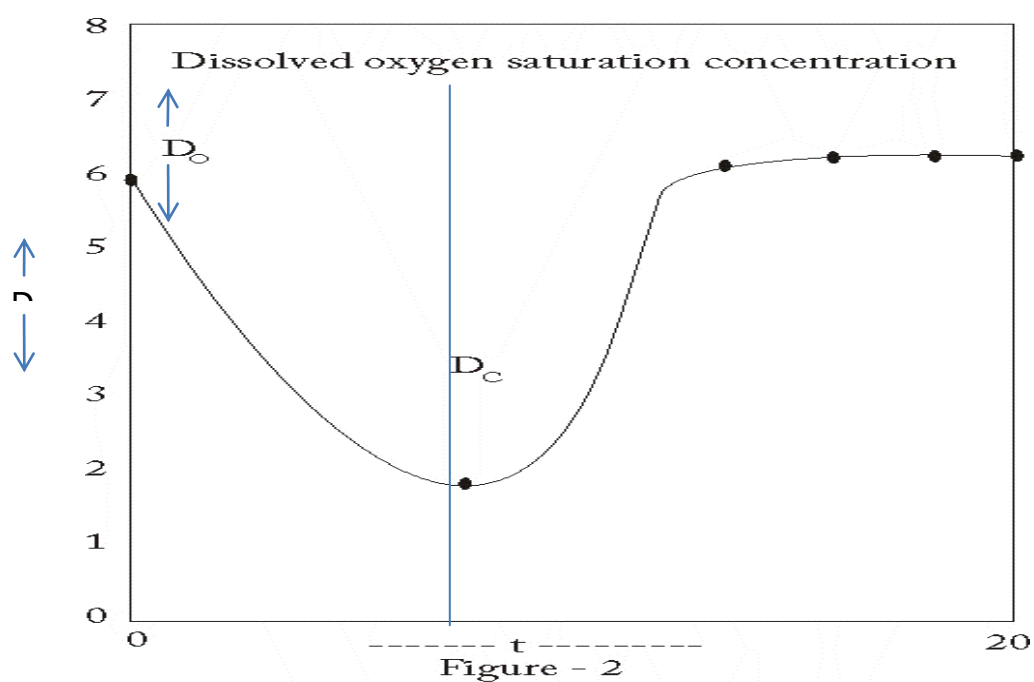


Figure - 2  
Dissolved oxygen Sag - Curve

Figure 2 Dissolved Oxygen Sag-Curve

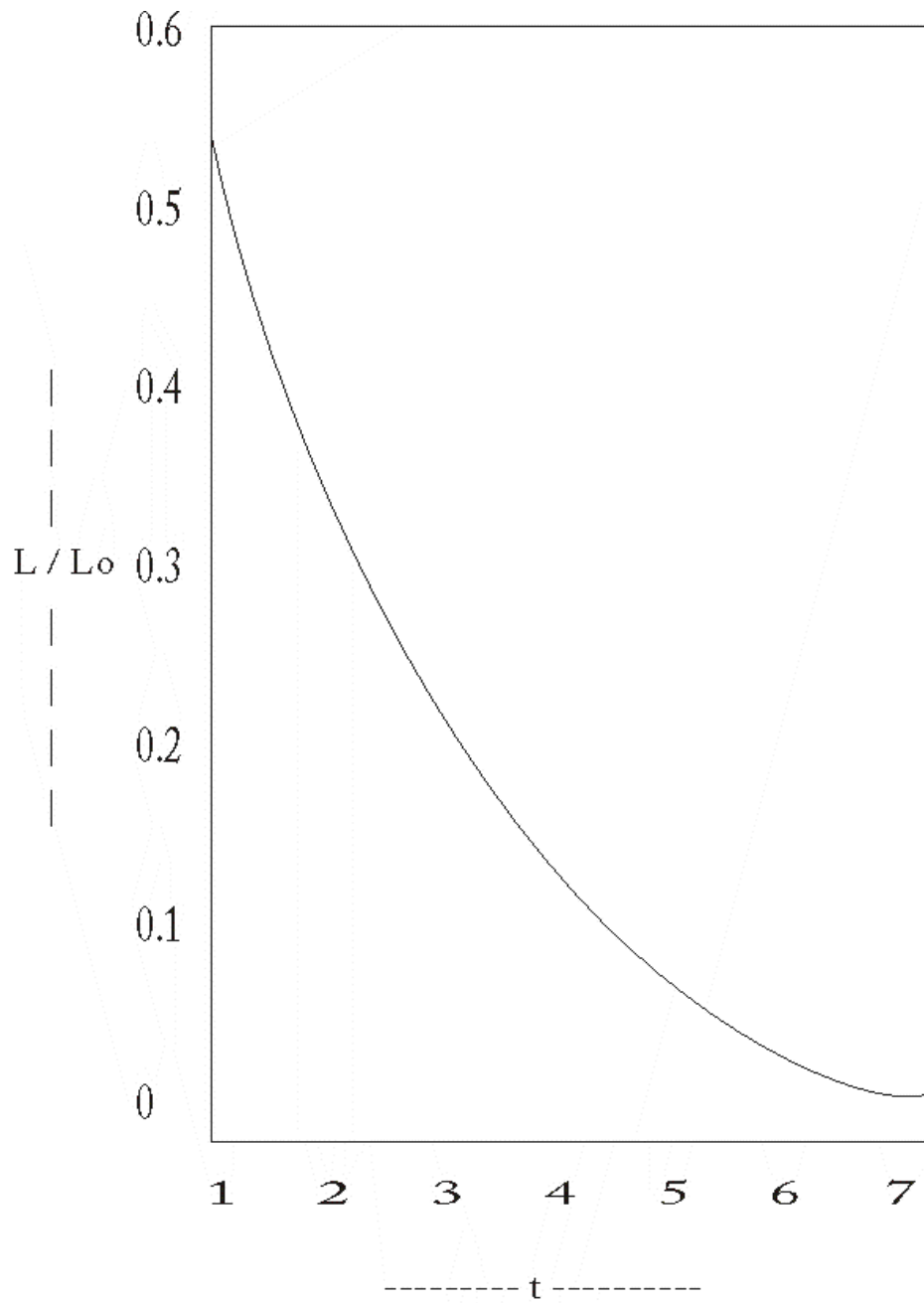


Figure - 3  
Relation Between  $D_c / L_0$  &  $k_1 / k_2$  Graph of

Figure 3 Graph of  $L/L_0$  against  $t$

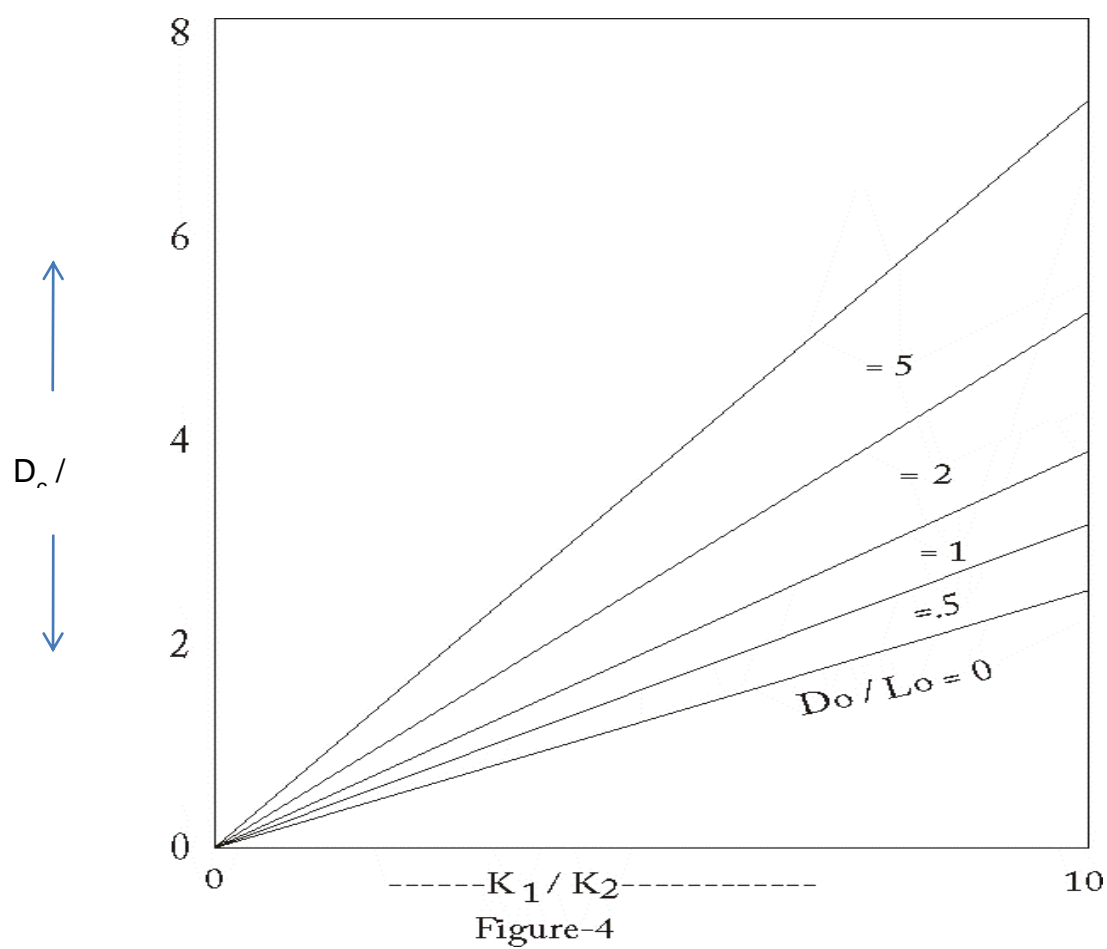


Figure 4 Relation Between  $D_c/L_o$  &  $K_1/K_2$

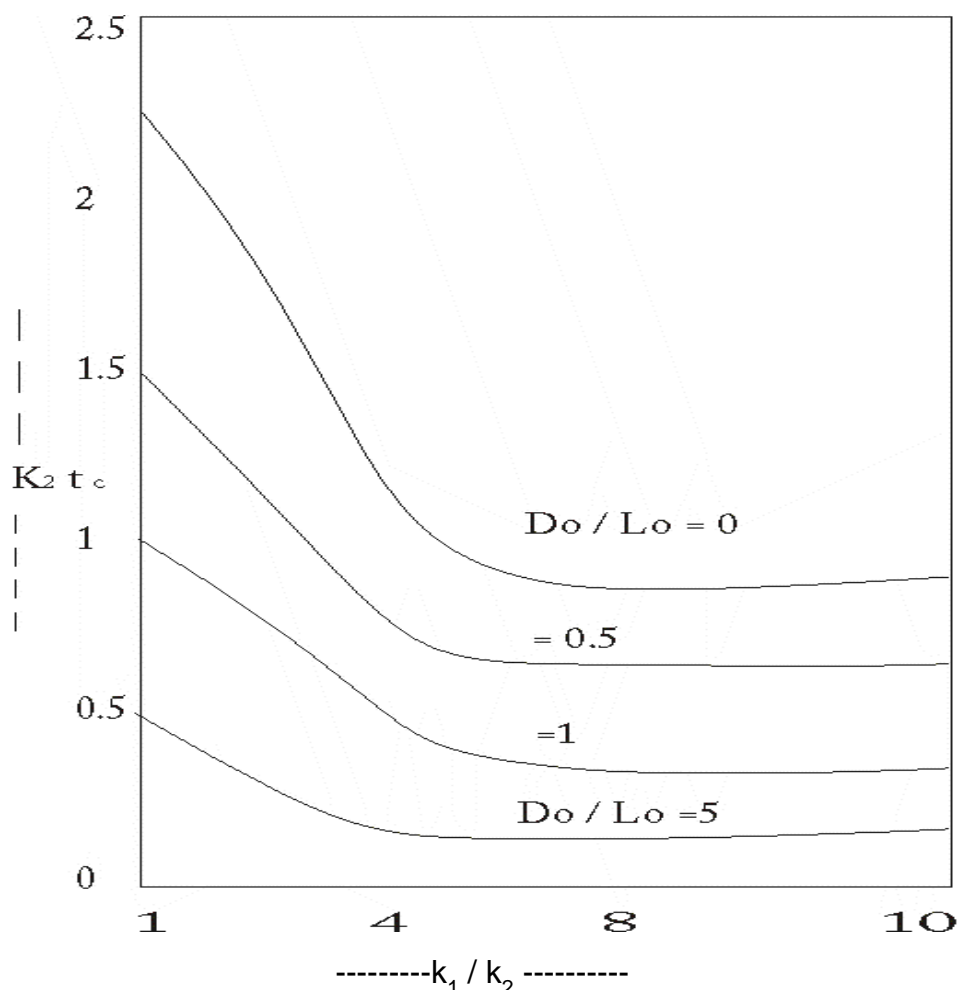


Figure- 5

Graph of  $k_2 t_c$  against  $k_1 / k_2$

Figure 5 Graph of  $K_2 t_c$  against  $k_1 / k_2$

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